Mechanisms for Generating Power-Law Size Distributions, Part 1

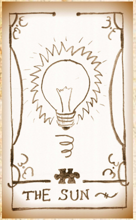
Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

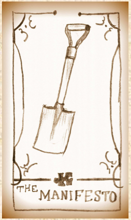
Prof. Peter Dodds | @peterdodds

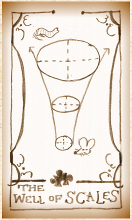
Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

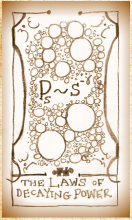
These slides are brought to you by:

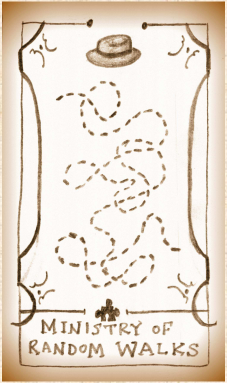
Outline


THE SUN

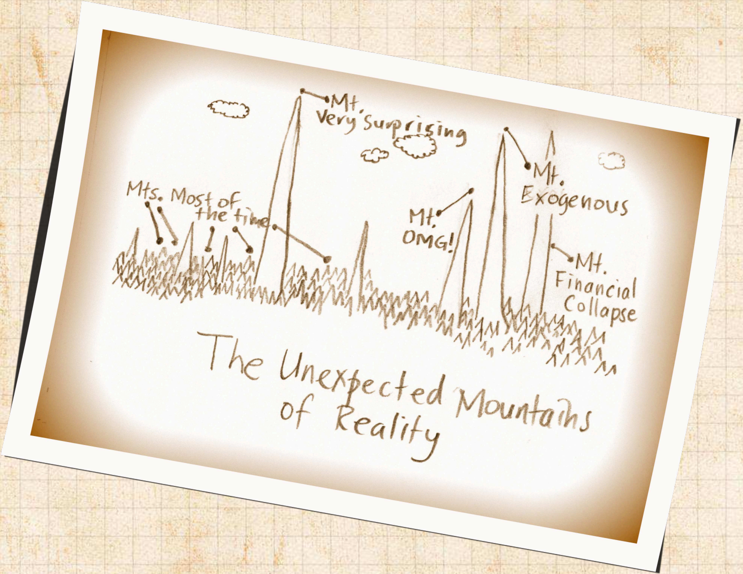

THE MANIFESTO


THE WELL OF SCALES


$P_s \sim s^{-\gamma}$
THE LAWS OF DECAYING POWER


MINISTRY OF RANDOM WALKS


Mt. Whatever
Mt. Zzzz
Mt. Who cares?
Mt. Meh
Mt. Quite boring
The Snormals


Mt. Very Surprising
Mts. Most of the time
Mt. OMG!
Mt. Exogenous
Mt. Financial Collapse
The Unexpected Mountains of Reality


MINISTRY OF RANDOM WALKS

Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

The UNIVERSITY of VERMONT

Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Mechanisms:

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Mechanisms:

A powerful story in the rise of complexity:

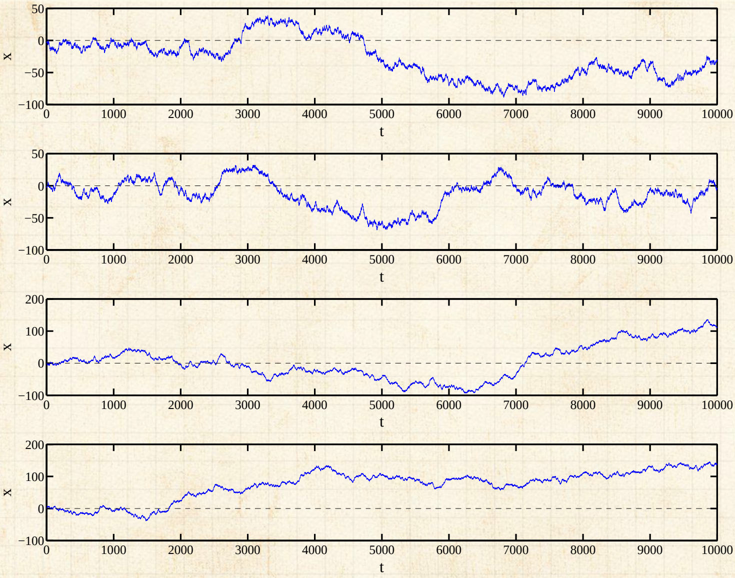
- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

A few random random walks:



Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Random walks:

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we ‘expect’ our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?

Variances sum:*

$$\text{Var}(x_t) = \text{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \text{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

A non-trivial scaling law arises out of additive aggregation or accumulation.

Variances sum: *

$$\mathrm{Var}(x_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

A non-trivial scaling law arises out of additive aggregation or accumulation.

Variances sum:*

$$\mathrm{Var}(x_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

A non-trivial scaling law arises out of additive aggregation or accumulation.

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Variances sum: *

$$\mathrm{Var}(x_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

A non-trivial scaling law arises out of additive aggregation or accumulation.

Variances sum:*

$$\text{Var}(x_t) = \text{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \text{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\boxed{\sigma = t^{1/2}}$$

A non-trivial scaling law arises out of additive aggregation or accumulation.

Variances sum:*

$$\text{Var}(x_t) = \text{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \text{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\boxed{\sigma = t^{1/2}}$$

- A non-trivial scaling law arises out of additive aggregation or accumulation.

Stock Market randomness:

Also known as the bean machine, the quincunx (simulation), and the Galton box.

Great moments in Televised Random Walks:

Plinko! from the Price is Right.

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 3

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 3

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i,j,t)$ as # distinct walks that start at $x=i$ and end at $x=j$ after t time steps.
- Random walk must displace by $+(j-i)$ after t steps.
- Insert question from assignment 3

$$N(i,j,t) = \binom{t}{(t+j-i)/2}$$

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 3

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 3

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i,j,t)$ as # distinct walks that start at $x=i$ and end at $x=j$ after t time steps.
- Random walk must displace by $+(j-i)$ after t steps.
- Insert question from assignment 3

$$N(i,j,t) = \binom{t}{(t+j-i)/2}$$

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 3

- The whole is different from the parts. #nutritious
- See also: Stable Distributions

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \ldots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 3 ↗

- The whole is different from the parts. #nutritious
- See also: Stable Distributions ↗

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

- Insert question from assignment 3
- The whole is different from the parts. #nutritious
- See also: Stable Distributions

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

- Insert question from assignment 3
- The whole is different from the parts. #nutritious
- See also: Stable Distributions

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 3

- The whole is different from the parts. #nutritious
- See also: Stable Distributions

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 3

- The whole is different from the parts. #nutritious
- See also: Stable Distributions

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 3

- The whole is different from the parts. #nutritious
- See also: Stable Distributions

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

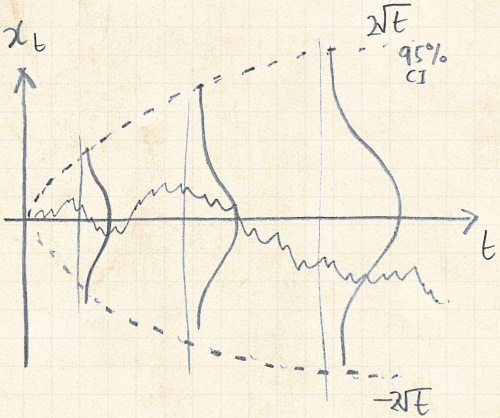
- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Insert question from assignment 3

- The whole is different from the parts. #nutritious
- See also: Stable Distributions

Universality is also not left-handed:



- This is Diffusion: the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.


t^{1/2}
t
The Snormals
The Random Road
through the Forests of Forgettable Events


MINISTRY OF
RANDOM WALKS


Mt. Whatever
Mt. Zzzz
Mt. Who cares?
Mt. Meh
Mt. Quite boring
The Snormals

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

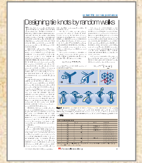
- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
- Even crazier:
  The expected time between tied scores = ∞

See Feller, Intro to Probability Theory, Volume I [3]

Applied knot theory:



"Designing tie knots by random walks"
Fink and Mao,
Nature, **398**, 31–32, 1999. [4]

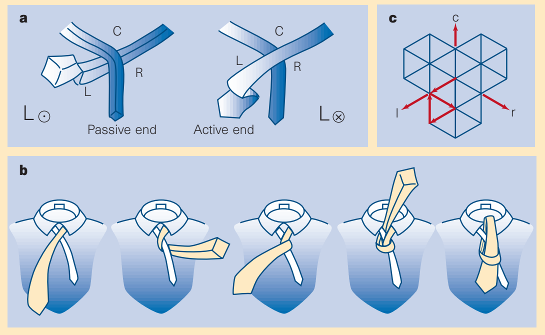


Figure 1 All diagrams are drawn in the frame of reference of the mirror image of the actual tie. **a**, The two ways of beginning a knot, $L_\odot$ and $L_\otimes$. For knots beginning with $L_\odot$, the tie must begin inside-out. **b**, The four-in-hand, denoted by the sequence $L_\otimes R_\odot L_\otimes C_\odot T$. **c**, A knot may be represented by a persistent random walk on a triangular lattice. The example shown is the four-in-hand, indicated by the walk $\hat{l}\,\hat{r}\,\hat{l}\,\hat{c}$.

Applied knot theory:

Table 1 **Aesthetic tie knots**

h	γ	γ/h	$K(h, \gamma)$	s	b	Name	Sequence
3	1	0.33	1	0	0		$L_\odot R_\otimes C_\odot T$
4	1	0.25	1	−1	1	Four-in-hand	$L_\otimes R_\odot L_\otimes C_\odot T$
5	2	0.40	2	−1	0	Pratt knot	$L_\odot C_\otimes R_\odot L_\otimes C_\odot T$
6	2	0.33	4	0	0	Half-Windsor	$L_\otimes R_\odot C_\otimes L_\odot R_\otimes C_\odot T$
7	2	0.29	6	−1	1		$L_\odot R_\otimes L_\odot C_\otimes R_\odot L_\otimes C_\odot T$
7	3	0.43	4	0	1		$L_\odot C_\otimes R_\odot C_\otimes L_\odot R_\otimes C_\odot T$
8	2	0.25	8	0	2		$L_\otimes R_\odot L_\otimes C_\odot R_\otimes L_\odot R_\otimes C_\odot T$
8	3	0.38	12	−1	0	Windsor	$L_\otimes C_\odot R_\otimes L_\odot C_\otimes R_\odot L_\otimes C_\odot T$
9	3	0.33	24	0	0		$L_\odot R_\otimes C_\odot L_\otimes R_\odot C_\otimes L_\odot R_\otimes C_\odot T$
9	4	0.44	8	−1	2		$L_\odot C_\otimes R_\odot C_\otimes L_\odot C_\otimes R_\odot L_\otimes C_\odot T$

Knots are characterized by half-winding number h, centre number γ, centre fraction γ/h, knots per class $K(h, \gamma)$, symmetry s, balance b, name and sequence.

- h = number of moves
- γ = number of center moves
- $K(h,\gamma) = 2^{\gamma-1}\binom{h-\gamma-2}{\gamma-1}$
- $s = \sum_{i=1}^{h} x_i$ where x = -1 for L and +1 for R.
- $b = \frac{1}{2}\sum_{i=2}^{h-1} |\omega_i + \omega_{i-1}|$ where $\omega = \pm 1$ represents winding direction.

Outline

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

Random walks #crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

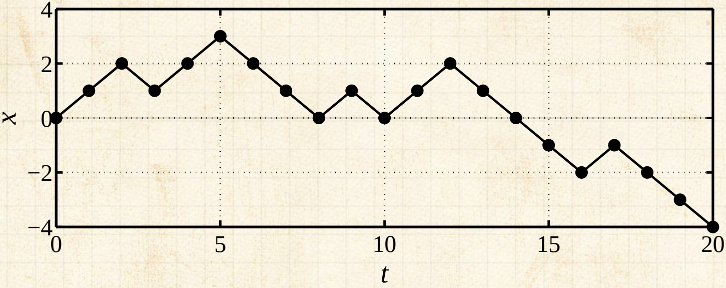
The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

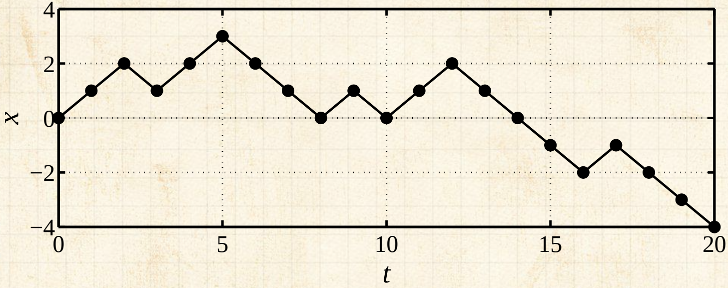
1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
3. We'll start to see how different scalings relate to each other.

For random walks in 1-d:



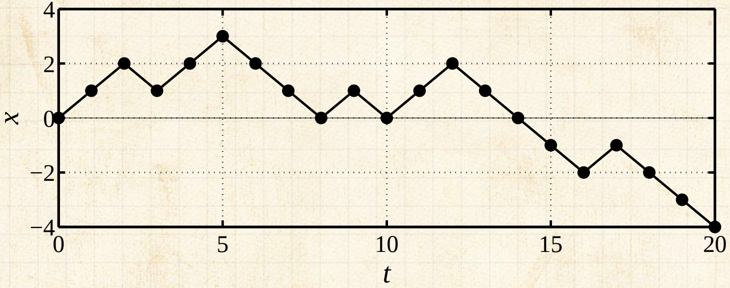
- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\mathrm{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

For random walks in 1-d:



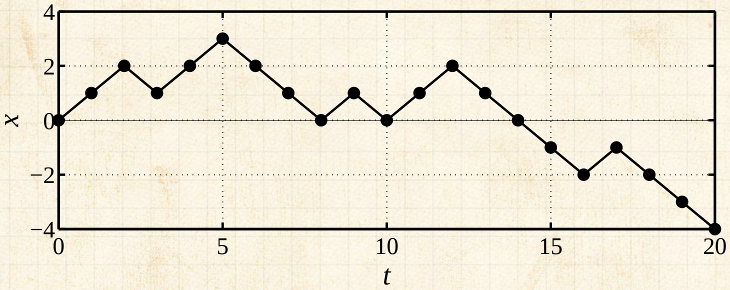
- A **return** to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\mathrm{fr}(2n)}$ the probability of **first return** at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

For random walks in 1-d:



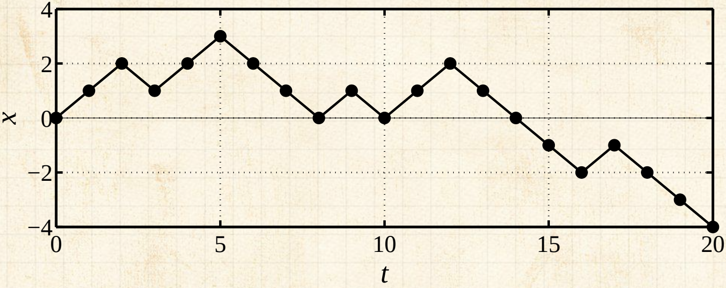
- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\text{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

For random walks in 1-d:



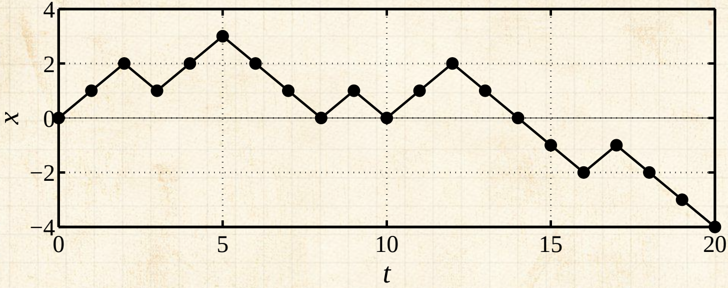
- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\text{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

For random walks in 1-d:

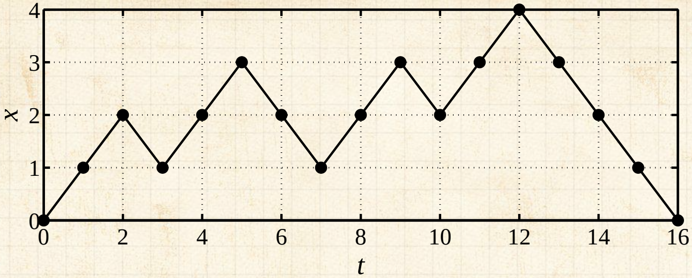


- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\mathrm{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

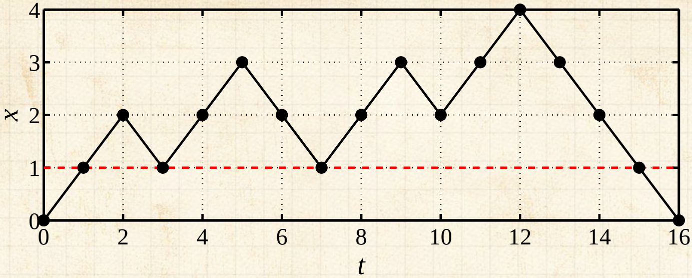
For random walks in 1-d:



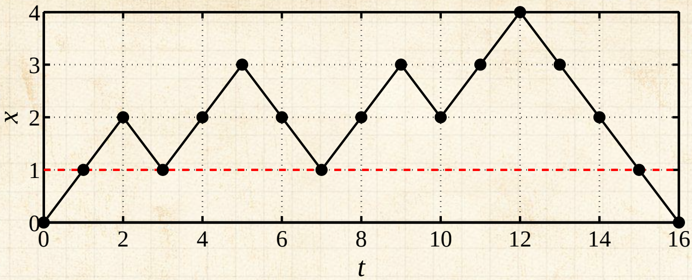
- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\text{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.



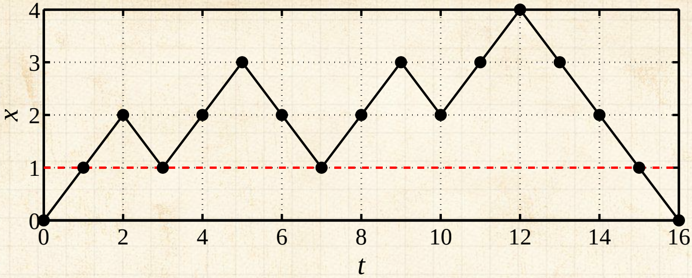
- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \le t \le 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\rm fr}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \ge 1, 1 \le t \le 2n-1,$ and $x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.



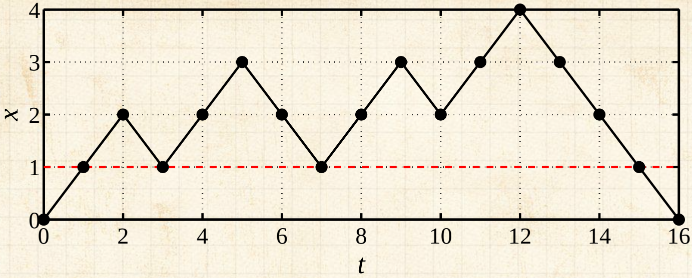
- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n - 1, \text{ and } x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.



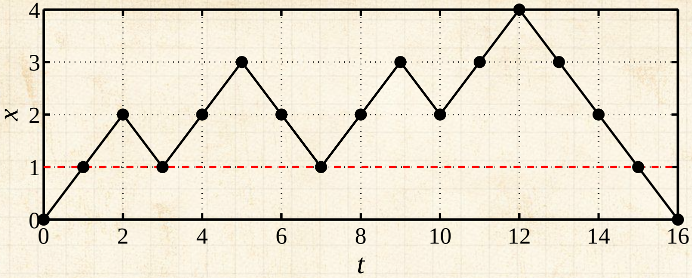
- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \le t \le 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \ge 1, 1 \le t \le 2n-1,$ and $x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.



- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n-1,$ and $x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.



- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n-1,$ and $x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.



- Can assume drunkard first lurches to $x = 1$.
- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
  $2 \cdot \frac{1}{2} Pr(x_t \geq 1, 1 \leq t \leq 2n-1,$ and $x_1 = x_{2n-1} = 1)$
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.

The UNIVERSITY of VERMONT

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i,j,t)$ is the # of possible walks between $x=i$ and $x=j$ taking t steps.
- Consider all paths starting at $x=1$ and ending at $x=1$ after $t=2n-2$ steps.
- Idea: If we can compute the number of walks that hit $x=0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x=1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Counting first returns:

Approach:

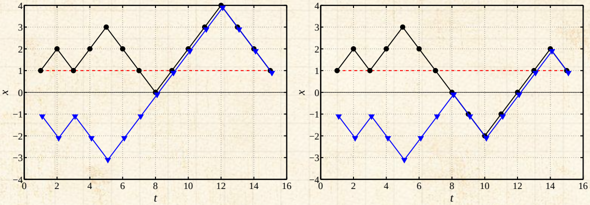
- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x = 1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

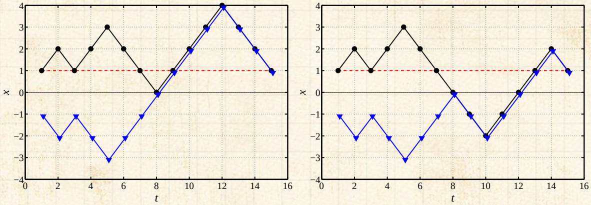
Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
- So $N_{\text{first return}}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$

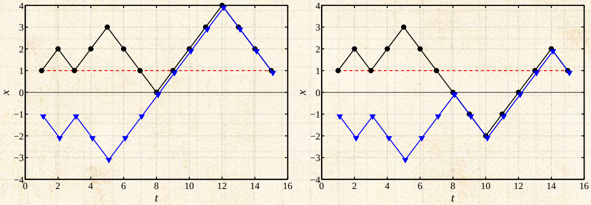
Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
- So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

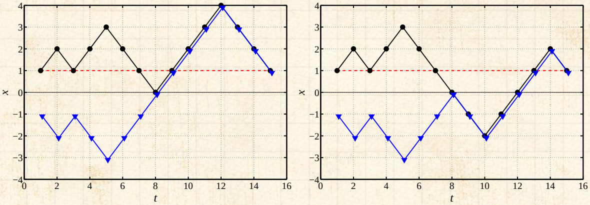
Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
- So $N_{\text{first return}}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$

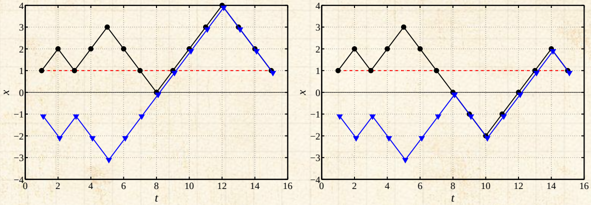
Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
  = # of t-step paths starting at $x=-1$ and ending at $x=1$
- So $N_{\text{first return}}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$

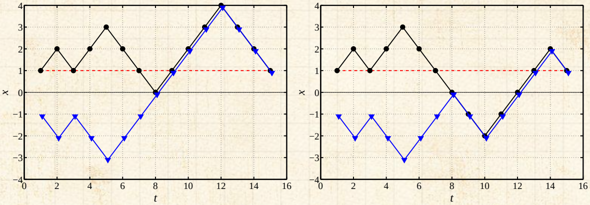
Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
  = # of t-step paths starting at $x=-1$ and ending at $x=1$ = $N(-1, 1, t)$
- So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

Examples of excluded walks:



Key observation for excluded walks:

- For any path starting at $x=1$ that hits 0, there is a unique matching path starting at $x=-1$.
- Matching path first mirrors and then tracks after first reaching $x=0$.
- # of t-step paths starting and ending at $x=1$ and hitting $x=0$ at least once
  = # of t-step paths starting at $x=-1$ and ending at $x=1$ = $N(-1, 1, t)$
- So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

Probability of first return:

Insert question from assignment 3 :

- Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}.$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.
-

$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

Probability of first return:

Insert question from assignment 3 :

- Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$

- Normalized number of paths gives probability.
- Total number of possible paths $= 2^{2n}$.

$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

Probability of first return:

Insert question from assignment 3 :

- Find

$$\boxed{N_{\mathrm{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.}$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.
-

$$P_{\mathrm{fr}}(2n) = \frac{1}{2^{2n}} N_{\mathrm{fr}}(2n)$$

Probability of first return:

Insert question from assignment 3 :

- Find

$$\boxed{N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}.}$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.

$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

Probability of first return:

Insert question from assignment 3 :

- Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.
-

$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}}(2n)^{-3/2} \propto t^{-3/2}$$

Probability of first return:

Insert question from assignment 3 :

- Find

$$N_{\mathsf{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.
-

$$P_{\mathsf{fr}}(2n) = \frac{1}{2^{2n}} N_{\mathsf{fr}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}$$

Probability of first return:

Insert question from assignment 3 :

- Find

$$N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.
-

$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}$$

Probability of first return:

Insert question from assignment 3 :

- Find

$$\boxed{N_{\text{fr}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}.}$$

- Normalized number of paths gives probability.
- Total number of possible paths = 2^{2n}.
-

$$P_{\text{fr}}(2n) = \frac{1}{2^{2n}} N_{\text{fr}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2} \propto t^{-3/2}.$$

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}, \; \gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}, \ \gamma = 3/2.$
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

- We have $P(t) \propto t^{-3/2}$, $\gamma = 3/2$.
- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite. #totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy

Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy

Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy

Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency ∝ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy

Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy

Random walks

On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy

Random walks

On finite spaces:

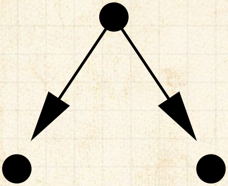
- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

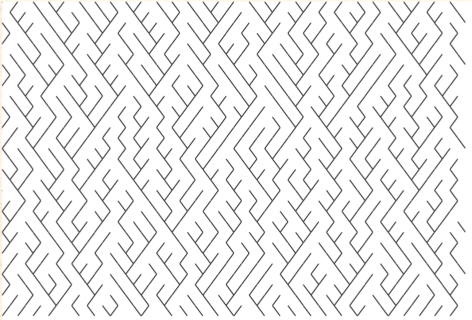
On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present:
  walkers traverse edges with equal frequency.
  #totallygroovy

Outline

Scheidegger Networks [9, 2]





- Random directed network on triangular lattice.
- Toy model of real networks.
- 'Flow' is southeast or southwest with equal probability.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

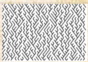
- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$
- For real river networks, generalize to $P(\ell) \propto \ell^{-\gamma}$.

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3} \mathrm{d}a$
- $\mathbf{Pr}$(basin area $= a$)da $= \mathbf{Pr}$(basin length $= \ell$)dℓ

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- d$\ell \propto$ d$(a^{2/3}) = 2/3 a^{-1/3}$da
- $\mathbf{Pr}$(basin area $= a$)da
  $= \mathbf{Pr}$(basin length $= \ell$)dℓ

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3a^{-1/3}\mathrm{d}a$
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3} \mathrm{d}a$
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- d$\ell \propto$ d$(a^{2/3}) = 2/3 a^{-1/3}$da
- $\mathbf{Pr}$(basin area $= a$)da
  $= \mathbf{Pr}$(basin length $= \ell$)dℓ
  $\propto \ell^{-3/2}$dℓ
  $\propto (a^{2/3})^{-3/2} a^{-1/3}da$
  $= a^{-4/3}$da
  $= a^{-\tau}$da

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- dℓ $\propto$ d$(a^{2/3}) = 2/3 a^{-1/3}$da
- $\mathbf{Pr}$(basin area $= a$)da
  $= \mathbf{Pr}$(basin length $= \ell$)dℓ
  $\propto \ell^{-3/2}$dℓ
  $\propto (a^{2/3})^{-3/2} a^{-1/3}da$
  $= a^{-4/3}$da
  $= a^{-\tau}$da

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3}\mathrm{d}a$
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-3/2}\mathrm{d}\ell$
  $\propto (a^{2/3})^{-3/2} a^{-1/3}\mathrm{d}a$
  $= a^{-4/3}\mathrm{d}a$
  $= a^{-\tau}\mathrm{d}a$

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3} \mathrm{d}a$
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-3/2}\mathrm{d}\ell$
  $\propto (a^{2/3})^{-3/2} a^{-1/3}\mathrm{d}a$
  $= a^{-4/3}\mathrm{d}a$
  $= a^{-\tau}\mathrm{d}a$

Connections between exponents:

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3} \mathrm{d}a$
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-3/2}\mathrm{d}\ell$
  $\propto (a^{2/3})^{-3/2} a^{-1/3}\mathrm{d}a$
  $= a^{-4/3}\mathrm{d}a$
  $= a^{-\tau}\mathrm{d}a$

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law[5]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law[5]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^h .$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^h .$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^h.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$

Generalize relationship between area and length:

- Hack's law [5]:

$$\ell \propto a^{h}.$$

- For real, large networks $h \simeq 0.5$
- Smaller basins possibly $h > 1/2$ (see: allometry).
- Models exist with interesting values of h.
- Plan: Redo calc with γ, τ, and h.

Connections between exponents:

- Given

$$\ell \propto a^h, \; P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
- $$\boxed{\tau = 1 + h(\gamma - 1)}$$
- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h, \; P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
-

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h, \; P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
-

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h,\ P(a) \propto a^{-\tau},\ \text{and}\ P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h,\ P(a) \propto a^{-\tau},\ \text{and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1}\mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
-

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h,\ P(a) \propto a^{-\tau},\ \text{and}\ P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h\,(\gamma-1))}\mathrm{d}a$

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h,\ P(a) \propto a^{-\tau},\ \text{and}\ P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h\,(\gamma-1))}\mathrm{d}a$
-
$$\boxed{\tau = 1 + h(\gamma - 1)}$$
- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

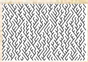
- Given

$$\ell \propto a^h, \ P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h\,(\gamma-1))}\mathrm{d}a$
-

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h, \ P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h\,(\gamma-1))}\mathrm{d}a$
-

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

- Given

$$\ell \propto a^h, \; P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- Find τ in terms of γ and h.
- $\mathbf{Pr}(\text{basin area} = a)\mathrm{d}a$
  $= \mathbf{Pr}(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h\,(\gamma-1))}\mathrm{d}a$
-

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.

Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to: [1]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take h).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality class with independent exponents.

Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to: [1]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take h).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality class with independent exponents.

Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to: [1]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take h).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality class with independent exponents.

Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to: [1]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take h).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality class with independent exponents.

Connections between exponents:

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies to: [1]

$$\boxed{\tau = 2 - h}$$

and

$$\boxed{\gamma = 1/h}$$

- Only one exponent is independent (take h).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality class with independent exponents.

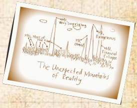
Other First Returns or First Passage Times:

Failure:

- A very simple model of failure/death: [11]
- x_t = entity's 'health' at time t
- Start with $x_0 > 0$.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.



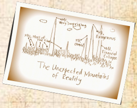
Other First Returns or First Passage Times:

Failure:

- A very simple model of failure/death: [11]
- x_t = entity's 'health' at time t
- Start with $x_0 > 0$.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

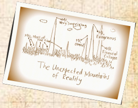


More than randomness

- Can generalize to Fractional Random Walks [7, 8, 6]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [6]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

$$\sigma \sim t^{\alpha}$$

- Extensive memory of path now matters...

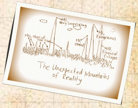


More than randomness

- Can generalize to Fractional Random Walks [7, 8, 6]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [6]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

$$\sigma \sim t^{\alpha}$$

- Extensive memory of path now matters...

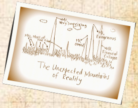


More than randomness

- Can generalize to Fractional Random Walks [7, 8, 6]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [6]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

$$\sigma \sim t^{\alpha}$$

- Extensive memory of path now matters...



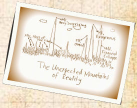
More than randomness

- Can generalize to Fractional Random Walks [7, 8, 6]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [6]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

$$\sigma \sim t^{\alpha}$$

$\alpha = 1/2$ — diffusive
$\alpha > 1/2$ — superdiffusive
$\alpha < 1/2$ — subdiffusive

- Extensive memory of path now matters...

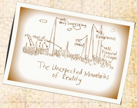


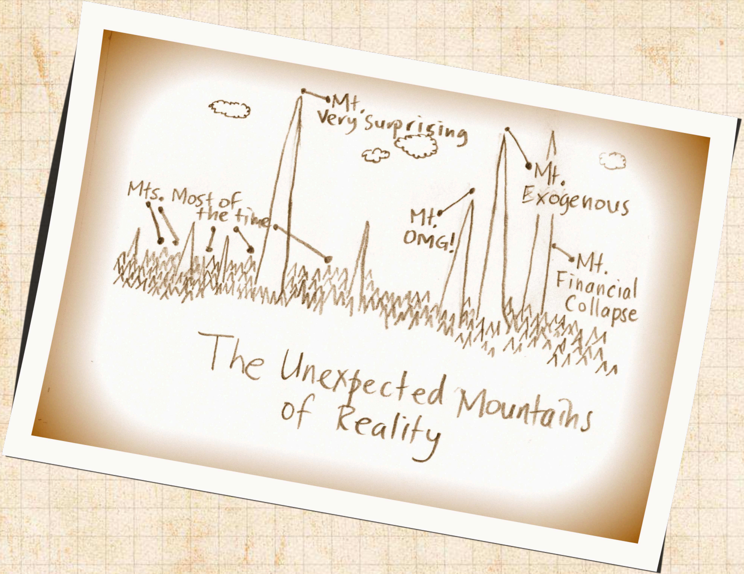
More than randomness

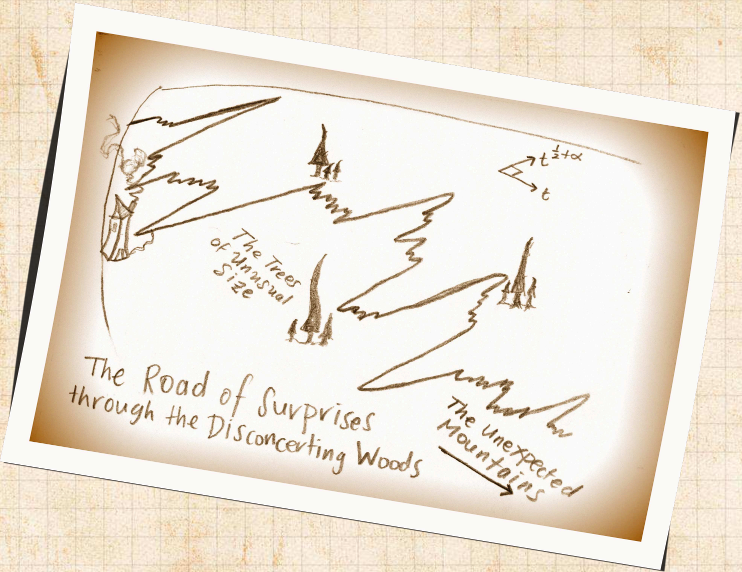
- Can generalize to Fractional Random Walks [7, 8, 6]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [6]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

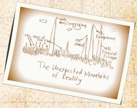
$$\sigma \sim t^{\alpha}$$

  $\alpha = 1/2$ — diffusive
  $\alpha > 1/2$ — superdiffusive
  $\alpha < 1/2$ — subdiffusive
- Extensive memory of path now matters...




Mt. Very Surprising
Mts. Most of the time
Mt. OMG!
Mt. Exogenous
Mt. Financial Collapse
The Unexpected Mountains of Reality

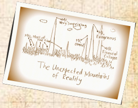

The Road of Surprises through the Disconcerting Woods
The Trees of Unusual Size
The Unexpected Mountains
$t^{\frac{1}{2}+\alpha}$
t

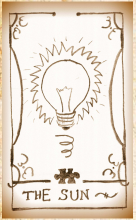

The Unexpected Mountains of Reality

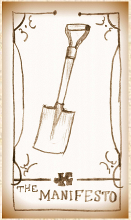
Neural reboot (NR):

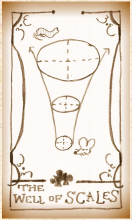
Desert rain frog/Squeaky toy:

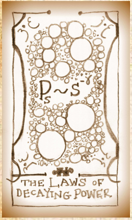
https://www.youtube.com/v/cBkWhkAZ9ds?rel=0

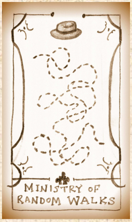


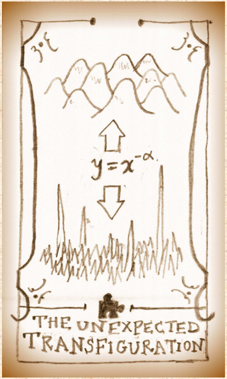

THE SUN

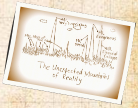

THE MANIFESTO


THE WELL OF SCALES

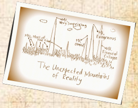

$P_s \sim s^{-\gamma}$
THE LAWS OF DECAYING POWER


MINISTRY OF RANDOM WALKS


$y = x^{-\alpha}$
THE UNEXPECTED TRANSFIGURATION


The Unexpected Mountains of Reality

Outline



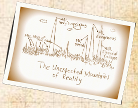
Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $$P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$
- Often easier to do by hand...



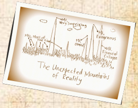
Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $$P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$
- Often easier to do by hand...

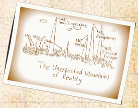


Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $$P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$
- Often easier to do by hand...

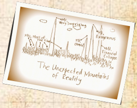


Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...

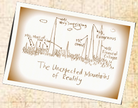


Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...



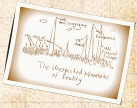
Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $$P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$
- Often easier to do by hand...



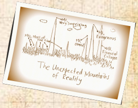
Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.

- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...



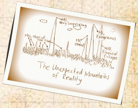
General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
-

$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

General Example

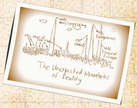
- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$



General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables:
  $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
-

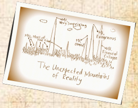
$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$



General Example

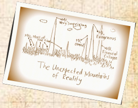
- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small

$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$



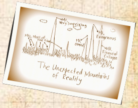
General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$



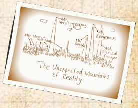
General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$
  $$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$



General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$
  $$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$
  $$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$



General Example

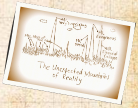
- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
-

$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$

$$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$

$$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$



General Example

- Assume relationship between x and y is 1-1.
- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
-

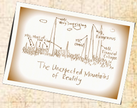
$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$

$$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$

$$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$

$$\mathrm{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$



Now make transformation:

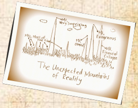
$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

- If $P_x(x) \to$ non-zero constant as $x \to 0$ then

$$P_x(y) \propto y^{-1-1/\alpha} \text{ as } y \to \infty.$$

- If $P_x(x) \to x^{\beta}$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty.$$



Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

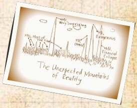
$$P_y(y)\mathrm{d}y = P_x\overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)}\overbrace{\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \to$ non-zero constant as $x \to 0$ then

$$P_x(y) \propto y^{-1-1/\alpha} \text{ as } y \to \infty.$$

- If $P_x(x) \to x^{\beta}$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty.$$



Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

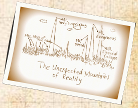
$$P_y(y)\mathrm{d}y = P_x \overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)} \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} \mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_x(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- If $P_x(x) \rightarrow x^{\beta}$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$



Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

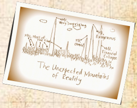
$$P_y(y)\mathrm{d}y = P_x \overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)} \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \to$ non-zero constant as $x \to 0$ then

$$P_x(y) \propto y^{-1-1/\alpha} \text{ as } y \to \infty.$$

- If $P_x(x) \to x^{\beta}$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty.$$



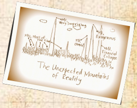
Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.



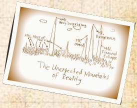
Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.



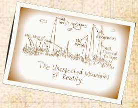
Example

Exponential distribution

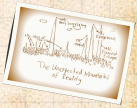
Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.



Outline



Gravity

- Select a random point in the universe $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

Gravity

- Select a random point in the universe $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

Gravity

- Select a random point in the universe $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
  $\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3 \mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^{3}\mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^{3}\mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3\mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert: $\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3 \mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3\mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Matter is concentrated in stars: [10]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3 \mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2}) F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F\,.$$

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\text{const} \times F^{-1/2}) F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F\,.$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F\,.$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F\,.$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F\,.$$

Transformation:

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

$$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$

$$\propto P_r(\mathrm{const} \times F^{-1/2})F^{-3/2}\mathrm{d}F$$

$$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$

$$= F^{-1-3/2}\mathrm{d}F$$

$$= F^{-5/2}\mathrm{d}F\,.$$

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

$$\gamma = 5/2$$

- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

$$\gamma = 5/2$$

- Mean is finite.
- Variance $= \infty$.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

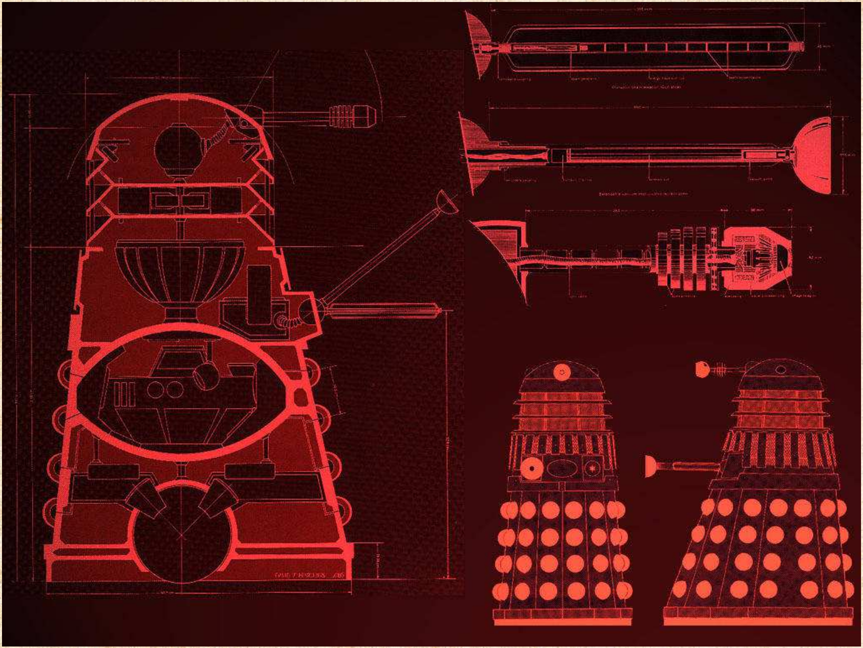
Gravity:

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite.
- Variance = ∞.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...

Doctorin' the Tardis

☐ Todo: Build Dalek army.



Outline

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument ...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is fine.
- In general: We need mechanisms!

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument ...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is fine.
- In general: We need mechanisms!

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is fine.
- In general: We need mechanisms!

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is fine.
- In general: We need mechanisms!

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is fine.
- In general: We need mechanisms!

Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is fine.
- In general: We need mechanisms!

Neural reboot (NR):

Zoomage in slow motion

https://www.youtube.com/v/axrTxEVQqN4?rel=0

References I

[1] P. S. Dodds and D. H. Rothman.
Unified view of scaling laws for river networks.
Physical Review E, 59(5):4865–4877, 1999. pdf

[2] P. S. Dodds and D. H. Rothman.
Scaling, universality, and geomorphology.
Annu. Rev. Earth Planet. Sci., 28:571–610, 2000.
pdf

[3] W. Feller.
An Introduction to Probability Theory and Its Applications, volume I.
John Wiley & Sons, New York, third edition, 1968.

[4] T. M. Fink and Y. Mao.
Designing tie knots by random walks.
Nature, 398:31–32, 1999. pdf

References II

[5] J. T. Hack.
Studies of longitudinal stream profiles in Virginia and Maryland.
United States Geological Survey Professional Paper, 294-B:45–97, 1957. pdf

[6] E. W. Montroll and M. F. Shlesinger.
On the wonderful world of random walks, volume XI of Studies in statistical mechanics, chapter 1, pages 1–121.
New-Holland, New York, 1984.

[7] E. W. Montroll and M. W. Shlesinger.
On $1/f$ noise and other distributions with long tails.
Proc. Natl. Acad. Sci., 79:3380–3383, 1982. pdf

References III

[8] E. W. Montroll and M. W. Shlesinger.
Maximum entropy formalism, fractals, scaling phenomena, and $1/f$ noise: a tale of tails.
J. Stat. Phys., 32:209–230, 1983.

[9] A. E. Scheidegger.
The algebra of stream-order numbers.
United States Geological Survey Professional Paper, 525-B:B187–B189, 1967. pdf

[10] D. Sornette.
Critical Phenomena in Natural Sciences.
Springer-Verlag, Berlin, 1st edition, 2003.

[11] J. S. Weitz and H. B. Fraser.
Explaining mortality rate plateaus.
Proc. Natl. Acad. Sci., 98:15383–15386, 2001. pdf