Biological Contagion

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2016 | #FallPoCS2016

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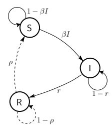

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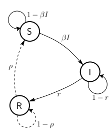
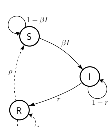
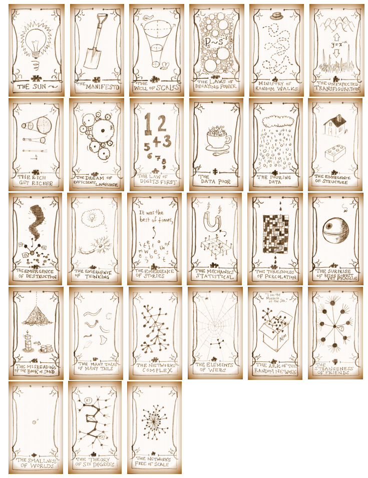
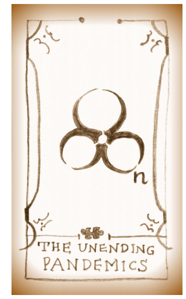
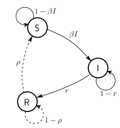


Outline

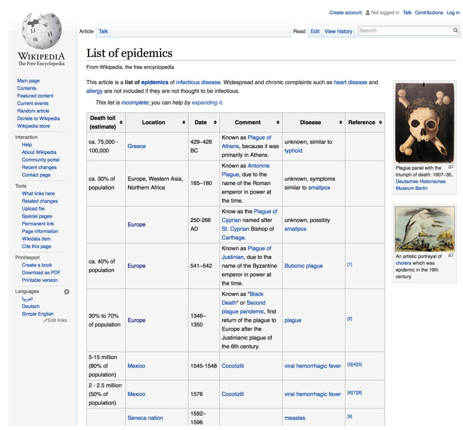








An awful recording: Wikipedia's list of epidemics from 430 BC on.



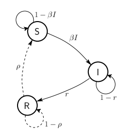
List of epidemics

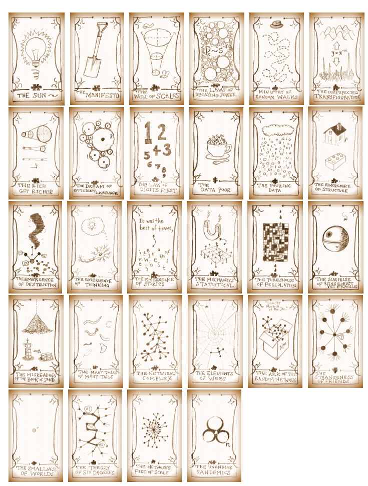
From Wikipedia, the free encyclopedia

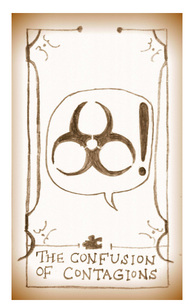
This article is a **list of epidemics** of infectious disease. Widespread and chronic complaints such as heart disease and allergy are not included if they are not thought to be infectious.

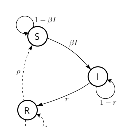
This list is incomplete; you can help by expanding it.

Death toll (estimate)	Location	Date	Comment	Disease	Reference
ca. 75,000 - 100,000	Greece	429–426 BC	Known as Plague of Athens, because it was primarily in Athens.	unknown, similar to typhoid	
ca. 30% of population	Europe, Western Asia, Northern Africa	165–180	Known as Antonine Plague, due to the name of the Roman emperor in power at the time.	unknown, symptoms similar to smallpox	
	Europe	250-266 AD	Know as the Plague of Cyprian named after St. Cyprian Bishop of Carthage.	unknown, possibly smallpox	
ca. 40% of population	Europe	541–542	Known as Plague of Justinian, due to the name of the Byzantine emperor in power at the time.	Bubonic plague	[1]
30% to 70% of population	Europe	1346–1350	Known as "Black Death" or Second plague pandemic, first return of the plague to Europe after the Justinianic plague of the 6th century.	plague	[2]
5-15 million (80% of population)	Mexico	1545-1548	Cocoliztli	viral hemorrhagic fever	[3][4][5]
2 - 2.5 million (50% of population)	Mexico	1576	Cocoliztli	viral hemorrhagic fever	[6][7][8]
	Seneca nation	1592–1596		measles	[9]









Contagion

A confusion of contagions:

- Is Harry Potter some kind of virus?
- What about the Da Vinci Code?
- Did Sudoku spread like a disease?
- Language? The alphabet? [9]
- Religion?
- Democracy...?

Contagion

Naturomorphisms

- "The feeling was contagious."
- "The news spread like wildfire."
- "Freedom is the most contagious virus known to man."
  —Hubert H. Humphrey, Johnson's vice president
- "Nothing is so contagious as enthusiasm."
  —Samuel Taylor Coleridge

Optimism according to Ambrose Bierce:

The doctrine that everything is beautiful, including what is ugly, everything good, especially the bad, and everything right that is wrong. ... It is hereditary, but fortunately not contagious.

Social contagion

Eric Hoffer, 1902–1983

There is a grandeur in the uniformity of the mass. When a fashion, a dance, a song, a slogan or a joke sweeps like wildfire from one end of the continent to the other, and a hundred million people roar with laughter, sway their bodies in unison, hum one song or break forth in anger and denunciation, there is the overpowering feeling that in this country we have come nearer the brotherhood of man than ever before.

- Hoffer was an interesting fellow...

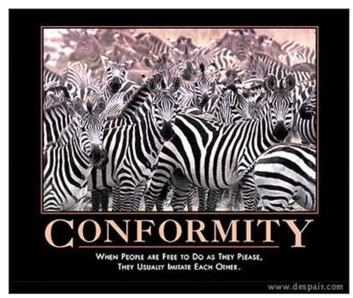
The spread of fanaticism

Hoffer's most famous work: "The True Believer: Thoughts On The Nature Of Mass Movements" (1951) [11]

Aphorisms-aplenty:

- "We can be absolutely certain only about things we do not understand."
- "Mass movements can rise and spread without belief in a God, but never without belief in a devil."
- "Where freedom is real, equality is the passion of the masses. Where equality is real, freedom is the passion of a small minority."

Imitation

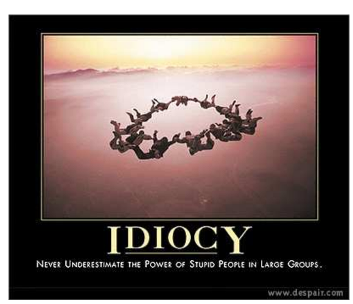


despair.com

"When people are free to do as they please, they usually imitate each other."

—Eric Hoffer "The Passionate State of Mind" [12]

The collective...



despair.com

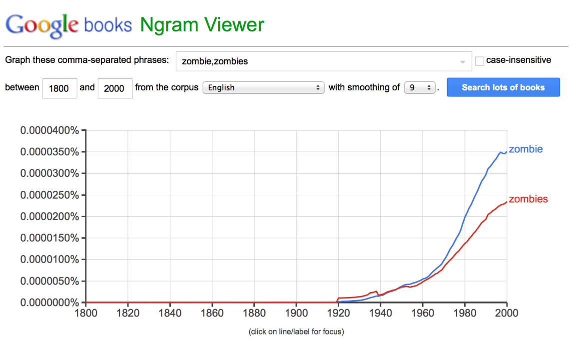
"Never Underestimate the Power of Stupid People in Large Groups."

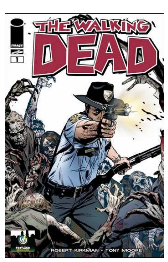
Examples of non-disease spreading:

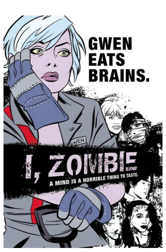
Interesting infections:

- Spreading of certain buildings in the US:
- 2008 Viral get-out-the-vote video.

The most terrifying contagious outbreak?







Contagion

Definitions

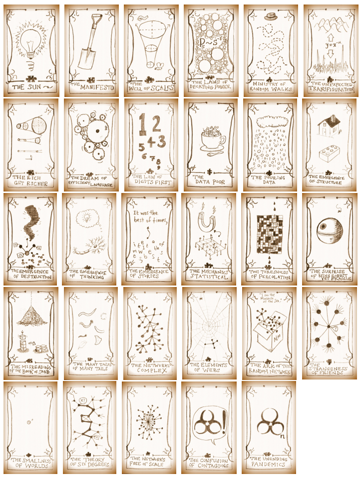
- (1) The spreading of a quality or quantity between individuals in a population.
- (2) A disease itself: the plague, a blight, the dreaded lurgi, ...
- from Latin: *con* = 'together with' + *tangere* 'to touch.'
- Contagion has unpleasant overtones...
- Just Spreading might be a more neutral word
- But contagion is kind of exciting...

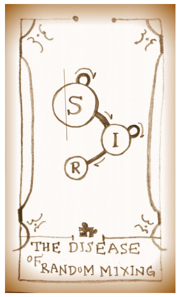
Contagions

Two main classes of contagion

1. Infectious diseases:
   tuberculosis, HIV, ebola, SARS, influenza, zombification, ...
2. Social contagion:
   fashion, word usage, rumors, uprisings, religion, stories about zombies, ...

An awful recording: Wikipedia's list of epidemics from 430 BC on.





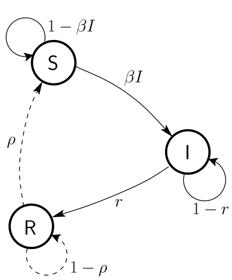
Mathematical Epidemiology

The standard SIR model [17]

- = basic model of disease contagion
- Three states:
  1. S = Susceptible
  2. I = Infective/Infectious
  3. R = Recovered or Removed or Refractory
- $S(t) + I(t) + R(t) = 1$
- Presumes random interactions (mass-action principle)
- Interactions are independent (no memory)
- Discrete and continuous time versions

Mathematical Epidemiology

Discrete time automata example:



Transition Probabilities:

β for being infected given contact with infected
r for recovery
ρ for loss of immunity

Mathematical Epidemiology

Original models attributed to

- 1920's: Reed and Frost
- 1920's/1930's: Kermack and McKendrick [13, 15, 14]
- Coupled differential equations with a mass-action principle

Independent Interaction models

Differential equations for continuous model

$$\frac{d}{dt}S = -\beta IS + \rho R$$

$$\frac{d}{dt}I = \beta IS - rI$$

$$\frac{d}{dt}R = rI - \rho R$$

β, r, and ρ are now rates.

Reproduction Number R_0

Reproduction Number R_0

- R_0 = expected number of infected individuals resulting from a single initial infective
- Epidemic threshold: If $R_0 > 1$, 'epidemic' occurs.
- Exponential take off: R_0^n where n is the number of generations.
- Fantastically awful notation convention: R_0 and the R in SIR.

Reproduction Number R_0

Discrete version:

- Set up: One Infective in a randomly mixing population of Susceptibles
- At time $t = 0$, single infective random bumps into a Susceptible
- Probability of transmission = β
- At time $t = 1$, single Infective remains infected with probability $1 - r$
- At time $t = k$, single Infective remains infected with probability $(1 - r)^k$

Reproduction Number R_0

Discrete version:

- Expected number infected by original infective:

$$R_0 = \beta + (1 - r)\beta + (1 - r)^2\beta + (1 - r)^3\beta + \ldots$$

$$= \beta\left(1 + (1 - r) + (1 - r)^2 + (1 - r)^3 + \ldots\right)$$

$$= \beta\frac{1}{1 - (1 - r)} = \beta/r$$

For $S(0) \simeq 1$ initial susceptibles
($1 - S(0) = R(0)$ = fraction initially immune):

$$R_0 = S(0)\beta/r$$

Independent Interaction models

For the continuous version

- Second equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}I = \beta SI - rI$$

$$\frac{\mathrm{d}}{\mathrm{d}t}I = (\beta S - r)I$$

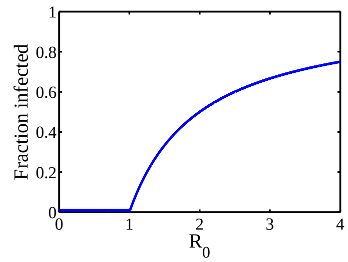
- Number of infectives grows initially if

$$\beta S(0) - r > 0 \Rightarrow \beta S(0) > r \Rightarrow \beta S(0)/r > 1$$

where $S(0) \simeq 1$.
- Same story as for discrete model.

Independent Interaction models

Example of epidemic threshold:



- Continuous phase transition.
- Fine idea from a simple model.

Independent Interaction models

Many variants of the SIR model:

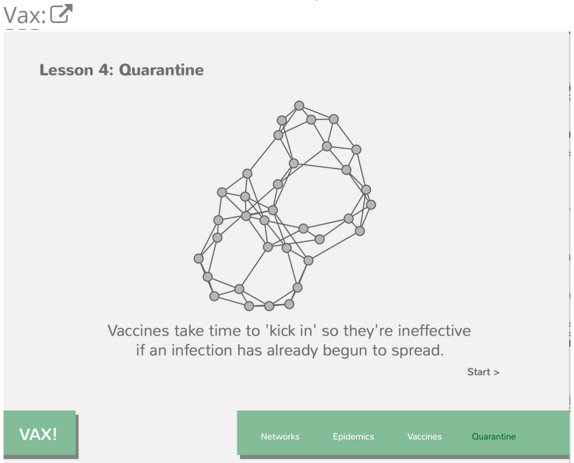
- SIS: susceptible-infective-susceptible
- SIRS: susceptible-infective-recovered-susceptible
- compartment models (age or gender partitions)
- more categories such as 'exposed' (SEIRS)
- recruitment (migration, birth)

Watch someone else pretend to save the world:

Save the world yourself:

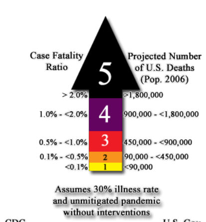
- And you can be the virus.
- Also contagious?: Cooperative games ...

Neural reboot—Save another pretend world with Vax:



Pandemic severity index (PSI)

- Classification during/post pandemic:



- Category based.
- 1–5 scale.
- Modeled on the Saffir-Simpson hurricane scale.

For novel diseases:

1. Can we predict the size of an epidemic?
2. How important is the reproduction number R_0?

R_0 approximately same for all of the following:

- 1918-19 “Spanish Flu” $\sim$ 75,000,000 world-wide, 500,000 deaths in US.
- 1957-58 “Asian Flu” $\sim$ 2,000,000 world-wide, 70,000 deaths in US.
- 1968-69 “Hong Kong Flu” $\sim$ 1,000,000 world-wide, 34,000 deaths in US.
- 2003 “SARS Epidemic” $\sim$ 800 deaths world-wide.

Size distributions

Size distributions are important elsewhere:

- earthquakes (Gutenberg-Richter law)
- city sizes, forest fires, war fatalities
- wealth distributions
- ‘popularity’ (books, music, websites, ideas)
- Epidemics?

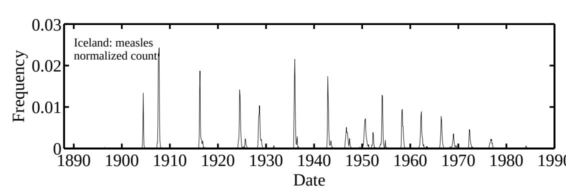
Power laws distributions are common but not obligatory...

Really, what about epidemics?

- Simply hasn’t attracted much attention.
- Data not as clean as for other phenomena.

Feeling Ill in Iceland

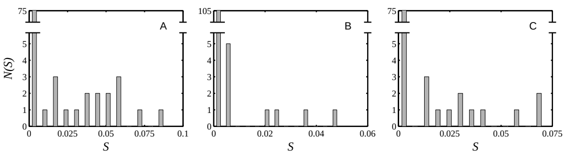
Caseload recorded monthly for range of diseases in Iceland, 1888-1990



- Treat outbreaks separated in time as ‘novel’ diseases.

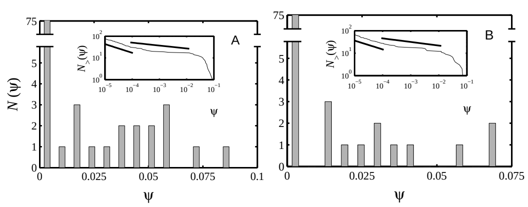
Really not so good at all in Iceland

Epidemic size distributions $N(S)$ for Measles, Rubella, and Whooping Cough.



Spike near $S = 0$, relatively flat otherwise.

Measles & Pertussis



Insert plots:
Complementary cumulative frequency distributions:

$$N(\Psi' > \Psi) \propto \Psi^{-\gamma+1}$$

Limited scaling with a possible break.

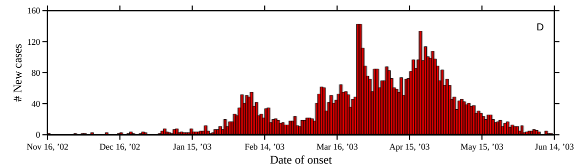
Power law distributions

Measured values of γ:

- measles: 1.40 (low Ψ) and 1.13 (high Ψ)
- pertussis: 1.39 (low Ψ) and 1.16 (high Ψ)

- Expect $2 \leq \gamma < 3$ (finite mean, infinite variance)
- When $\gamma < 1$, can't normalize
- Distribution is quite flat.

Resurgence—example of SARS

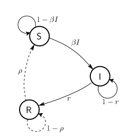


- Epidemic slows...
  then an infective moves to a new context.
- Epidemic discovers new 'pools' of susceptibles: Resurgence.
- Importance of rare, stochastic events.

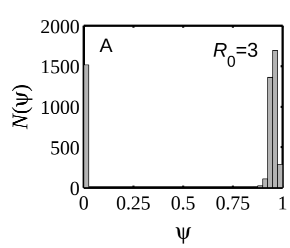
The challenge

So... can a simple model produce

1. broad epidemic distributions
   and
2. resurgence ?

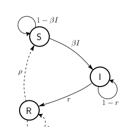


Size distributions



Simple models typically produce bimodal or unimodal size distributions.

- This includes network models:
  random, small-world, scale-free, ...
- Exceptions:
  1. Forest fire models
  2. Sophisticated metapopulation models



Burning through the population

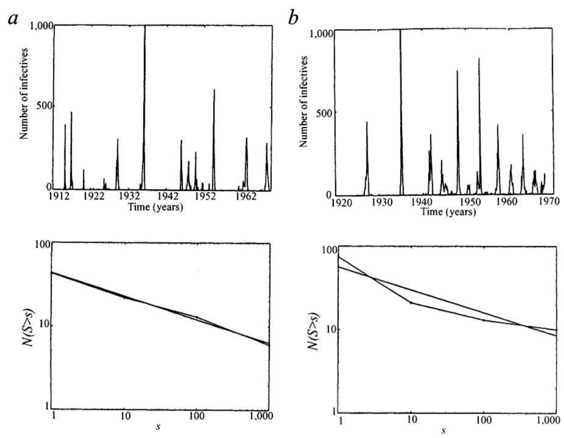
Forest fire models: [18]

- Rhodes & Anderson, 1996
- The physicist's approach:
  "if it works for magnets, it'll work for people..."

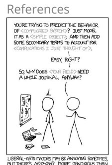
A bit of a stretch:

1. Epidemics ≡ forest fires spreading on 3-d and 5-d lattices.
2. Claim Iceland and Faroe Islands exhibit power law distributions for outbreaks.
3. Original forest fire model not completely understood.

Size distributions

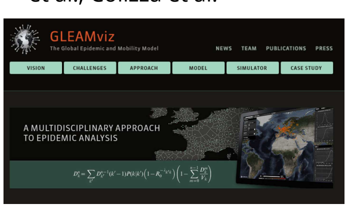


From Rhodes and Anderson, 1996.

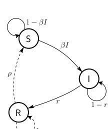


Sophisticated metapopulation models:

- Multiscale models suggested earlier by others but not formalized (Bailey [1], Cliff and Haggett [5], Ferguson et al.)
- Community based mixing (two scales)—Longini. [16]
- Eubank et al.'s EpiSims/TRANSIMS—city simulations. [8]
- Spreading through countries—Airlines: Germann et al., Colizza et al. [6]

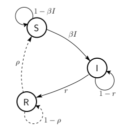


- GLEAM: Global pandemic simulations by Vespignani et al.



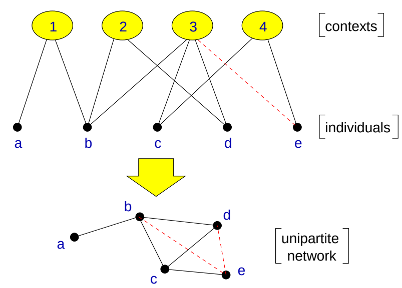
Size distributions

- Vital work but perhaps hard to generalize from...
- ⇒ Create a simple model involving multiscale travel
- Very big question: What is N?
- Should we model SARS in Hong Kong as spreading in a neighborhood, in Hong Kong, Asia, or the world?
- For simple models, we need to know the final size beforehand...

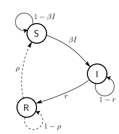


Improving simple models

Contexts and Identities—Bipartite networks



- boards of directors
- movies
- transportation modes (subway)



Improving simple models

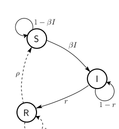
Idea for social networks: incorporate identity

Identity is formed from attributes such as:

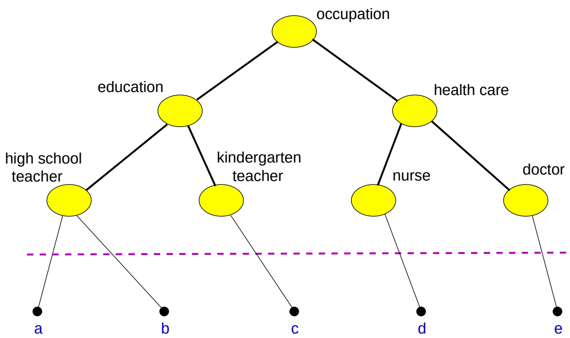
- Geographic location
- Type of employment
- Age
- Recreational activities

Groups are crucial...

- formed by people with at least one similar attribute
- Attributes ⇔ Contexts ⇔ Interactions ⇔ Networks. [22]

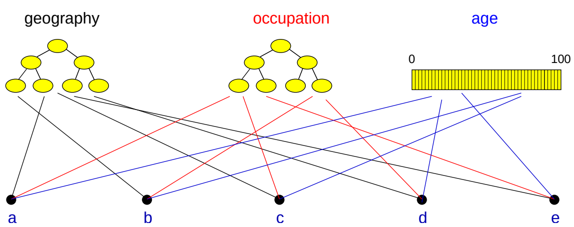


Infer interactions/network from identities



Distance makes sense in identity/context space.

Generalized context space



(Blau & Schwartz [3], Simmel [19], Breiger [4])

A toy agent-based model:

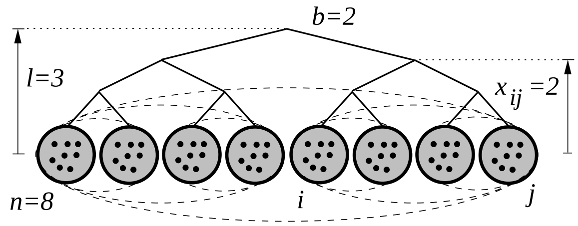
"Multiscale, resurgent epidemics in a hierarchcial metapopulation model"
Watts et al.,
Proc. Natl. Acad. Sci., **102**, 11157–11162, 2005. [23]

Geography: allow people to move between contexts

- Locally: standard SIR model with random mixing
- discrete time simulation
- β = infection probability
- γ = recovery probability
- P = probability of travel
- Movement distance: $\Pr(d) \propto \exp(-d/\xi)$
- ξ = typical travel distance

A toy agent-based model

Schematic:

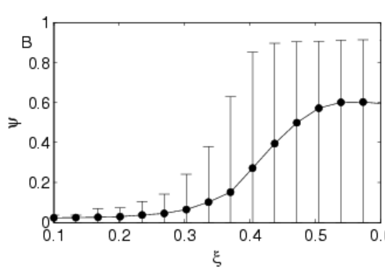


Model output

- Define P_0 = Expected number of infected individuals leaving initially infected context.
- Need $P_0 > 1$ for disease to spread (independent of R_0).
- Limit epidemic size by restricting frequency of travel and/or range

Model output

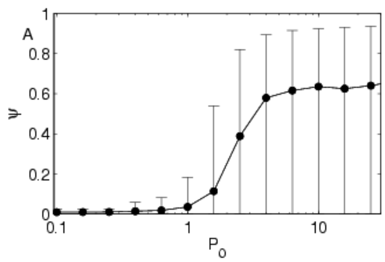
Varying ξ:



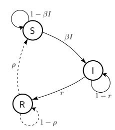
- Transition in expected final size based on typical movement distance (sensible)

Model output

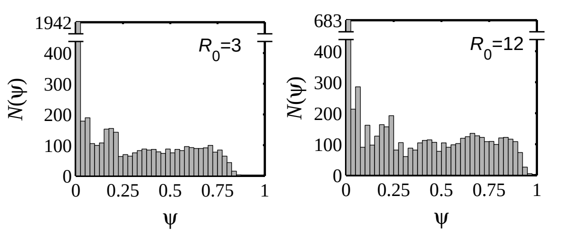
Varying P_0:



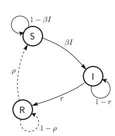
- Transition in expected final size based on typical number of infectives leaving first group (also sensible)
- Travel advisories: ξ has larger effect than P_0.



Example model output: size distributions

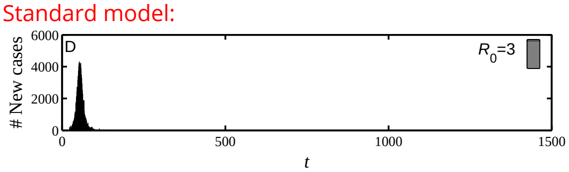


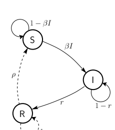
- Flat distributions are possible for certain ξ and P.
- Different R_0's may produce similar distributions
- Same epidemic sizes may arise from different R_0's



Model output—resurgence

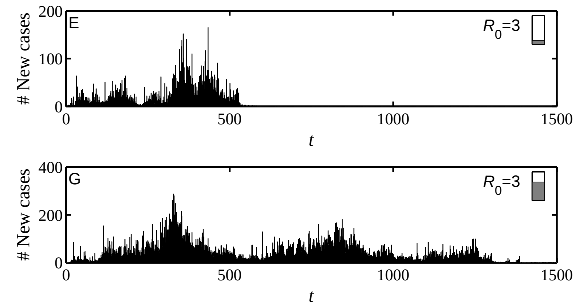
Standard model:

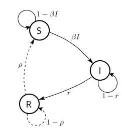




Model output—resurgence

Standard model with transport:



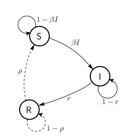


The upshot

Simple multiscale population structure
+
stochasticity

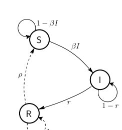
leads to

resurgence
+
broad epidemic size distributions



Nutshelling

- For the hierarchical movement model, epidemic size is highly unpredictable
- Model is more complicated than SIR but still simple.
- We haven't even included normal social responses such as travel bans and self-quarantine.
- The reproduction number R_0 is not terribly useful.
- R_0, however measured, is not informative about
  1. how likely the observed epidemic size was,
  2. and how likely future epidemics will be.
- Problem: R_0 summarises one epidemic after the fact and enfolds movement, the price of bananas, everything.



Conclusions

- Disease's spread is highly sensitive to population structure.
- Rare events may matter enormously: e.g., an infected individual taking an international flight.
- More support for controlling population movement:
  e.g., travel advisories, quarantine

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Nutshelling

What to do:

- Need to separate movement from disease
- R_0 needs a friend or two.
- Need $R_0 > 1$ and $P_0 > 1$ and ξ sufficiently large for disease to have a chance of spreading
- And in general: keep building up the kitchen sink models.

More wondering:

- Exactly how important are rare events in disease spreading?
- Again, what is N?

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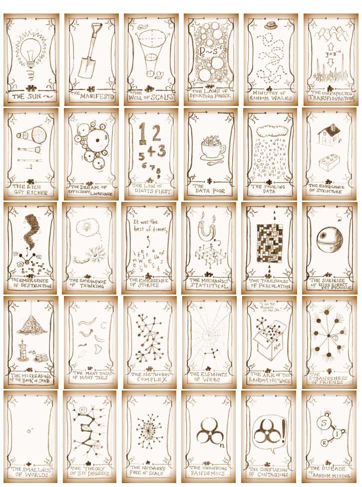

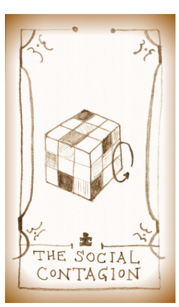

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Krugman, 1998: "Why most economists' predictions are wrong."

"The growth of the Internet will slow drastically, as the flaw in "Metcalfe's law"—which states that the number of potential connections in a network is proportional to the square of the number of participants—becomes apparent: most people have nothing to say to each other! By 2005 or so, it will become clear that the Internet's impact on the economy has been no greater than the fax machine's."[1]

[1] http://www.redherring.com/mag/issue55/economics.html

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Economics, Schmeconomics

Alan Greenspan (September 18, 2007):

"I've been dealing with these big mathematical models of forecasting the economy ...

If I could figure out a way to determine whether or not people are more fearful or changing to more euphoric,

I don't need any of this other stuff.

I could forecast the economy better than any way I know."

http://wikipedia.org

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Economics, Schmeconomics

Greenspan continues:

"The trouble is that we can't figure that out. I've been in the forecasting business for 50 years. I'm no better than I ever was, and nobody else is. Forecasting 50 years ago was as good or as bad as it is today. And the reason is that human nature hasn't changed. We can't improve ourselves."

Jon Stewart:

"You just bummed the @*!# out of me."

wildbluffmedia.com

- From the Daily Show (September 18, 2007)
- The full inteview is here.

Predicting social catastrophe isn't easy...

"Greenspan Concedes Error on Regulation"

- ...humbled Mr. Greenspan admitted that he had put too much faith in the self-correcting power of free markets ...
- "Those of us who have looked to the self-interest of lending institutions to protect shareholders' equity, myself included, are in a state of shocked disbelief"
- Rep. Henry A. Waxman: "Do you feel that your ideology pushed you to make decisions that you wish you had not made?"
- Mr. Greenspan conceded: "Yes, I've found a flaw. I don't know how significant or permanent it is. But I've been very distressed by that fact."

New York Times, October 23, 2008

Economics, Schmeconomics

James K. Galbraith:

NYT But there are at least 15,000 professional economists in this country, and you're saying only two or three of them foresaw the mortgage crisis?
[JKG] Ten or 12 would be closer than two or three.

NYT What does that say about the field of economics, which claims to be a science? [JKG] It's an enormous blot on the reputation of the profession. There are thousands of economists. Most of them teach. And most of them teach a theoretical framework that has been shown to be fundamentally useless.

From the New York Times, 11/02/2008

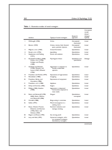
Other attempts to use SIR and co. elsewhere:

- Adoption of ideas/beliefs (Goffman & Newell, 1964) [10]
- Spread of rumors (Daley & Kendall, 1965) [7]
- Diffusion of innovations (Bass, 1969) [2]
- Spread of fanatical behavior (Castillo-Chávez & Song, 2003)
- Spread of Feynmann diagrams (Bettencourt et al., 2006)

Social contagion:

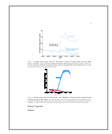
- SIR may apply sometimes ...
- But we need new fundamental models.
- Next up: Thresholds.

We really should know social contagion is different but ...



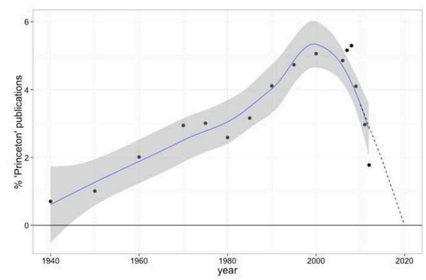
"It's contagious: Rethinking a metaphor dialogically"
Warren and Power,
Culture & Psychology, **21**, 359–379, 2015. [21]

- "Facebook will lose 80% of users by 2017, say Princeton researchers" (Guardian, 2014)



"Epidemiological modeling of online social network dynamics"
Spechler and Cannarella,
Availabe online at
`http://arxiv.org/abs/1401.4208`,
2014. [20]

The Facebook Data Science team's response:



- Mike Develin, Lada Adamic, and Sean Taylor.

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