

Scale-free networks

Principles of Complex Systems | @pocsvox
 CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

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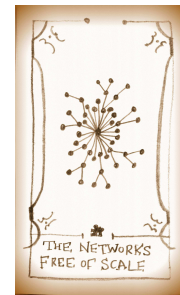
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Scale-free networks

- Networks with power-law degree distributions have become known as **scale-free** networks.
- Scale-free refers specifically to the **degree distribution** having a **power-law decay** in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

- One of the seminal works in complex networks:



"Emergence of scaling in random networks" Barabási and Albert, Science, **286**, 509-511, 1999. ^[2]

Times cited: **~ 23,532** (as of October 8, 2015)

- Somewhat misleading nomenclature...

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Scale-free networks

- Scale-free networks are **not fractal** in any sense.
- Usually talking about networks whose links are **abstract, relational, informational, ...**(non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...

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Some real data (we are feeling brave):

From Barabási and Albert's original paper [2]:

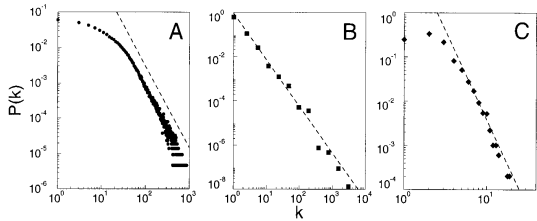


Fig. 1. The distribution function of connectivities for various large networks. (A) Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$. (B) WWW, $N = 325,729$, $\langle k \rangle = 5.46$ (6). (C) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

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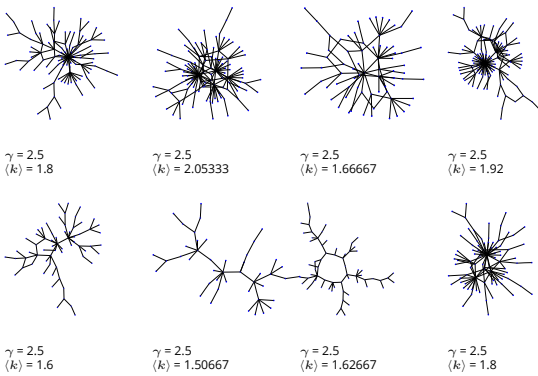
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Random networks: largest components



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Scale-free networks

The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

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BA model

- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
 - Growth—a new node appears at each time step $t = 0, 1, 2, \dots$
 - Each new node makes m links to nodes already present.
 - Preferential attachment—Probability of connecting to i th node is $\propto k_i$.
- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

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BA model

- Definition: A_k is the attachment kernel for a node with degree k .
- For the original model:

$$A_k = k$$
- Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t .

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Approximate analysis

- When $(N + 1)$ th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{d}{dt} k_{i,t}$:

$$\frac{d}{dt} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.

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- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{d}{dt} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

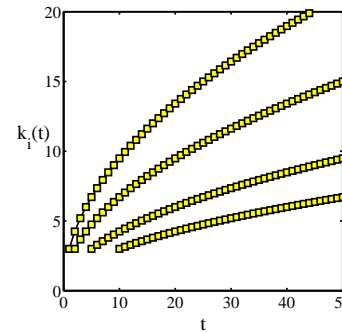
- Rearrange and solve:

$$\frac{dk_i(t)}{k_i(t)} = \frac{dt}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}}$$

- Next find c_i ...



Approximate analysis:



- $m = 3$
- $t_{i,start} = 1, 2, 5, \text{ and } 10.$



Approximate analysis

- Know i th node appears at time

$$t_{i,start} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,start}} \right)^{1/2} \text{ for } t \geq t_{i,start}$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,start}$ which flattens out growth curve.
- First-mover advantage: Early nodes do best.
- Clearly, a Ponzi scheme.



Degree distribution

- So what's the degree distribution at time t ?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t :

$$\Pr(t_{i,start}) dt_{i,start} \approx \frac{dt_{i,start}}{t}$$

- Also use

$$k_i(t) = m \left(\frac{t}{t_{i,start}} \right)^{1/2} \Rightarrow t_{i,start} = \frac{m^2 t}{k_i(t)^2}$$

Transform variables—Jacobian:

$$\frac{dt_{i,start}}{dk_i} = -2 \frac{m^2 t}{k_i(t)^3}$$



Approximate analysis

We are already at the Zipf distribution:

- Degree of node i is the size of the i th ranked node:

$$k_i(t) = m \left(\frac{t}{t_{i,start}} \right)^{1/2} \text{ for } t \geq t_{i,start}$$

- From before:

$$t_{i,start} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

so $t_{i,start} \sim i$ which is the rank.

- We then have:

$$k_i \propto i^{-1/2} = i^{-\alpha}$$

- Our connection $\alpha = 1/(\gamma - 1)$ or $\gamma = 1 + 1/\alpha$ then gives

$$\boxed{\gamma = 1 + 1/(1/2) = 3.}$$



Degree distribution

$$\Pr(k_i) dk_i = \Pr(t_{i,start}) dt_{i,start}$$

$$= \Pr(t_{i,start}) dk_i \left| \frac{dt_{i,start}}{dk_i} \right|$$

$$= \frac{1}{t} dk_i 2 \frac{m^2 t}{k_i(t)^3}$$

$$= 2 \frac{m^2}{k_i(t)^3} dk_i$$

$$\propto k_i^{-3} dk_i$$



Degree distribution

- We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- $\gamma > 3$: finite mean and variance (mild)

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Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter? Maybe ...

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Back to that real data:

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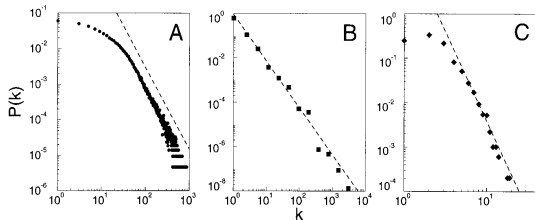


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Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have **complete knowledge** of the existing network's degree distribution.
- For example: If $P_{attach}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- PA is \therefore an **outrageous** assumption of node capability.
- But a **very simple mechanism** saves the day...

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Examples

Web	$\gamma \approx 2.1$ for in-degree
Web	$\gamma \approx 2.45$ for out-degree
Movie actors	$\gamma \approx 2.3$
Words (synonyms)	$\gamma \approx 2.8$

The Internet is a different business...

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Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an **extra step**: new nodes then connect to some of their friends' friends.
- Can also do this **at random**.
- Assuming the existing network is random, we know probability of a **random friend** having degree k is

$$Q_k \propto kP_k$$

- So **rich-gets-richer** scheme can now be seen to work in a natural way.

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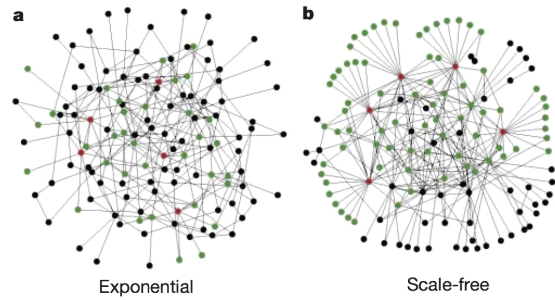
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Robustness

- Albert et al., Nature, 2000: "Error and attack tolerance of complex networks" [1]
- Standard random networks (Erdős-Rényi) versus Scale-free networks:



from Albert et al., 2000

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Robustness

Not a robust paper:



"The "Robust yet Fragile" nature of the Internet" [3]
Doyle et al.,
Proc. Natl. Acad. Sci., 2005, 14497-14502, 2005. [3]

- HOT networks versus scale-free networks
- Same degree distributions, different arrangements.
- Doyle *et al.* take a look at the actual Internet.
- Excellent project material.

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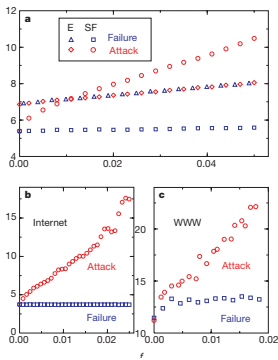
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Robustness



from Albert et al., 2000

- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

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Generalized model

Fooling with the mechanism:

- 2001: Krapivsky & Redner (KR) [4] explored the **general attachment kernel**:

$$\Pr(\text{attach to node } i) \propto A_k = k_i^\nu$$

where A_k is the attachment kernel and $\nu > 0$.

- KR also looked at changing the details of the attachment kernel.
- KR model will be fully studied in CoNKS.

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Robustness

- Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.
- All very reasonable: Hubs are a big deal.
- But**: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - Physically larger nodes that may be harder to "target"
 - or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

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Generalized model

- We'll follow KR's approach using rate equations [5].
- Here's the set up:

$$\frac{dN_k}{dt} = \frac{1}{A} [A_{k-1}N_{k-1} - A_kN_k] + \delta_{k1}$$

where N_k is the number of nodes of degree k .

- One node with one link is added per unit time.
- The **first term** corresponds to degree $k - 1$ nodes becoming degree k nodes.
- The **second term** corresponds to degree k nodes becoming degree $k - 1$ nodes.
- A is the correct normalization (coming up).
- Seed with some initial network (e.g., a connected pair)
- Detail: $A_0 = 0$

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Generalized model

- In general, probability of attaching to a **specific node** of degree k at time t is

$$\Pr(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.

- For $A_k = k$, we have

$$A(t) = \sum_{k=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

- Detail: we are ignoring initial seed network's edges.

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- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \approx 2t \text{ for large } t.$$

- We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- We assume that $A = \mu t$
- We'll find μ later and make sure that our assumption is consistent.
- As before, also assume $N_k(t) = n_k t$.

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Generalized model

- So now

$$\frac{dN_k}{dt} = \frac{1}{A} [A_{k-1} N_{k-1} - A_k N_k] + \delta_{k1}$$

becomes

$$\frac{dN_k}{dt} = \frac{1}{2t} [(k-1)N_{k-1} - kN_k] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- We replace dN_k/dt with $dn_k t/dt = n_k$.
- We arrive at a difference equation:

$$n_k = \frac{1}{2t} [(k-1)n_{k-1}t - kn_k t] + \delta_{k1}$$

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- For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

- This now becomes

$$n_k = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_k n_k] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1} n_{k-1} + \mu \delta_{k1}$$

- Again two cases:

$$k=1 : n_1 = \frac{\mu}{\mu + A_1}; \quad k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k ?
- Again, we're asking if the result $\gamma = 3$ universal?
- KR's natural modification: $A_k = k^\nu$ with $\nu \neq 1$.
- But we'll first explore a more subtle modification of A_k made by Krapivsky/Redner^[4]
- Keep A_k linear in k but tweak details.
- Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \rightarrow \infty$.

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Universality?

- Time for pure excitement: Find **asymptotic behavior** of n_k given $A_k \rightarrow k$ as $k \rightarrow \infty$.

- For large k , we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu-1}$$

- Since μ depends on A_k , **details matter...**

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Universality?

- Now we need to find μ .
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- Closed form expression for μ .
- We can solve for μ in some cases.
- Our assumption that $A = \mu t$ looks to be not too horrible.

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Sublinear attachment kernels

Details:

- For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu} - 2^{1-\nu}}{1-\nu} \right)}$$

- For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

- And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.

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Universality?

- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- Again, we can find $\gamma = \mu + 1$ by finding μ .
- Closed form expression for μ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun

- $\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}$.

- Since $\gamma = \mu + 1$, we have

$$0 \leq \alpha < \infty \Rightarrow 2 \leq \gamma < \infty$$

- Craziness...

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Superlinear attachment kernels

- Rich-get-much-richer:

$$A_k \sim k^\nu \text{ with } \nu > 1.$$

- Now a **winner-take-all** mechanism.
- One single node ends up being connected to almost all other nodes.
- For $\nu > 2$, all but a finite # of nodes connect to one node.

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Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^\nu \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality**: now details of kernel **do not matter**.
- Distribution of degree is universal providing $\nu < 1$.

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Nutshell:

Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
 - Description**: Characterizing very large networks
 - Explanation**: Micro story \Rightarrow Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics... **#excitement**

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