Lognormals and friends Principles of Complex Systems | @pocsvox CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

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Dept. of Mathematics & Statistics | Vermont Complex Systems Center Vermont Advanced Computing Core | University of Vermont









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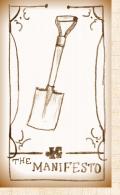
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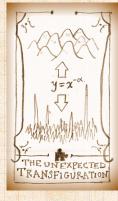
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THE UNEXPECTED TRANSFIGURATION





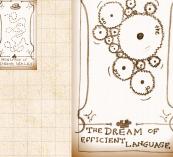


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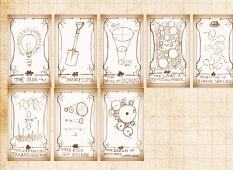
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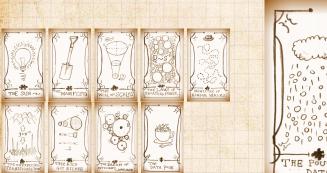








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THE REMERSION



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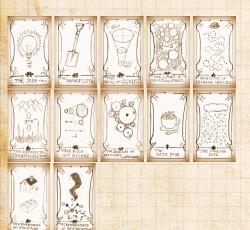
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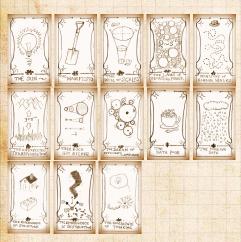


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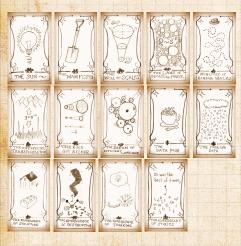
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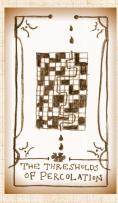
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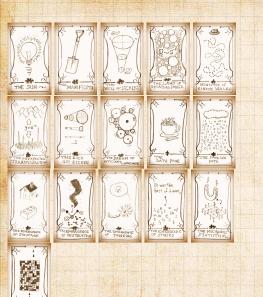
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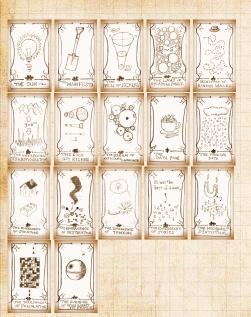
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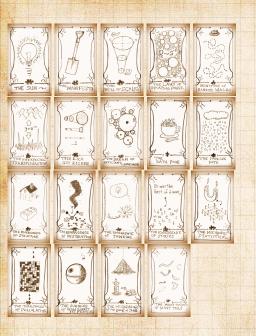
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Alternative distributions

There are other 'heavy-tailed' distributions:1. The Log-normal distribution 了

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

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$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions

$$P(x) dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}} dx$$

 $CCDF = stretched exponential \square$.





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CCDF = stretched exponential C.3. Gamma distributions C, and more.



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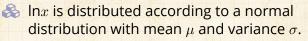
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The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$



Appears in economics and biology where growth increments are distributed normally.





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Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals.

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Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

🗞 For lognormals:

$$\begin{split} \mu_{\rm lognormal} &= e^{\mu + \frac{1}{2}\sigma^2}, & {\rm median}_{\rm lognormal} = e^{\mu}, \\ \sigma_{\rm lognormal} &= (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, & {\rm mode}_{\rm lognormal} = e^{\mu - \sigma^2}. \end{split}$$





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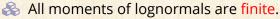
Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

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$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

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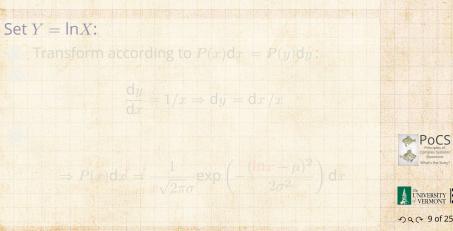
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$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

Set $Y = \ln X$:

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Solution Transform according to P(x)dx = P(y)dy:

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$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

Set $Y = \ln X$: Set $Y = \ln X$: Transform according to P(x)dx = P(y)dy:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$

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Derivation from a normal distribution Take *Y* as distributed normally:

2

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$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

Set $Y = \ln X$: \Im Transform according to P(x)dx = P(y)dy:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$

$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

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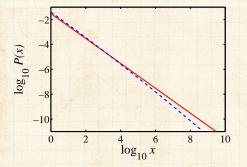
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Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude! PoCS | @pocsvox

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Solution For lognormal (blue), $\mu = 0$ and $\sigma = 10$. For power law (red), $\gamma = 1$ and c = 0.03.





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Confusion What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

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What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

If the first term is relatively small

 $\ln P(x) \sim -\left(1 + rac{\mu}{\sigma^2}\right)\ln x + ext{const}$

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$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

 $\ln P(x) \sim -\left(1 + \frac{\mu}{\sigma^2}\right)\ln x + \text{const.}$

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If the first term is relatively small,

$$\ln P(x) \sim -\left(1 - \frac{\mu}{\sigma^2}\right) \ln x + \text{const.}$$

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What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

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$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\ln P(x) \sim -\left(1 - \frac{\mu}{\sigma^2}\right)\ln x + \text{const.} \Rightarrow \gamma = 1 - \frac{\mu}{\sigma^2}$$

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Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

 $\begin{aligned} &-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x\\ \Rightarrow &\log_{10}x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10}x \end{aligned}$

 \Im If $\mu < 0$, $\gamma > 1$ which is totally cool.

If you find a -1 exponent, you may have a lognormal distribution..





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Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

 $-\frac{1}{2\sigma^2}(\ln r)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln r$

⇒ If you find a -1 exponent, you may have a lognormal distribution.

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So If $\mu < 0$, $\gamma > 1$ which is totally cool. So If $\mu > 0$, $\gamma < 1$, not so much. So If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Expect -1 scaling to hold until $(\ln x)^2$ term become significant compared to $(\ln x)$: $-\frac{1}{2\sigma^2}(\ln e)^2 \approx 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$ $\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$ $\Rightarrow \text{ If you find a -1 exponent,}$ you may have a lognormal distribution...

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Solution Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-rac{1}{2\sigma^2}({\mathsf{In}} x)^2\simeq 0.05\left(rac{\mu}{\sigma^2}-1
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If you find a -1 exponent, you may have a lognormal distribution...

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$$-rac{1}{2\sigma^2}(\mathsf{ln} x)^2\simeq 0.05\left(rac{\mu}{\sigma^2}-1
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$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

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Solution & Solutio

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Random multiplicative growth:

where r > 0 is a random growth variable (Shrinkage is allowed) In log space, growth is by addition: $\ln x_{n+1} = \ln r + \ln x_n$ $\Rightarrow \ln x_n$ is normally distributed $\Rightarrow x_n$ is lognormally distributed

 $x_{n+1} = rx_n$

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Random multiplicative growth:

$$x_{n+1} = rx_n$$

where r > 0 is a random growth variable (Shrinkage is allowed)

 $\ln x_{n+1} = \ln r + \ln r$ $\Rightarrow \ln x_n$ is normally distributed

r_n is lognormally distributed

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Random multiplicative growth:

 $x_{n+1} = rx_n$

where r > 0 is a random growth variable (Shrinkage is allowed) In log space, growth is by addition:

 $\ln x_{n+1} = \ln r + \ln x_n$

 $\Rightarrow \ln x_n$ is normally distributed $\Rightarrow x_n$ is lognormally distributed

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3

Random multiplicative growth:

 $x_{n+1} = rx_n$

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Random multiplicative growth:

 $x_{n+1} = rx_n$

where r > 0 is a random growth variable (Shrinkage is allowed) (Shrinkage, growth is by addition:

 $\ln x_{n+1} = \ln r + \ln x_n$

 $\Rightarrow \ln x_n \text{ is normally distributed}$ $\Rightarrow x_n \text{ is lognormally distributed}$

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Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).

But Robert Axtell (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!) Problem of data censusing (missing small firms). PoCS | @pocsvox

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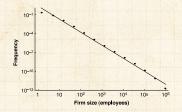
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 $\frac{\mathrm{Freq} \propto (\mathrm{size})^{-\gamma}}{\gamma \simeq 2}$

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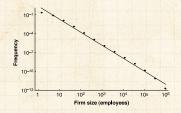
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why power laws appear with exponent $\gamma \simeq 2$ The set up: N entities with size $x_i(t)$ Generally: $x_i(t+1) = rx_i(t)$ where r is drawn from some happy distribution Same as for lognormal but one extra piece. Each x_i cannot drop too low with respect to the other sizes:

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Some math later... Insert question from assignment 7 C

where γ is implicitly given by

N = total number of firms.

Now, if $c/N \ll 1$ and $\gamma > 2$ $N = \frac{CT}{2}$

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Some math later...

3

Insert question from assignment 7 🖸

Find $P(x) \sim x^{-\gamma}$

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Some math later...

3

Insert question from assignment 7 🖸

Find $P(x) \sim x^{-\gamma}$

 \circledast where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N =total number of firms.

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Some math later...

3

2

Insert question from assignment 7 🖸

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Some math later...

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 \bigcirc Groovy... $c \text{ small} \Rightarrow \gamma \simeq 2$

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Ages of firms/people/... may not be the same

Allow the number of updates for each size x_i vary

Example: $P(t)dt = ae^{-at}dt$ where t = ageBack to no bottom limit: each x_i follows a lognormal Sizes are distributed as

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$ Now averaging different lognormal distributions

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🚳 Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

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Averaging lognormals

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 $P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) \mathrm{d}t$

Insert fabulous calculation (team is spared) Some enjoyable suffering leads to:





Averaging lognormals

3

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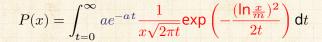
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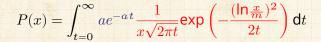
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Insert fabulous calculation (team is spared).
 Some enjoyable suffering leads to:

 $P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln \frac{x}{m})^2}$





2

 $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x)^2}}$

Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ $\frac{x}{m} < 1$. $P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \end{cases}$

Break in scaling (not uncommon) Double-Frieto destribution Ch First noticed by Montroll and Shlesinger Later: Huberman and Adamic 1: Number pages per website

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$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

Solution Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

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Lognormals and power laws can be awfully similar
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 Enforcing a minimum size leads to a power law tail
 With no minimum size but a distribution of lifetimes, the double Pareto distribution appears

Take-home message: Be careful out there...





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