Mechanisms for Generating Power-Law Size Distributions, Part 1 Principles of Complex Systems | @pocsvox CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

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Variable transformation Basics Holtsmark's Distribution PLIPLO

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## Outline

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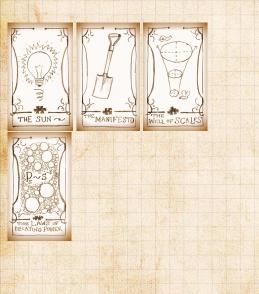
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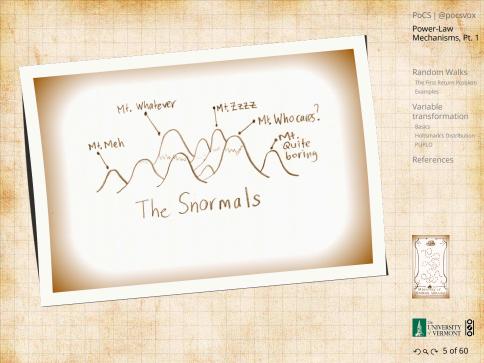


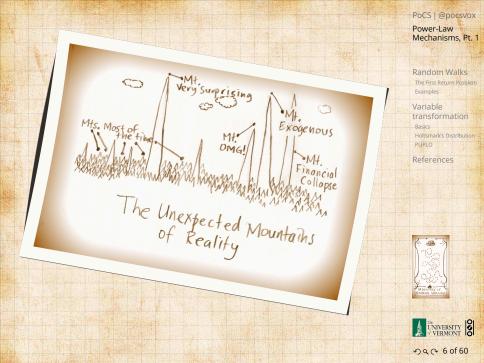


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### A powerful story in the rise of complexity:

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A powerful story in the rise of complexity: structure arises out of randomness.

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A powerful story in the rise of complexity: structure arises out of randomness. Exhibit A: Random walks.

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A powerful story in the rise of complexity:
Structure arises out of randomness.
Exhibit A: Random walks. C

The essential random walk:

🚳 One spatial dimension.

Time and space are discrete Random walker (e.g., a drunk) starts at origin x = 0.

> +1 with probability 1/2 +1 with probability 1/2

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A powerful story in the rise of complexity: structure arises out of randomness. Exhibit A: Random walks.

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A powerful story in the rise of complexity: structure arises out of randomness. Exhibit A: Random walks.

### The essential random walk:

- 🚳 One spatial dimension.
- 🚳 Time and space are discrete
- Random walker (e.g., a drunk) starts at origin x = 0.

### $\mathfrak{B}$ Step at time t is $\epsilon_t$ :

 $\epsilon_t = \begin{cases} +1 & \text{with probability 1/2} \\ -1 & \text{with probability 1/2} \end{cases}$ 

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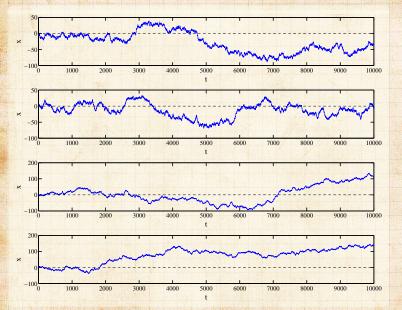
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### A few random random walks:



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Displacement after *t* steps:

 $x_t = \sum_{i=1}^{\iota} \epsilon_i$ 

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Displacement after *t* steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle$$

At any time step, we 'expect' our drunkard to b back at the pub. Obviously fails for odd number of steps... But as time goes on, the chance of our drunkar lurching back to the pub must diminish, right?

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At any time step, we 'expect' our drunkard to be back at the pub.

- Obviously fails for odd number of steps... 2
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$$\mathsf{Var}(x_t) = \mathsf{Var}\left(\sum_{i=1}^t \epsilon_i\right)$$

\* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

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$$\operatorname{Var}(x_t) = \operatorname{Var}\left(\sum_{i=1}^t \epsilon_i\right)$$

$$=\sum_{i=1}^{\iota} \operatorname{Var}\left(\epsilon_{i}\right) =$$

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$$= \sum_{i=1}^{t} \operatorname{Var}\left(\epsilon_{i}\right) = \sum_{i=1}^{t} 1$$

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So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

A non-trivial scaling law arises out o

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So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

A non-trivial scaling law arises out of additive aggregation or accumulation.

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### Stock Market randomness:

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Also known as the bean machine **C**, the quincunx (simulation) **C**, and the Galton box.



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### Great moments in Televised Random Walks:

Plinko! T from the Price is Right.

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### Counting random walks:

Each specific random walk of length t appears with a chance  $1/2^t$ . We'll be more interested in how many random walks end up at the same place. Define N(i, j, t) as # distinct walks that start at x = i and end at x = j after t time steps. Random walk must displace by  $\pm (j - i)$  after tsteps.

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- Random walk must displace by +(j-i) after t steps.

lnsert question from assignment 3 🗹

$$N(i,j,t) = {t \choose (t+j-i)/2}$$

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How does  $P(x_t)$  behave for large t? Take time t = 2n to help ourselves.  $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$   $x_{2n}$  is even so set  $x_{2n} = 2k$ . Using our expression N(i, j, t) with i = 0, j = 2n, we have

For large *p*, the binomial deliciously approaches the Normal Distribution of Snoredom:

The whole is different from the parts

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 $rac{1}{3}$  Take time t = 2n to help ourselves.

 $x_{2n}$  is even so set  $x_{2n} = 2k$ . Using our expression N(i, j, t) with i = 0, j = 2, and t = 2n, we have

For large *n*, the binomial deliciously approaches the Normal Distribution of Snoredom:

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3 Take time t = 2n to help ourselves.  $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$ 

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 $\begin{array}{l} & \textbf{Take time } t=2n \text{ to help ourselves.} \\ & \texttt{\&} \quad x_{2n} \in \{0,\pm 2,\pm 4,\ldots,\pm 2n\} \\ & \texttt{\&} \quad x_{2n} \text{ is even so set } x_{2n}=2k. \end{array}$ 

For large *n*, the binomial deliciously approaches the Normal Distribution of Shoredom:

The whole is different from the parts

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 $rac{2}{8}$  Take time t = 2n to help ourselves.

- $\textcircled{3} x_{2n} \in \{0,\pm 2,\pm 4,\ldots,\pm 2n\}$
- $x_{2n}$  is even so set  $x_{2n} = 2k$ .

Solution Using our expression N(i, j, t) with i = 0, j = 2k, and t = 2n, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto {\binom{2n}{n+k}}$$

For large n, the binomial deliciously approaches the Normal Distribution of Shoredom:

The whole is different from the parts

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For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

Insert question from assignment 3 The whole is different from the parts. See also

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Insert question from assignment 3 <sup>C</sup>
The whole is different from the parts. #nutritious
See also: Stable Distributions <sup>C</sup>

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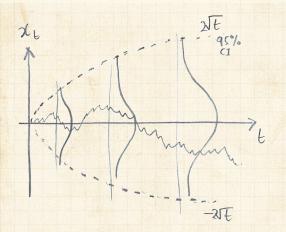
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## Universality C is also not left-handed:



This is Diffusion C: the most essential kind of spreading (more later).

🗞 View as Random Additive Growth Mechanism.

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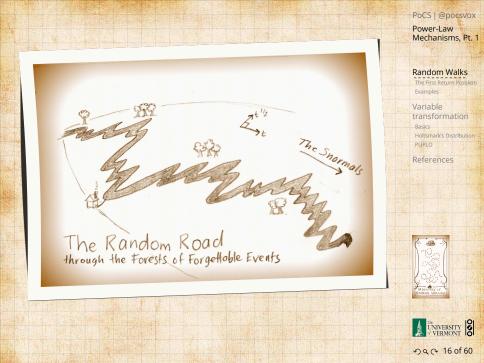
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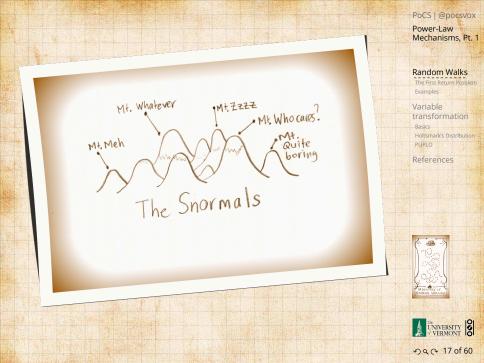
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 $\xi_{r,t}$  = the probability that by time step t, a random walk has crossed the origin r times. Think of a coin flip game with ten thousand tosses If you are behind early on, what are the chances you will make a comeback? The most likely number of lead changes is... In fact:  $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$ Even crazier: The expected time between tied scores =  $\infty$ 

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$${}_{\bigotimes}$$
 In fact:  $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdot$ 

\lambda Even crazier:

The expected time between tied scores =  $\infty$ 

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 In fact:  $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$ 

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See Feller, Intro to Probability Theory, Volume I<sup>[3]</sup>

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## Applied knot theory:



"Designing tie knots by random walks" Fink and Mao, Nature, **398**, 31–32, 1999.<sup>[4]</sup>





Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie. **a**, The two ways of beginning a knot,  $L_{\rm o}$  and  $L_{\rm o}$ . For knots beginning with  $L_{\rm o}$ , the tie must begin inside-out. **b**, The four-in-hand, denoted by the sequence  $L_{\rm o}$  R<sub>0</sub>  $L_{\rm o}$  C<sub>0</sub> T. **c**, A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk 1116.

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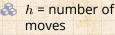
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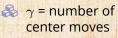
## Applied knot theory:

Table 1	Aesthetic	tie knots
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h γ	$\gamma/h$	K(h, γ)	S	b	Name	Sequence	1	
31	0.33	1	0	0		L₀R⊗C₀T	N.W.	
4 1	0.25	1	- 1	1	Four-in-hand	L <sub>⊗</sub> R <sub>☉</sub> L <sub>⊗</sub> C <sub>☉</sub> T		
5 2	0.40	2	- 1	0	Pratt knot	$L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$		
6 2	0.33	4	0	0	Half-Windsor	$L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$		
7 2	0.29	6	- 1	1		$L_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$	1000	
7 3	0.43	4	0	1		$L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$		
8 2	0.25	8	0	2		$L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$	1925	
8 3	0.38	12	- 1	0	Windsor	$L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$		
9 3	0.33	24	0	0		$L_{\odot}R_{\otimes}C_{\odot}L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$	100	
9 4	0.44	8	- 1	2		$L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$	1	

Knots are characterized by half-winding number h, centre number  $\gamma$ , centre fraction  $\gamma/h$ , knots per class  $K(h, \gamma)$ , symmetry s, balance b, name and sequence.





 $\bigotimes K(h,\gamma) =$  $2^{\gamma-1} \binom{h-\gamma-2}{\gamma-1}$ 

 $s = \sum_{i=1}^{h} x_i \text{ where } x = -1$ for L and +1 for R.

 $b = \frac{1}{2} \sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where  $\omega = \pm 1$ represents winding
direction.

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## Outline

### Random Walks The First Return Problem

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## The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the firs time after *t* steps?
- Will our drunkard always return to the origin What about higher dimensions?

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### Reasons for caring:

- We will find a power-law size distribution with an interesting exponent.
- Some physical structures may result from random walks.
- We'll start to see how different scalings relate each other.

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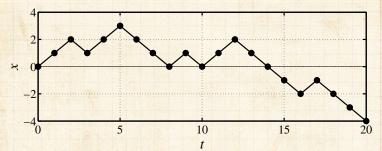
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### For random walks in 1-d:



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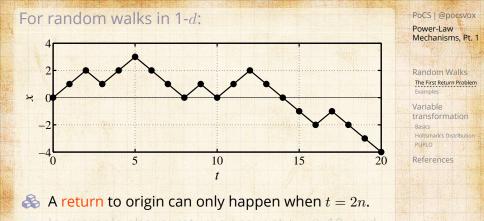
MINISTRY OF RADING WALKS

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A return to origin can only happen when t = 2n. In example above, returns occur at t = 8, 10, and14.

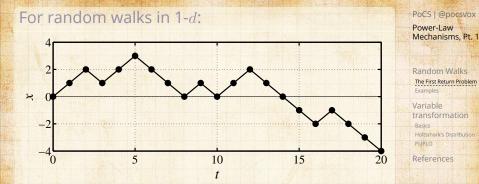
Probability calculation  $\equiv$  Counting problem (combinatorics/statistical mechanics). Idea: Transform first return problem into an easier return problem.



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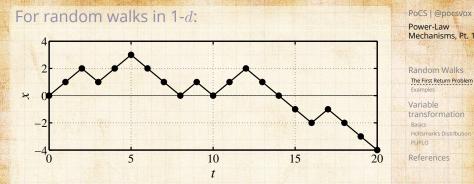


A return to origin can only happen when t = 2n. In example above, returns occur at t = 8, 10, and 14.

Call  $P_{fr(2n)}$  the probability of first return at t =Probability calculation = Counting problem (combinatorics/statistical mechanics). Idea: Transform first return problem into an easier return problem.







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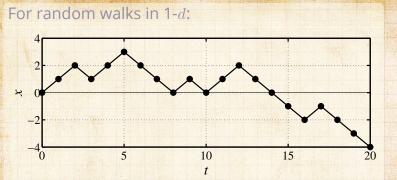
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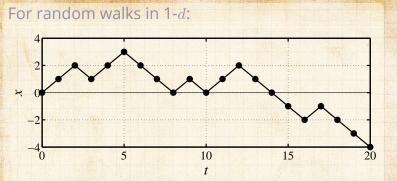
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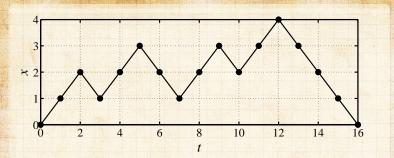
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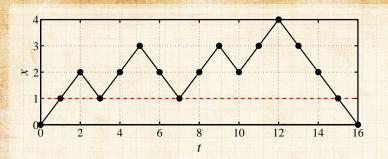
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Solution Can assume drunkard first lurches to x = 1.

Observe walk first returning at t = 16 stays at or above x = 1 for  $1 \le t \le 15$  (dashed red line). Now want walks that can return many times to x = 1.  $P_{\rm fr}(2n) =$  $2 \quad \frac{1}{2} Pr(x_t \ge 1, 1 \le t \le 2n - 1, \text{ and } x_1 = x_{2n-1} = 1)$ The  $\frac{1}{2}$  accounts for  $x_{2n} = 2$  instead of 0. The 2 accounts for drunkards that first lurch to x = -1



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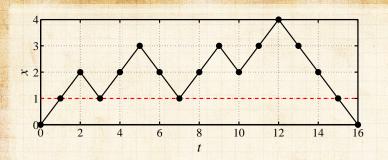
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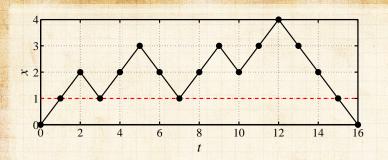
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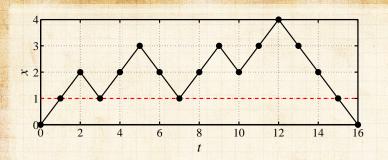
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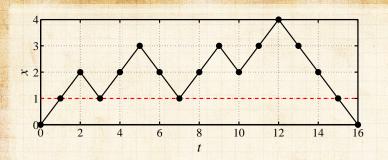
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## Approach:

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### Approach:



Move to counting numbers of walks.

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### Approach:



Move to counting numbers of walks. Return to probability at end.

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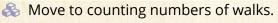
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## Approach:



- 🚳 Return to probability at end.
- Again, N(i, j, t) is the # of possible walks between x = i and x = j taking t steps.

Consider all paths starting at x = 1 and ending at x = 1 after t = 2n - 2 steps.

Idea: If we can compute the number of walks tha hit x = 0 at least once, then we can subtract this from the total number to find the ones that maintain  $x \ge 1$ .

Call walks that drop below x = 1 excluded walks We'll use a method of images to identify these excluded walks:

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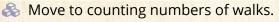
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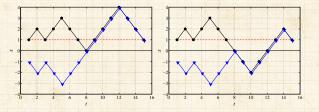
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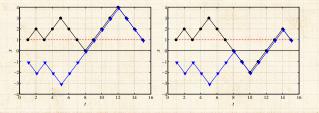


Key observation for excluded walks:

Solution For any path starting at x=1 that hits 0, there is a unique matching path starting at x=-1.

Matching path first mirrors and then tracks after first reaching x=0.

# of t-step paths starting and ending at x=1 a hitting x=0 at least once



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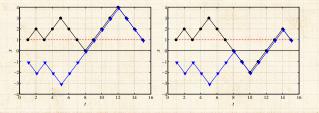
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Key observation for excluded walks:

- Solution For any path starting at x=1 that hits 0, there is a unique matching path starting at x=-1.
- Matching path first mirrors and then tracks after first reaching x=0.
  - # of *t*-step paths starting and ending at x=1 and hitting x=0 at least once



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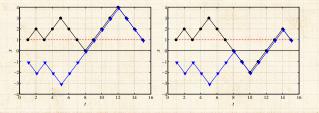
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### Key observation for excluded walks:

- So For any path starting at x=1 that hits 0, there is a unique matching path starting at x=-1.
- Matching path first mirrors and then tracks after first reaching x=0.
- $\Rightarrow$  # of *t*-step paths starting and ending at x=1 and hitting x=0 at least once







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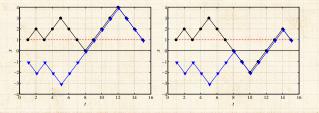
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# of t-step paths starting and ending at x=1 and hitting x=0 at least once = # of t-step paths starting at x=-1 and ending at x=1







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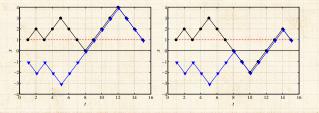
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# of t-step paths starting and ending at x=1 and hitting x=0 at least once = # of t-step paths starting at x=-1 and ending at x=1 = N(-1,1,t)



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 $\textcircled{So } N_{\rm first \ return}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$ 





## Probability of first return: Insert question from assignment 3 🗗 : Find

Normalized number of paths gives probabilit Total number of possible paths =  $2^{2n}$ 

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## Probability of first return: Insert question from assignment 3 🗹 : 🍣 Find

$$\boxed{N_{\rm fr}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.}$$

Normalized number of paths gives probability Total number of possible paths =  $2^{2n}$ .

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$$N_{\rm fr}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$



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Probability of first return: Insert question from assignment 3 🖸 :

🝰 Find

$$N_{\rm fr}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$

Solution Normalized number of paths gives probability. Total number of possible paths =  $2^{2n}$ .

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Probability of first return: Insert question from assignment 3 🖙 :

💑 Find

2

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## We have $P(t) \propto t^{-3/2}$ , $\gamma = 3/2$ . Same scaling holds for continuous space/time wal P(t) is normalizable.

- Recurrence: Random walker always returns to origi But mean, variance, and all higher moments are
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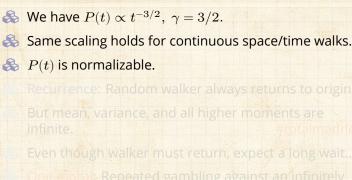
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Higher dimensions C:

Walker in d = 2 dimensions must also return. Walker may not return in d > 3 dimensions PoCS | @pocsvox

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## Higher dimensions C:



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### On finite spaces:

In any finite homogeneous space, a random walker will visit every site with equal probabilit Call this probability the Invariant Density of a dynamical system Non-trivial Invariant Densities arise in chaotic systems

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### On networks:

On networks, a random walker visits each nod with frequency  $\propto$  node degree for Equal probability still present: walkers traverse edges with equal frequency.

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Image: Second systemImage: Second syste

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#totallygroovy

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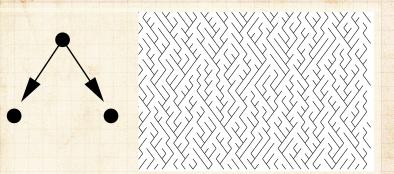
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# Scheidegger Networks <sup>[9, 2]</sup>



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MINISTRY OF RADON WALKS



Random directed network on triangular lattice.
 Toy model of real networks.
 'Flow' is southeast or southwest with equal

probability.

🚳 Creates basins with random walk boundaries.

Observe that subtracting one random walk fro another gives random walk with increments:

+1 with probability 1/4
0 with probability 1/2
-1 with probability 1/4

Random walk with probabilistic pauses. Basin termination = first return random wa problem. Basin length  $\ell$  distribution:  $P(\ell) \propto \ell^{-3/2}$ For real river networks, generalize to  $P(\ell) \propto$  PoCS | @pocsvox

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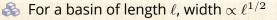
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Invert:  $\ell \propto a^{2/3}$   $d\ell \propto d(a^{2/3}) = 2/3a^{-1/3}da$  Pr(basin area = a)da= Pr(basin length = l)dl

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Solution For a basin of length  $\ell$ , width  $\propto \ell^{1/2}$ Basin area  $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$ 

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Both basin area and length obey power law distributions Observed for real river networks Reportedly:  $1.3 < \tau < 1.5$  and  $1.5 < \gamma < 2$ 

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Generalize relationship between area and length: Hack's law  $\ell \propto a^h$ . For real, large networks  $h \simeq 0.5$ Smaller basins possibly h > 1/2 (see: allometry). Models exist with interesting values of h. Plan: Redo calc with  $\gamma$ ,  $\tau$ , and h.

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 $\ell \propto a^h$ ,  $P(a) \propto a^{-\tau}$ ,  $d\ell \propto d(a^h) = ha^{h-1}da$ Find  $\tau$  in terms of  $\gamma$  and h. Pr(basin area = a)da $= Pr(basin length = \ell)d\ell$  PoCS | @pocsvox

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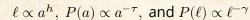
 $\wedge$ 

Excellent example of the Scaling Relations found between exponents describing power laws for many systems.



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Find au in terms of  $\gamma$  and h  $\mathbf{Pr}(basin area = a)da$  $= \mathbf{Pr}(basin length = \ell)d\ell$  PoCS | @pocsvox

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Excellent example of the Scaling Relations foun between exponents describing power laws for many systems.



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 $\ell \propto a^h$ ,  $P(a) \propto a^{-\tau}$ , and  $P(\ell) \propto \ell^{-\gamma}$ 

 $\mathfrak{A}$  d $\ell \propto \mathsf{d}(a^h) = ha^{h-1}\mathsf{d}a$ 

 $\mathbf{Pr}(basin area = a)da$ =  $\mathbf{Pr}(basin length = \ell)d\ell$ 

Excellent example of the Scaling Relations four between exponents describing power laws for many systems.

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🖂 Given

$$\ell \propto a^{m h}, \ P(a) \propto a^{- au}, \ {
m and} \ P(\ell) \propto \ell^{-\gamma}$$

 $\mathfrak{G} d\ell \propto \mathsf{d}(a^h) = ha^{h-1}\mathsf{d}a$ Solution Find  $\tau$  in terms of  $\gamma$  and h.

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$$\tau = 1 + h(\gamma - 1)$$

Excellent example of the Scaling Relations fou between exponents describing power laws for many systems.

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With more detailed description of network structure,  $\tau = 1 + h(\gamma - 1)$  simplifies to:<sup>[1]</sup>

$$\tau = 2 - h$$

and

$$\gamma = 1/h$$

Only one exponent is independent (take *h*). Simplifies system description. Expect Scaling Relations where power laws ar found. Need only characterize to be class w independent exponents.

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# Other First Returns or First Passage Times:

### Failure:

A very simple model of failure/death: <sup>[11]</sup>  $x_t = \text{entity's 'health' at time } t$ Start with  $x_0 > 0$ . Entity fails when x hits 0.

Dispersion of suspended sediments in streams Long times for clearing. PoCS | @pocsvox

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# Other First Returns or First Passage Times:

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### Streams

Dispersion of suspended sediments in streams.
 Long times for clearing.

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🙈 Can generalize to Fractional Random Walks<sup>[7, 8, 6]</sup>

Levy flights, Fractional Brownian Motion See Montroll and Shlesinger for example: <sup>61</sup> "On 1/f noise and other distributions with lo tails." Proc. Natl. Acad. Sci., 1982.

Extensive memory of path now matters..

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 $\begin{array}{l} \alpha = 1/2 - \text{diffusive} \\ \alpha > 1/2 - \text{superdiffusive} \\ \alpha < 1/2 - \text{subdiffusive} \end{array}$ 

Extensive memory of path now matters.

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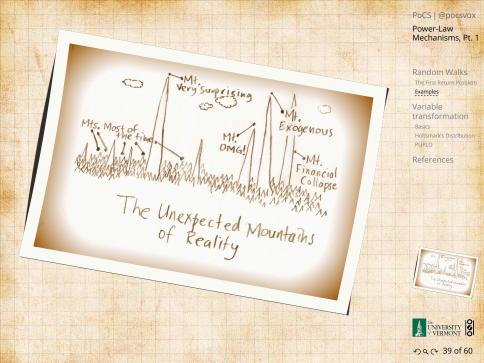
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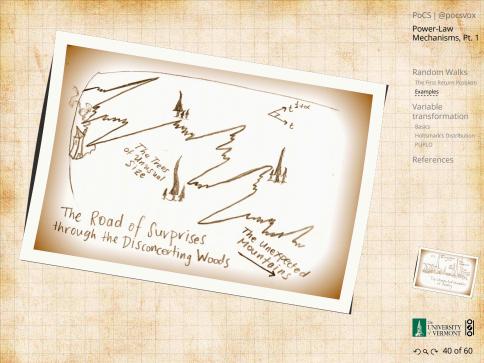
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## Neural reboot (NR):

Desert rain frog/Squeaky toy:

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https://www.youtube.com/v/cBkWhkAZ9ds?rel=0



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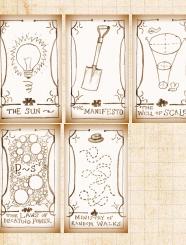
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Understand power laws as arising from

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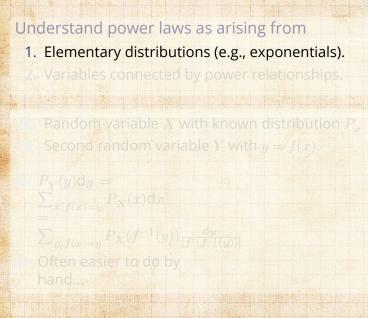
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### Understand power laws as arising from

- 1. Elementary distributions (e.g., exponentials).
- 2. Variables connected by power relationships.

Random variable *X* with known distribution *F* Second random variable *Y* with y = f(x). PoCS | @pocsvox

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 $rac{2}{8}$  Random variable X with known distribution  $P_x$ 

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Random variable X with known distribution  $P_x$ Second random variable Y with y = f(x). PoCS | @pocsvox

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$$\begin{array}{ll} P_Y(y) \mathrm{d}y &= \\ \sum_{x \mid f(x) = y} P_X(x) \mathrm{d}x \\ &= \\ \sum_{y \mid f(x) = y} P_X(f^{-1}(y)) \frac{\mathrm{d}y}{\mid f'(f^{-1}(y))} \\ \end{array}$$
 Often easier to do by hand.

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### General Example

Assume relationship between x and y is 1-Power-law relationship between variables:  $y = cx^{-\alpha}, \alpha > 0$ Look at y large and x small

 $dy = d(cx^{-\alpha})$ 

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 $\bigotimes$  Assume relationship between x and y is 1-1.

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Assume relationship between x and y is 1-1. Power-law relationship between variables:  $y = cx^{-\alpha}, \alpha > 0$ 

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Assume relationship between x and y is 1-1. Power-law relationship between variables:  $y = cx^{-\alpha}$ ,  $\alpha > 0$ 

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2

Assume relationship between x and y is 1-1. Power-law relationship between variables:  $y = cx^{-\alpha}$ ,  $\alpha > 0$ 

 $\bigotimes$  Look at y large and x small

$$\mathsf{d}y = \mathsf{d}\left(cx^{-\alpha}\right)$$

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2

Assume relationship between x and y is 1-1. Power-law relationship between variables:  $y = cx^{-\alpha}$ ,  $\alpha > 0$ 

 $\bigotimes$  Look at y large and x small

$$\mathsf{d}y = \mathsf{d}\left(cx^{-\alpha}\right)$$

$$= c(-\alpha)x^{-\alpha-1}\mathsf{d}x$$

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invert: 
$$dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

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$$\mathsf{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathsf{d}y$$

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$$P_y(y)\mathsf{d} y = P_x(x)\mathsf{d} x$$

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$$P_y(y)\mathsf{d} y = P_x(x)\mathsf{d} x$$

$$P_y(y)\mathsf{d} y = P_x \underbrace{\overline{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)} \underbrace{\frac{\mathsf{d} x}{c^{1/\alpha}}}_{\alpha} y^{-1-1/\alpha} \mathsf{d} y}^{(x)}$$

If  $P_{\alpha}(x) \rightarrow$  non-zero constant as  $x \rightarrow 0$  the

 $(x) \propto y^{-1-1/\alpha}$  as  $y \rightarrow 0$ 

If  $P_x(x) o x^eta$  as x o 0 then

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$$P_y(y) dy = P_x(x) dx$$

$$P_y(y)\mathsf{d} y = P_x \underbrace{\overline{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}}^{(x)} \underbrace{\frac{\mathsf{d} x}{c^{1/\alpha}}}_{\alpha} y^{-1-1/\alpha} \mathsf{d} y}$$

 $\mathfrak{s}$  If  $P_x(x) \rightarrow$  non-zero constant as  $x \rightarrow 0$  then

$$P_x(y) \propto y^{-1-1/\alpha}$$
 as  $y \to \infty$ .

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$$P_y(y)dy = P_x(x)dx$$

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$$\begin{split} & \& \ \, \text{If} \ \, P_x(x) \to \text{non-zero constant as} \ \, x \to 0 \ \text{then} \\ & P_x(y) \propto y^{-1-1/\alpha} \ \, \text{as} \ \, y \to \infty. \\ & \& \ \, \text{If} \ \, P_x(x) \to x^\beta \ \, \text{as} \ \, x \to 0 \ \text{then} \\ & P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \ \, \text{as} \ \, y \to \infty. \end{split}$$

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## Example

Exponential distribution Given  $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then  $P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$ 

> Exponentials arise from randomness (easy More later when we cover robustness.

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## Example

Exponential distribution Given  $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then  $P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$ 

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## Example

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Exponential distribution Given  $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then  $P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$ 

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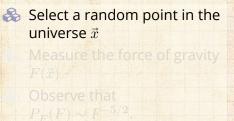
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Measure the force of gravity  $F(\vec{x})$ 

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lacktrian Select a random point in the universe  $\vec{x}$ 

Measure the force of gravity  $F(\vec{x})$ 



Observe that  $P_{F}(F) \sim F^{-5/2}$ .



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Random Walks

Variable transformation Basics

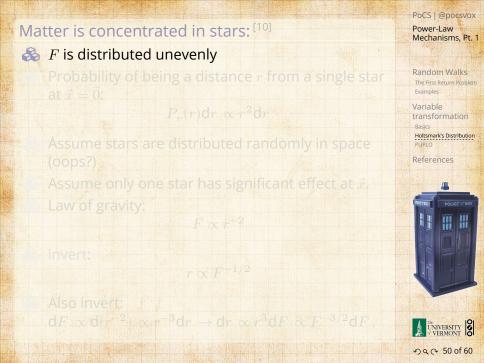
Holtsmark's Distribution

References





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Solution F is distributed unevenly  $\Im$  Probability of being a distance r from a single star at  $\vec{x} = \vec{0}$ :

$$P_r(r) \mathrm{d}r \propto r^2 \mathrm{d}r$$

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- $\underset{F}{\bigotimes}$  F is distributed unevenly
- Probability of being a distance r from a single star at  $\vec{x} = \vec{0}$ :

$$P_r(r) \mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
  - Assume only one star has significant effect at Law of gravity:

Also invert:  $dF \propto d(r^{-2}) \propto r^{-3} dr \rightarrow dr \propto r^{3} dF \propto$  PoCS | @pocsvox

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- $\underset{F}{\bigotimes}$  F is distributed unevenly
- Probability of being a distance r from a single star at  $\vec{x} = \vec{0}$ :

 $P_r(r) \mathrm{d}r \propto r^2 \mathrm{d}r$ 

- Assume stars are distributed randomly in space (oops?)
- $\mathfrak{B}$  Assume only one star has significant effect at  $\vec{x}$ .

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### Matter is concentrated in stars: [10]

- left for the second sec
- Probability of being a distance r from a single star at  $\vec{x} = \vec{0}$ :

$$P_r(r) \mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Solution  $\mathbf{s}_{\mathbf{s}}$  Assume only one star has significant effect at  $\vec{x}$ . Solution  $\mathbf{s}_{\mathbf{s}}$  Law of gravity:

$$F \propto r^{-2}$$

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- 🗞 F is distributed unevenly
- Probability of being a distance r from a single star at  $\vec{x} = \vec{0}$ :

 $P_r(r) \mathrm{d}r \propto r^2 \mathrm{d}r$ 

- Assume stars are distributed randomly in space (oops?)
- Solution Assume only one star has significant effect at  $\vec{x}$ . As Law of gravity:

 $F \propto r^{-2}$ 

🚳 invert:

$$r \propto F^{-1/2}$$

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- 🚳 F is distributed unevenly
- Probability of being a distance r from a single star at  $\vec{x} = \vec{0}$ :

 $P_r(r) {\rm d} r \propto r^2 {\rm d} r$ 

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at  $\vec{x}$ .
  Law of gravity:  $F \propto r^{-2}$

1

🚳 invert:

$$r \propto F^{-1/2}$$

 $\begin{aligned} & \& \text{Also invert:} \\ & \mathsf{d}F \propto \mathsf{d}(r^{-2}) \propto r^{-3} \mathsf{d}r \to \mathsf{d}r \propto r^3 \mathsf{d}F \propto F^{-3/2} \mathsf{d}F. \end{aligned}$ 

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Using 
$$r \propto F^{-1/2}$$
,  $dr \propto F^{-3/2}dF$ , and  $P_r(r) \propto r^2$   
 $P_F(F)dF = P_r(r)dr$   
 $\propto P_r(const \times F^{-1/2})F^{-3/2}dF$   
 $\propto (P^{-1/2})^2F^{-3/2}dF$   
 $= F^{-1-3/2}dF$ 

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3

Using 
$$r \propto F^{-1/2}$$
 ,  $dr \propto F^{-3/2} dF$  , and  $P_r(r) \propto r^2$ 

 $P_F(F)\mathsf{d} F = P_r(r)\mathsf{d} r$ 

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2

2

Using 
$$r \propto F^{-1/2}$$
 ,  $dr \propto F^{-3/2} dF$  , and  $P_r(r) \propto r^2$ 

 $P_F(F)\mathsf{d} F = P_r(r)\mathsf{d} r$ 

 $\propto P_r({\rm const} imes F^{-1/2})F^{-3/2}{\rm d}F$ 

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2

2

2

Using 
$$\boxed{r\propto F^{-1/2}}$$
 ,  $\boxed{{\rm d}r\,\propto F^{-3/2}{\rm d}F}$  , and  $\boxed{P_r(r)\propto r^2}$ 

 $P_F(F) \mathrm{d}F = P_r(r) \mathrm{d}r$ 

 $\propto P_r({\rm const} imes F^{-1/2})F^{-3/2}{\rm d}F$ 

$$\propto \left(F^{-1/2}
ight)^2 F^{-3/2} \mathsf{d} F$$

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3

3

Using 
$$r \propto F^{-1/2}$$
 ,  $dr \propto F^{-3/2} dF$  , and  $P_r(r) \propto r^2$ 

$$P_F(F) \mathrm{d}F = P_r(r) \mathrm{d}r$$

$$\propto P_r({\rm const} imes F^{-1/2})F^{-3/2}{\rm d}F$$

$$\propto \left(F^{-1/2}
ight)^2 F^{-3/2} \mathsf{d} F$$

$$= F^{-1-3/2} \mathsf{d} F$$

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3

Using 
$$r \propto F^{-1/2}$$
 ,  $dr \propto F^{-3/2} dF$  , and  $P_r(r) \propto r^2$ 

$$P_F(F) \mathrm{d}F = P_r(r) \mathrm{d}r$$

$$\propto P_r({
m const} imes F^{-1/2})F^{-3/2}{
m d}F$$

$$\propto \left(F^{-1/2}
ight)^2 F^{-3/2} \mathsf{d} F$$

$$= F^{-1-3/2} \mathsf{d} F$$

 $= F^{-5/2} \mathrm{d} F \, .$ 

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 $P_F(F)=F^{-5/2}\mathrm{d} F$ 

Mean is finite. Variance =  $\infty$ . A wild distribution. Upshot: Random sampling of space usually s but can end badly...

2

## $P_F(F)=F^{-5/2}\mathrm{d} F$

$$\gamma = 5/2$$

Mean is finite. Variance =  $\infty$ . A wild distribution. Upshot: Random sampling of space usually s but can end badly...

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23

## $P_F(F)=F^{-5/2}\mathrm{d} F$

$$\gamma = 5/2$$

### 🚳 Mean is finite.

A wild distribution. Upshot: Random sampling of space usually sa but can end badly...

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2

## $P_F(F)=F^{-5/2}\mathrm{d} F$

$$\gamma = 5/2$$

 $\bigotimes$  Mean is finite.  $\bigotimes$  Variance =  $\infty$ .

> A wild distribution. Upshot! Random sampling of space usually sa but can end badly...

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# Gravity:

2

## $P_F(F)=F^{-5/2}\mathrm{d} F$

$$\gamma = 5/2$$

Mean is finite.
Variance =  $\infty$ .
A wild distribution.

Upshot Random sampling of space usually sat but can end badly...

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# Gravity:

2

## $P_F(F)=F^{-5/2}\mathrm{d} F$

$$\gamma = 5/2$$

- 🚳 Mean is finite.
- $\clubsuit$  Variance =  $\infty$ .
- \lambda wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

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### Doctorin' the Tardis

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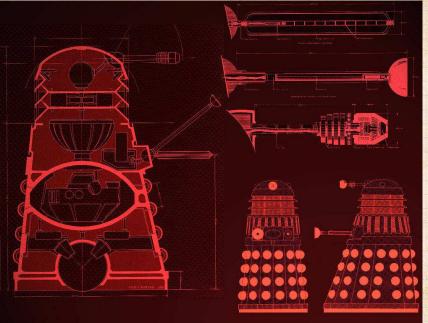
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# □ Todo: Build Dalek army.



# Outline

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PLIPLO





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### PLIPLO = Power law in, power law out

Explain a power law as resulting from anot unexplained power law.
Yet another homoculus argument.
Don't do this!!! (slap, slap)
MIWO = Mild in Wild out is fine.
In general: We need mechanisms!

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- 🚳 In general: We need mechanisms!

# Neural reboot (NR):

## Zoomage in slow motion

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 $\bigotimes$ 

https://www.youtube.com/v/axrTxEVQqN4?rel=0



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