Principles of Complex Systems | @pocsvox CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

# Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center Vermont Advanced Computing Core | University of Vermont























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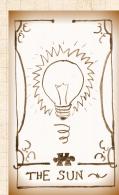
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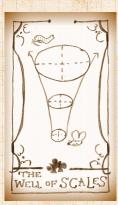
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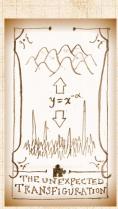
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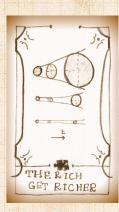
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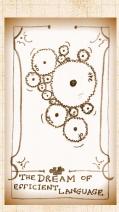
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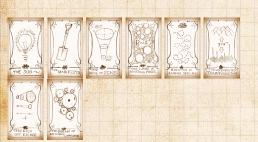
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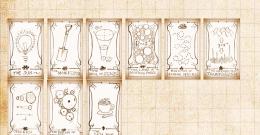
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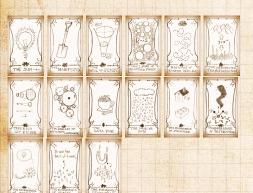
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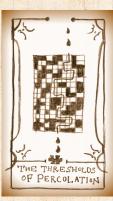
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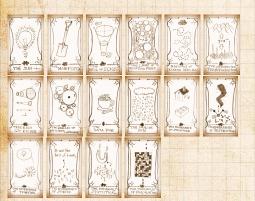
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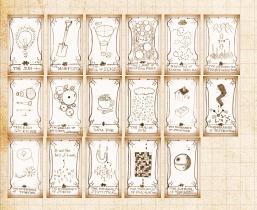
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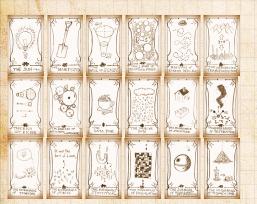
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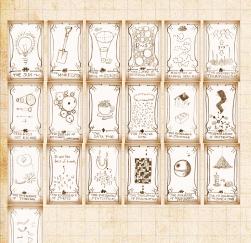
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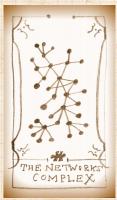
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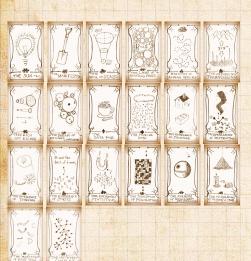
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Networks with power-law degree distributions have become known as scale-free networks.



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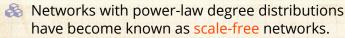
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Scale-free refers specifically to the degree distribution having a power-law decay in its tail:



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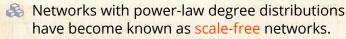
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Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

 $P_k \sim k^{-\gamma}$  for 'large' k



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One of the seminal works in complex networks:



"Emergence of scaling in random networks" 
Barabási and Albert,
Science, **286**, 509–511, 1999. [2]

Times cited:  $\sim 23,532$  (as of October 8, 2015)

Somewhat misleading nomenclature...

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### Scale-free networks are not fractal in any sense.

Usually talking about networks whose links are abstrace, relational, informational, ...(non-physical Primary example: hyperlink network of the Web Much arguing about whether or networks are 'scale-free' or not.

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# Some real data (we are feeling brave):

# From Barabási and Albert's original paper [2]:

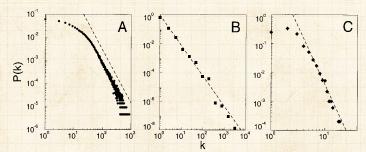


Fig. 1. The distribution function of connectivities for various large networks. (A) Actor collaboration graph with N=212,250 vertices and average connectivity  $\langle k \rangle=28.78$ . (B) WWW, N=325,729,  $\langle k \rangle=5.46$  (c) Power grid data, N=4941,  $\langle k \rangle=2.67$ . The dashed lines have slopes (A)  $\gamma_{\rm actor} = 2.3$ , (B)  $\gamma_{\rm www} = 2.1$  and (C)  $\gamma_{\rm power} = 4$ .

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# Random networks: largest components









$$\gamma = 2.5$$
 $\langle k \rangle = 1.8$ 

 $\gamma = 2.5$  $\langle k \rangle = 2.05333$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.66667$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.92$ 







 $\gamma = 2.5$  $\langle k \rangle = 1.62667$ 



 $\gamma = 2.5$  $\langle k \rangle = 1.8$ 

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### The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are. PoCS | @pocsvox

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## The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

## A big deal for scale-free networks:

 $\ref{how}$  How does the exponent  $\gamma$  depend on the mechanism?

Do the mechanism details matter?

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## The big deal:

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### Barabási-Albert model = BA model.

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Barabási-Albert model = BA model.



Key ingredients:

Growth and Preferential Attachment (PA).

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- 🙈 Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- $\clubsuit$  Step 1: start with  $m_0$  disconnected nodes.
- Step 2:

  1. Growth—a new node appears at each time step
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to *i*th node is  $\propto k_i$ .

In essence, we have a rich gets-richer scheme. Yes, we've seen this all before in Simon's mode

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### $\bigotimes$ Definition: $A_k$ is the attachment kernel for a node with degree k.

$$A_k = k$$

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

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For the original model:

$$A_k = k$$

 $Arr Definition: P_{\mathsf{attach}}(k,t)$  is the attachment probability.

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 $Arr Definition: P_{\mathsf{attach}}(k,t)$  is the attachment probability.

For the original model:

$$P_{\mathrm{attach}}(\mathsf{node}\ i,t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\mathrm{max}}(t)} k_j(t)} =$$

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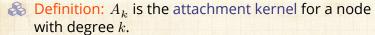
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For the original model:

$$A_k = k$$

- $ightharpoonup extstyle{ } extstyle{ }$
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\text{max}}} k_j(t)} = \frac{k_i(t)}{$$

where  $N(t) = m_0 + t$  is # nodes at time t

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- Definition:  $A_k$  is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- ightharpoonup Definition:  $P_{\mathsf{attach}}(k,t)$  is the attachment probability.
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where  $N(t) = m_0 + t$  is # nodes at time t and  $N_k(t)$  is # degree k nodes at time t.

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 $\aleph$  When (N+1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

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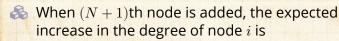
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$$E(k_{i,\,N+1}-k_{i,\,N}) \simeq m \frac{k_{i,\,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

Assumes probability of being connected to is small.

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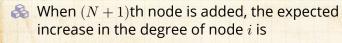
Universality?

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$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_{j}(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.

Approximate  $k_{i,N+1} - k_{i,N}$  with  $\frac{d}{dt}k_{i,N}$ 

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- $\Leftrightarrow$  Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ :

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)}$$

where  $t = N(t) - m_0$ .

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Deal with denominator: each added node brings m new edges.

$$\sum_{i=1}^{N(t)} k_j(t) = 2tm$$

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$$

$$\frac{\mathsf{d}k_i(t)}{k_i(t)} = \frac{\mathsf{d}t}{2t}$$

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$$\div \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

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Rearrange and solve

$$\frac{\mathsf{d}k_i(t)}{k_i(t)} = \frac{\mathsf{d}t}{2t}$$

Next find

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Next find

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 $\red {\mathbb R}$  Next find  $c_i$  ...

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### Know ith node appears at time

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

$$k_i(t) = m \left(rac{t}{t_i \, ext{start}}
ight)^{1/2} \, ext{for } t \geq t_i \, ext{start}$$

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3 So for  $i > m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\,\mathrm{start}}}\right)^{1/2} \, \mathrm{for} \, t \geq t_{i,\,\mathrm{start}}.$$

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All node degrees grow as  $t^{1/2}$  but later nodes have

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- All node degrees grow as  $t^{1/2}$  but later nodes have larger  $t_{i,\text{start}}$  which flattens out growth curve.
- First-mover advantage: Early nodes do best.

Clearly, a

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### We are already at the Zipf distribution:



 $\mathbb{R}$  Degree of node i is the size of the ith ranked node:

$$k_i(t) = m \left(\frac{t}{t_{i, \mathrm{start}}}\right)^{1/2} \ \mathrm{for} \ t \geq t_{i, \mathrm{start}}.$$

$$t_{i, \text{start}} = \left\{ \begin{array}{ll} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{array} \right.$$

$$k_i \propto i^{-1/2} = i^{-\alpha} \qquad -$$

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From before:

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

so  $t_{i,\text{start}} \sim i$  which is the rank.

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### We are already at the Zipf distribution:

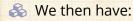
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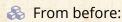


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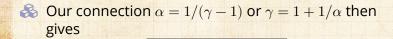
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We then have:

$$k_i \propto i^{-1/2} = i^{-\alpha}$$
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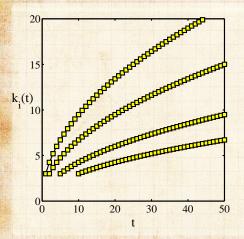
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m=3



 $\& t_{i,\text{start}} =$ 1, 2, 5, and 10.

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### & So what's the degree distribution at time t?

$$\mathbf{Pr}(t_{i, \mathsf{start}}) \mathsf{d}t_{i, \mathsf{start}} \simeq \frac{\mathsf{d}t_{i, \mathsf{start}}}{t}$$

$$k_i(t) = m \left( \begin{array}{c|c} t \\ \hline t_i & \text{start} \end{array} \right)^{1/2} \rightarrow t_i, \text{start} = \frac{m^2 t}{k_i(t)^2}$$

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& So what's the degree distribution at time t?



Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}t_{i, \mathrm{start}} \simeq \frac{\mathrm{d}t_{i, \mathrm{start}}}{t}$$

$$k_i(t) = m \left(\frac{t}{t_{i, \, \mathrm{start}}}\right)^{1/2} + t_i$$

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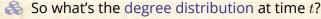
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🙈 Also use

$$k_i(t) = m \left(\frac{t}{t_{i, \mathrm{start}}}\right)^{1/2} \Rightarrow t_{i, \mathrm{start}} = \frac{m^2 t}{k_i(t)^2}.$$

Transform variables—lacobian:

$$\frac{\mathrm{d}t_{i,\mathrm{start}}}{\mathrm{d}k_i} = -2\frac{m^2t}{k_i(t)^3}$$

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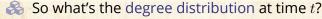
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$$\mathbf{Pr}(k_i) \mathrm{d}k_i = \mathbf{Pr}(t_{i,\mathrm{start}}) \mathrm{d}t_{i,\mathrm{start}}$$

$$= \mathbf{Pr}(t_{i, \, \mathsf{start}}) \mathsf{d} k_i \, \left| \frac{\mathsf{d} t_{i, \, \mathsf{start}}}{\mathsf{d} k_i} \right|$$

$$\frac{1}{t} dk_i 2 \frac{m^2 t}{k_i (t)^3}$$

$$=2\frac{m^2}{k_i(t)^3}\mathsf{d}k_i$$

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$$=\frac{1}{t}\mathrm{d}k_i\,2\frac{m^2t}{k_i(t)^3}$$

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$$=\frac{1}{t}\mathrm{d}k_i\,2\frac{m^2t}{k_i(t)^3}$$



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$$\mathbf{Pr}(k_i) \mathrm{d}k_i = \mathbf{Pr}(t_{i,\mathrm{start}}) \mathrm{d}t_{i,\mathrm{start}}$$



$$= \mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}k_i \left| \frac{\mathrm{d}t_{i, \mathrm{start}}}{\mathrm{d}k_i} \right|$$



$$=\frac{1}{t}\mathrm{d}k_i\,2\frac{m^2t}{k_i(t)^3}$$



$$=2\frac{m^2}{k_i(t)^3}\mathsf{d}k_i$$



 $\propto k_i^{-3} \mathrm{d} k_i$ .

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We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .

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In practice,  $\gamma < 3$  means variance is governed by upper cutoff.

3: finite mean and variance

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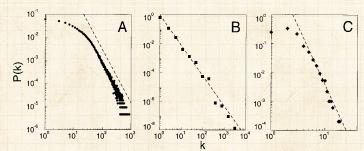
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### Back to that real data:

### From Barabási and Albert's original paper [2]:



**Fig. 1.** The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with N=212,250 vertices and average connectivity  $\langle k \rangle = 28.78$ . **(B)** WWW, N=325,729,  $\langle k \rangle = 5.46$  **(6)**. **(C)** Power grid data, N=4941,  $\langle k \rangle = 2.67$ . The dashed lines have slopes (A)  $\gamma_{\rm actor} = 2.3$ , (B)  $\gamma_{\rm www} = 2.1$  and (C)  $\gamma_{\rm power} = 4$ .

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## Examples

 $\gamma \simeq 2.1$  for in-degree Web  $\gamma \simeq 2.45$  for out-degree Web Movie actors  $\gamma \simeq 2.3$ Words (synonyms)  $\gamma \simeq 2.8$ 

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The Internets is a different business...

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# Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring

Deal with directed versus undirected networks.

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## Things to do and questions



Vary attachment kernel.



Vary mechanisms:

- 1. Add edge deletion
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Deal with directed versus undirected networks.



Important Q.: Are there distinct universality classes for these networks?

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- $\bigcirc$  Q.: How does changing the model affect  $\gamma$ ?

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## Things to do and questions

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- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: Do we need preferential attachment and growth?

Do model details matter?

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## Things to do and questions

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Vary attachment kernel.

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Vary mechanisms:

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2. Add node deletion

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Important Q.: Are there distinct universality classes for these networks?

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 $\mathfrak{S} = \mathbb{Q}$ .: How does changing the model affect  $\gamma$ ?

References

Q.: Do we need preferential attachment and growth?

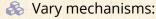


💫 Q.: Do model details matter? Maybe 🗔



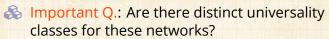
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### 🚵 Let's look at preferential attachment (PA) a little more closely.

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PA implies arriving nodes have complete knowledge of the existing network's degree distribution.

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For example: If  $P_{\rm attach}(k) \propto k$ , we need to determine the constant of proportionality.

We need to know what everyone's degree is

PA is an outrageous assumption of node capability.

But a very simple mechanism saves the day.

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Instead of attaching preferentially, allow new nodes to attach randomly.

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Instead of attaching preferentially, allow new nodes to attach randomly.



Now add an extra step: new nodes then connect to some of their friends' friends.

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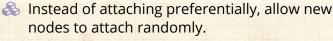
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- Instead of attaching preferentially, allow new nodes to attach randomly.
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 $Q_k \propto kP_k$ 

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So rich-gets-richer scheme can now be seen to work in a natural way.

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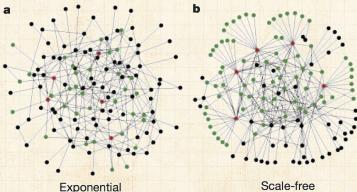






Albert et al., Nature, 2000: "Error and attack tolerance of complex networks" [1]

Standard random networks (Erdős-Rényi) versus Scale-free networks:



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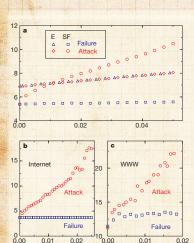
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from Albert et al., 2000

Plots of network diameter as a function of fraction of nodes

Erdős-Rényi versus scale-free networks

removed

- blue symbols = random removal
  - red symbols = targeted removal (most connected first)

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### Scale-free networks are thus robust to random failures yet fragile to targeted ones.

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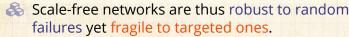
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Scale-free networks

All very reasonable: Hubs are a big deal.

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But: next issue is whether hubs are vulnerable or

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Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)

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Need to explore cost of various targeting schemes.



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1. Physically larger nodes that may be harder to 'target'

2 or subnetworks of smaller, normal-sized nodes.

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### Not a robust paper:



"The "Robust yet Fragile" nature of the Internet"

Doyle et al., Proc. Natl. Acad. Sci., 2005, 14497-14502, 2005. [3]

- HOT networks versus scale-free networks
- Same degree distributions, different arrangements.
- Doyle et al. take a look at the actual Internet.
- Excellent project material.

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### Fooling with the mechanism:

2001: Krapivsky & Redner (KR)<sup>[4]</sup> explored the general attachment kernel:

 $\mathbf{Pr}(\mathsf{attach}\ \mathsf{to}\ \mathsf{node}\ i) \propto A_k =$ 

where  $A_k$  is the attachment kernel and  $\nu > 0$ . KR also looked at changing the details of the attachment kernel.

KR model will be fully studied in CoNKS.

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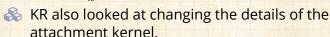


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We'll follow KR's approach using rate equations 
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We'll follow KR's approach using rate equations 

C.



Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

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- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.

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- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.
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We'll follow KR's approach using rate equations 
✓.



Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

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- 6. Detail:  $A_0 = 0$

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In general, probability of attaching to a specific node of degree k at time t is

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$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

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 $\clubsuit$  E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .

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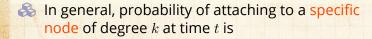
#### Analysis

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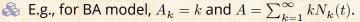






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.



$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2$$

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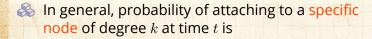
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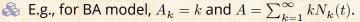






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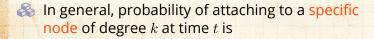
#### Analysis

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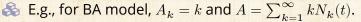






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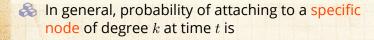
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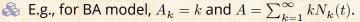






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since one edge is being added per unit time.

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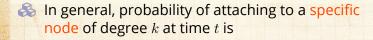
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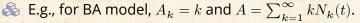






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since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

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So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

$$k = \frac{1}{2!} \left[ (k-1)n_{k+1} / - kn_k / \right] + \delta_{k1}$$

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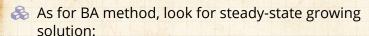


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As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .



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 $\frac{\mathrm{d}N_{k}}{\mathrm{d}t} = \frac{1}{4} \left[ A_{k-1} N_{k-1} - A_{k} N_{k} \right] + \delta_{k1}$ 

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .
- $Arr N_k$  We replace  $dN_k/dt$  with  $dn_k t/dt = n_k$ .
- We arrive at a difference equation:

$$n_{k} = \frac{1}{2 \textcolor{red}{t}} \left[ (k-1) n_{k-1} \textcolor{red}{t} - k n_{k} \textcolor{red}{t} \right] + \delta_{k1}$$

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As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

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But we'll first explore a more subtle modification of  $A_k$  made by Krapivsky/Redner Keep  $A_k$  linear in k but tweak details.

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- & Keep  $A_k$  linear in k but tweak details.
- $\clubsuit$  Idea: Relax from  $A_k = k$  to  $A_k \sim k$  as  $k \to \infty$ .

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Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

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where we only know the asymptotic behavior of  $A_k$ .

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assumption is consistent.

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where we only know the asymptotic behavior of  $A_k$ .

- $\red{\$}$  We assume that  $A = \mu t$
- $\clubsuit$  As before, also assume  $N_k(t) = n_k t$ .

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 $\Re$  For  $A_k = k$  we had

$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

$$n_k = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_k n_k] + \delta_k$$

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$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

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$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

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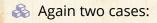


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$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$



$$\frac{k}{k} = 1 : n_1 = \frac{\mu}{\mu + A_1};$$

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$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$



Again two cases:

$$\frac{k=1}{\mu+A_1}; \qquad \frac{k}{k}>1: n_k=n_{k-1}\frac{A_{k-1}}{\mu+A_k}.$$

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Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .

$$\hat{n_k} = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \circ$$

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Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu - 1}$$

Since  $\mu$  depends on  $A_k$ , details matter

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- $\triangle$  Our assumption that  $A = \mu t$  looks to be not too horrible.

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### $A_1 = \alpha$ and $A_k = k$ for $k \ge 2$ .

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}$$

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

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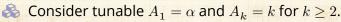
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Craziness

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Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

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$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + {\rm correction\ terms}}$$
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Universality: now details of kernel do not matter. Distribution of degree is universal providing v <

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### Details:



**Solution** For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1+\nu} + \frac{\mu^2}{2} \frac{k^{1-2}}{1+2\nu}}$$

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### Details:



 $\Re$  For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$



 $\Leftrightarrow$  For  $1/3 < \nu < 1/2$ :

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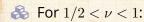
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 $And for 1/(r+1) < \nu < 1/r$ , we have r pieces in exponential.

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One single node ends up being connected to almost all other nodes.

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Rich-get-much-richer:

 $A_{\nu} \sim k^{\nu}$  with  $\nu > 1$ .

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- $\Rightarrow$  For  $\nu > 2$ , all but a finite # of nodes connect to one node.

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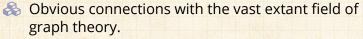
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## Overview Key Points for Models of Networks:



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## Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.

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  - 1. Description: Characterizing very large networks
  - 2. Explanation: Micro story ⇒ Macro features

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- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...

Still much work to be done, especially with respect to dynamics.

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# Neural reboot (NR):

Turning the corner:

https://www.youtube.com/v/axrTxEVQqN4?rel=0 2

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