Lognormals and friends

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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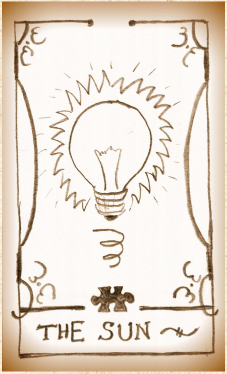
Outline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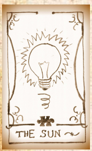
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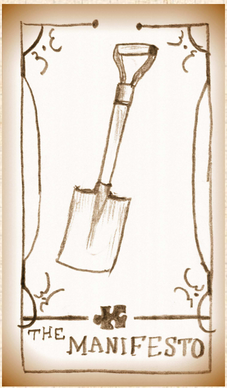
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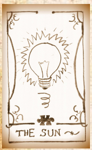
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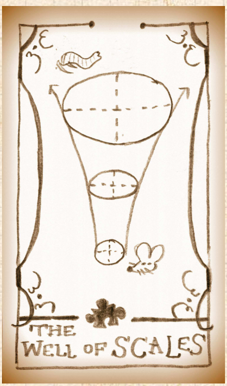
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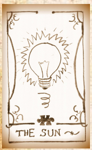
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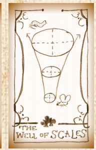
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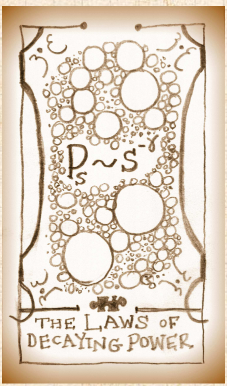
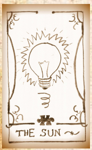

$P_s \sim S^{-\gamma}$
THE LAWS OF DECAYING POWER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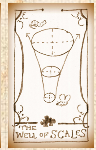
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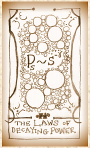
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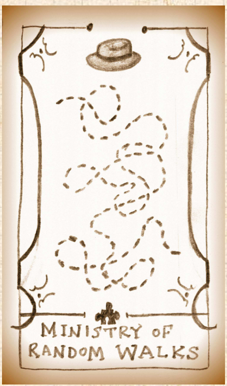
MINISTRY OF RANDOM WALKS


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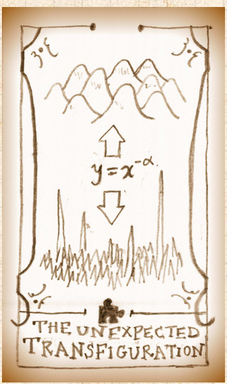
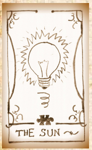

$y = x^{-\alpha}$
THE UNEXPECTED TRANSFIGURATION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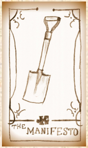
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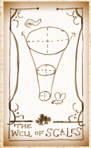
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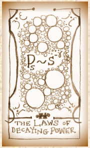
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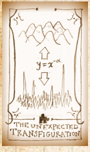
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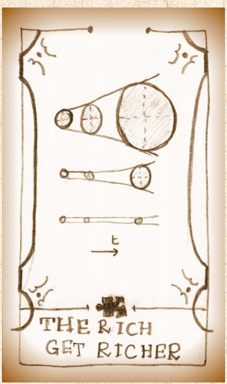
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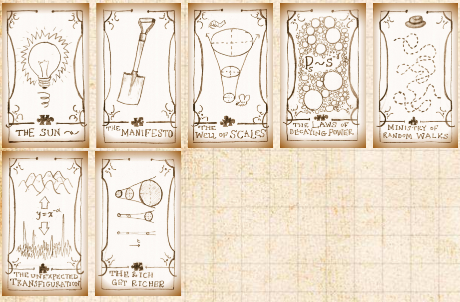
THE RICH GET RICHER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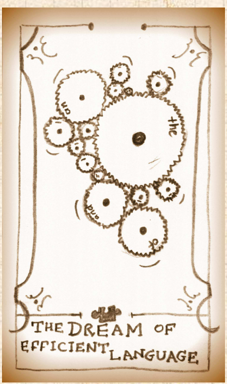
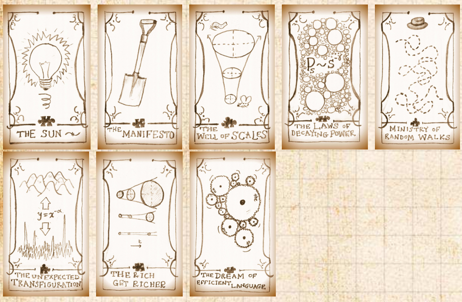

THE DREAM OF EFFICIENT LANGUAGE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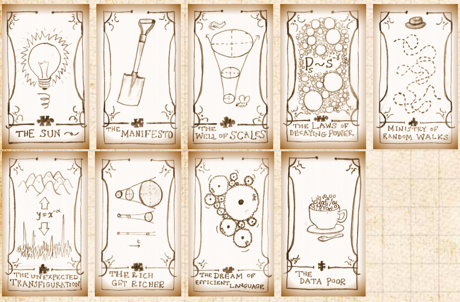
THE DATA POOR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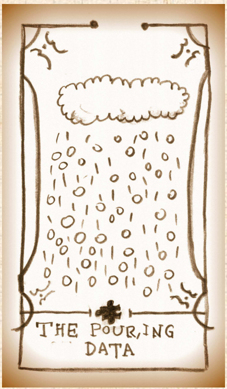
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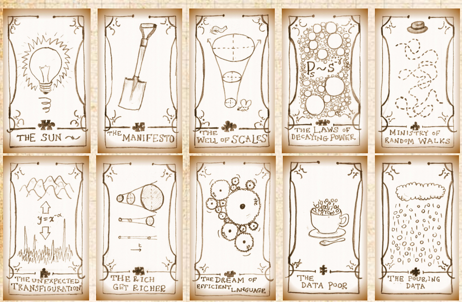
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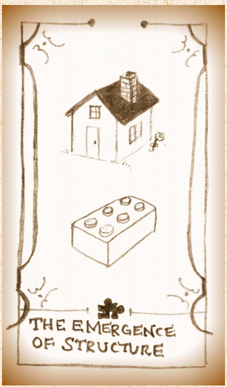
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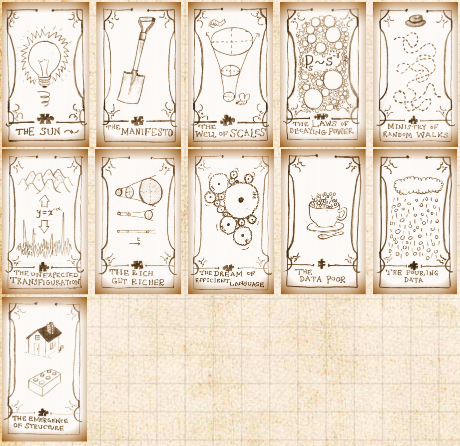
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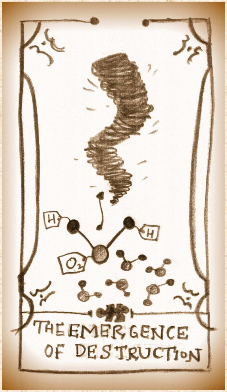
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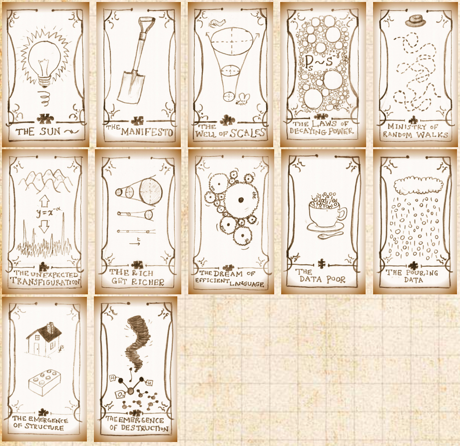
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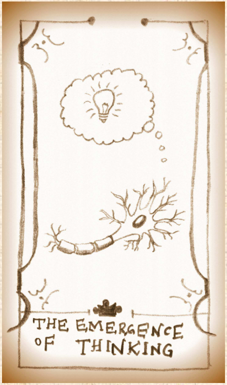
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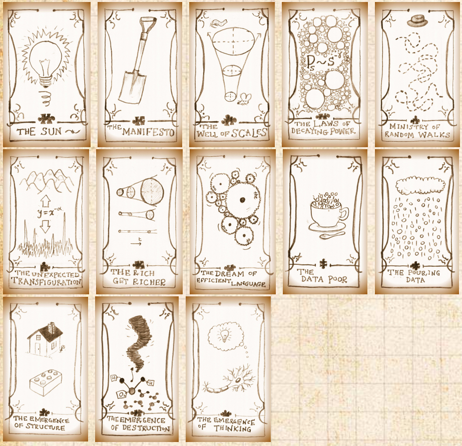
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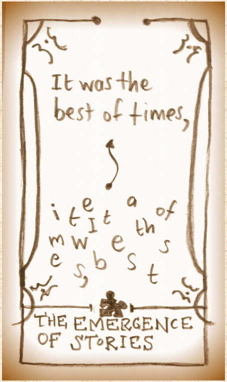
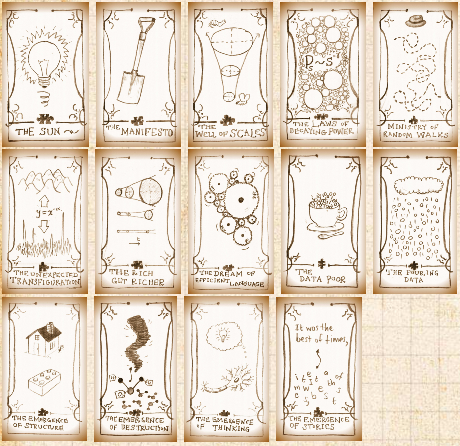

It was the best of times,
THE EMERGENCE OF STORIES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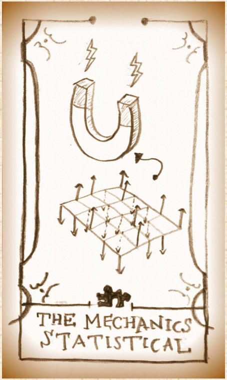
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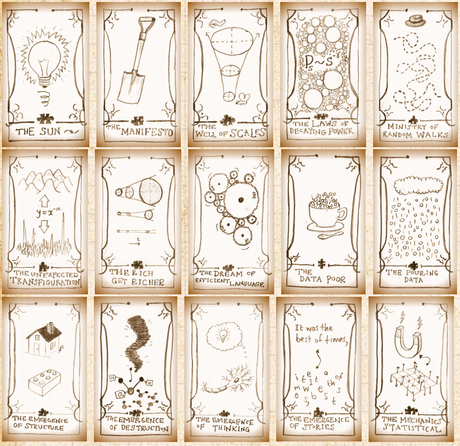
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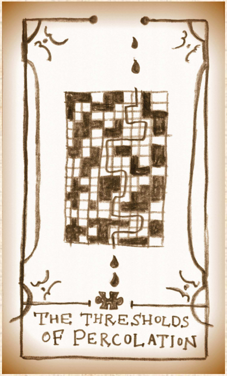
THE THRESHOLDS OF PERCOLATION


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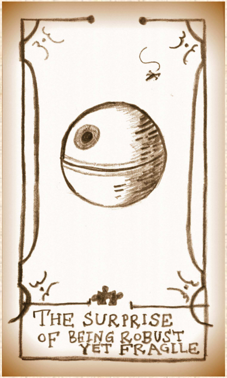
THE SURPRISE OF BEING ROBUST YET FRAGILE


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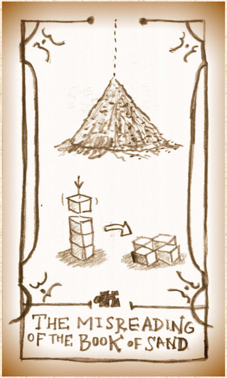
THE MISREADING OF THE BOOK OF SAND


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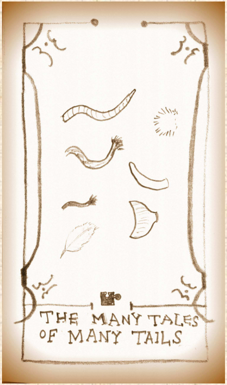
THE MANY TALES OF MANY TAILS


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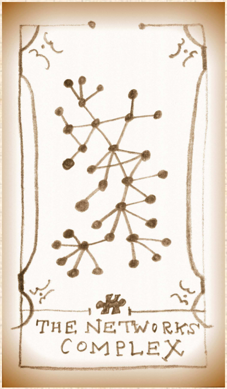
THE NETWORKS COMPLEX

Outline

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}} \mathrm{d}x$$

   CCDF = stretched exponential.

3. Gamma distributions, and more.

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lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \text{exp}\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma}\text{exp}\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu+\frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

- All moments of lognormals are finite.

lognormals

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- All moments of lognormals are finite.

Derivation from a normal distribution

Take Y as distributed normally:

$$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$

$$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

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- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

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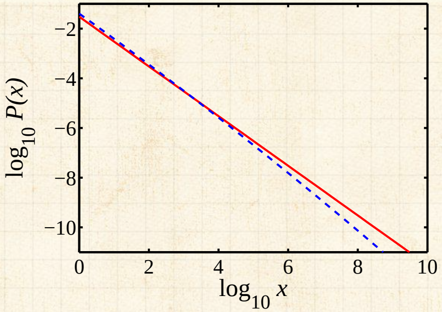
Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

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- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

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Outline

Generating lognormals:

Random multiplicative growth:

$$x_{n+1} = r x_n$$

where $r > 0$ is a random growth variable

- (Shrinkage is allowed)
- In log space, growth is by addition:

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- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
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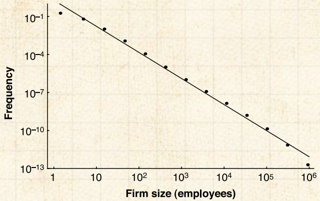

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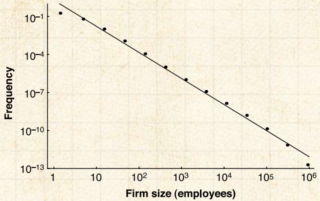

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An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

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Insert question from assignment 7

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$$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{(c/N)^{\gamma-1}-1}{(c/N)^{\gamma-1}-(c/N)}\right]$$

N = total number of firms.

Now, if $c/N \ll 1$ and $\gamma > 2$ $N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$

Which gives $\gamma \sim 1 + \frac{1}{1-c}$

Groovy: c small $\rightarrow \gamma \simeq 2$

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Outline

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

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(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

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- Insert fabulous calculation (team is spared).
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$

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- Depends on sign of $\ln\frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
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- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

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