

Optimal Supply Networks III: Redistribution

Complex Networks | @networksvox
 CSYS/MATH 303, Spring, 2016

Prof. Peter Dodds | @peterdodds

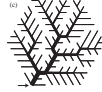
Dept. of Mathematics & Statistics | Vermont Complex Systems Center
 Vermont Advanced Computing Core | University of Vermont



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 References



1 of 47

Many sources, many sinks

How do we distribute sources?

- 🧩 Focus on 2-d (results generalize to higher dimensions).
- 🧩 Sources = hospitals, post offices, pubs, ...
- 🧩 **Key problem:** How do we cope with uneven population densities?
- 🧩 Obvious: if density is uniform then sources are best distributed **uniformly**.
- 🧩 Which lattice is optimal? The **hexagonal lattice**
- 🧩 **Q2:** Given population density is uneven, what do we do?
- 🧩 We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um *et al.* (2009) [6], and work cited by them.

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 Size-density law
 Cartograms
 A reasonable derivation
 Global redistribution
 Public versus Private
 References



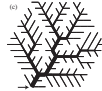
4 of 47

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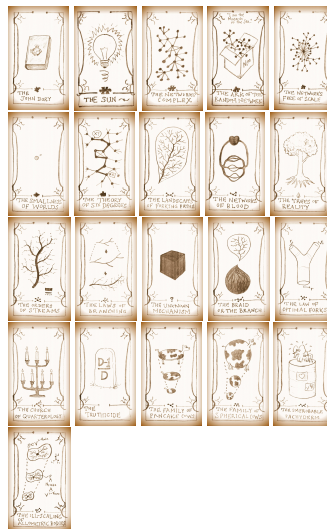


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Distributed Sources
 Size-density law
 Cartograms
 A reasonable derivation
 Global redistribution
 Public versus Private
 References



2 of 47



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Distributed Sources
 Size-density law
 Cartograms
 A reasonable derivation
 Global redistribution
 Public versus Private
 References



5 of 47

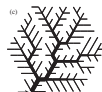
Outline

- Distributed Sources
- Size-density law
- Cartograms
- A reasonable derivation
- Global redistribution
- Public versus Private

References

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 Size-density law
 Cartograms
 A reasonable derivation
 Global redistribution
 Public versus Private
 References



3 of 47

Optimal source allocation

Solidifying the basic problem

- 🧩 Given a region with some population distribution ρ , most likely uneven.
- 🧩 Given resources to build and maintain N facilities.
- 🧩 **Q:** How do we locate these N facilities so as to **minimize the average distance** between an individual's residence and the nearest facility?

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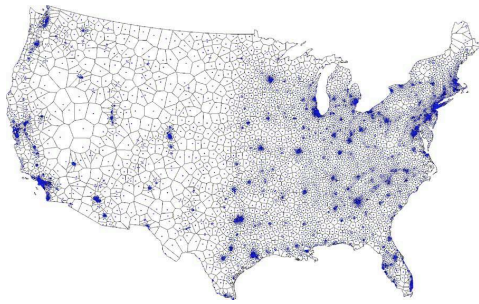
Distributed Sources
 Size-density law
 Cartograms
 A reasonable derivation
 Global redistribution
 Public versus Private
 References



6 of 47

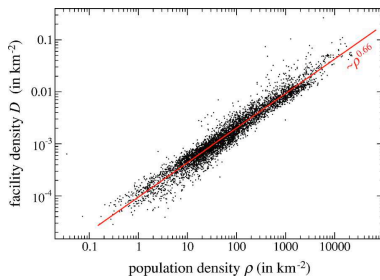


"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

Optimal source allocation



- Optimal facility density ρ_{fac} vs. population density ρ_{pop} .
- Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.
- Looking good for a $2/3$ power ...

Optimal source allocation

Size-density law:



$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{2/3}$$

- Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region.

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Distributed Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



7 of 47

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Distributed Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



8 of 47

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Distributed Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



10 of 47

Optimal source allocation



"Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries"
G. Edward Stephan,
Science, **196**, 523-524, 1977. [4]

- We first examine Stephan's treatment (1977) [4, 5]
- "Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries" (Science, 1977)
- Zipf-like approach: invokes **principle of minimal effort**.
- Also known as the Homer Simpson principle.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



11 of 47

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Optimal source allocation

- Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to **access and maintain center**.
- Write **average travel distance** to center as \bar{d} and assume **average speed of travel** is \bar{v} .
- Assume **isometry**: average travel distance \bar{d} will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



12 of 47

Optimal source allocation

- Next assume facility requires regular maintenance (person-hours per day).
- Call this quantity τ .
- If burden of maintenance is shared then average cost per person is τ/P where P = population.
- Replace P by $\rho_{\text{pop}}A$ where ρ_{pop} is density.
- Important assumption: uniform density.
- Total average time cost per person:

$$T = \bar{d}/\bar{v} + \tau/(\rho_{\text{pop}}A) = cA^{1/2}/\bar{v} + \tau/(\rho_{\text{pop}}A).$$

- Now Minimize with respect to A ...

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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



13 of 47

Optimal source allocation

☞ Differentiating ...

$$\begin{aligned} \frac{\partial T}{\partial A} &= \frac{\partial}{\partial A} (cA^{1/2}/\bar{v} + \tau/(\rho_{\text{pop}}A)) \\ &= \frac{c}{2\bar{v}A^{1/2}} - \frac{\tau}{\rho_{\text{pop}}A^2} = 0 \end{aligned}$$

☞ Rearrange:

$$A = \left(\frac{2\bar{v}\tau}{c\rho_{\text{pop}}} \right)^{2/3} \propto \rho_{\text{pop}}^{-2/3}$$

☞ # facilities per unit area ρ_{fac} :

$$\rho_{\text{fac}} \propto A^{-1} \propto \rho_{\text{pop}}^{2/3}$$

☞ Groovy ...

Optimal source allocation

An issue:

☞ Maintenance (τ) is assumed to be **independent** of population and area (P and A)

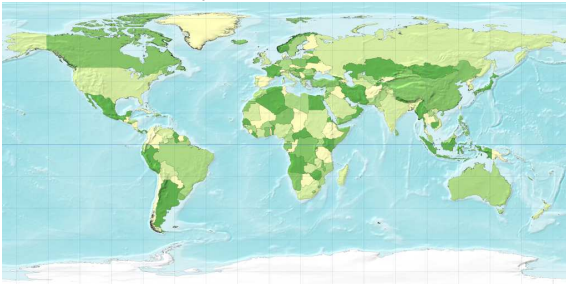
☞ Stephan's online book "[The Division of Territory in Society](#)" is [here](#).

☞ (It used to be [here](#).)

☞ The [Readme](#) is well worth reading (1995).

Cartograms

Standard world map:



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Cartograms
A reasonable derivation
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Public versus Private
References



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14 of 47

Cartograms

Cartogram of countries 'rescaled' by population:



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Cartograms
A reasonable derivation
Global redistribution
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Cartograms

Diffusion-based cartograms:

☞ Idea of cartograms is to **distort areas** to more accurately represent some local density ρ_{pop} (e.g. population).

☞ Many methods put forward—typically involve some kind of physical analogy to **spreading or repulsion**.

☞ Algorithm due to Gastner and Newman (2004)^[1] is based on **standard diffusion**:

$$\nabla^2 \rho_{\text{pop}} - \frac{\partial \rho_{\text{pop}}}{\partial t} = 0.$$

☞ Allow density to diffuse and trace the movement of individual elements and boundaries.

☞ Diffusion is constrained by boundary condition of surrounding area having density $\bar{\rho}_{\text{pop}}$.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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15 of 47

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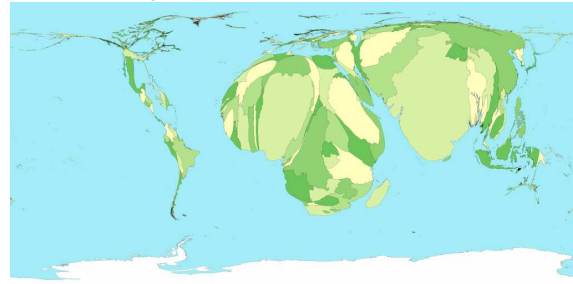
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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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19 of 47

Cartograms

Child mortality:



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Cartograms
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17 of 47

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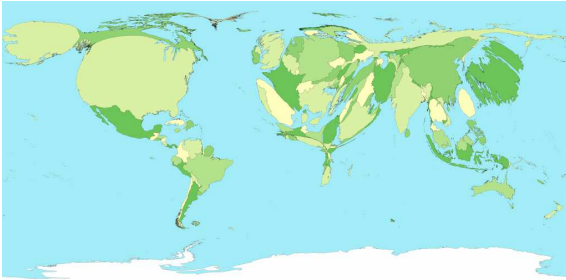
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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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20 of 47

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Energy consumption:



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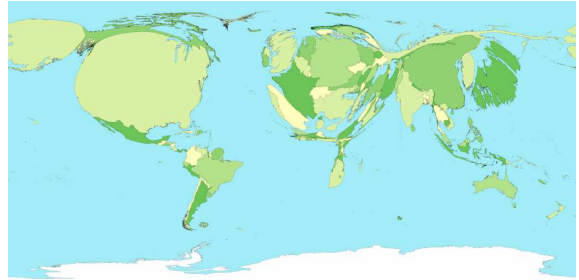
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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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21 of 47

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Spending on healthcare:



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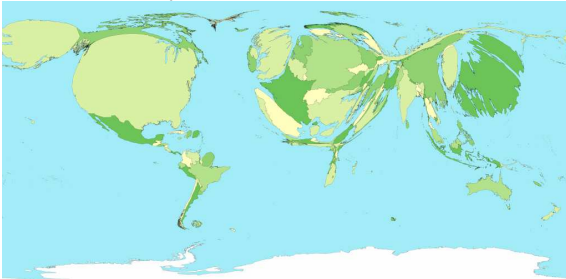
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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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24 of 47

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Gross domestic product:



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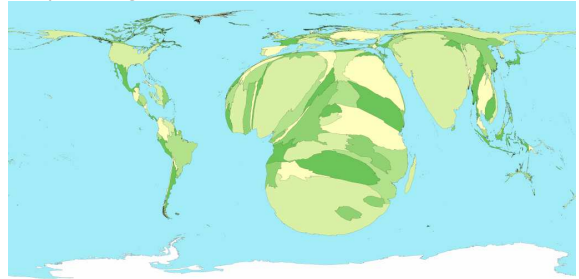
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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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22 of 47

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People living with HIV:



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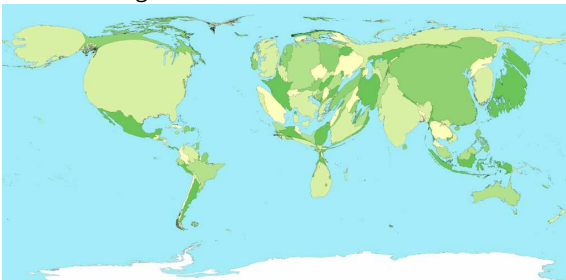
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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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25 of 47

Cartograms

Greenhouse gas emissions:



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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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23 of 47

Cartograms

The preceding sampling of Gastner & Newman's cartograms lives [here](#).

A larger collection can be found at worldmapper.org.

WORLDMAPPER The world as you've never seen it before

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References

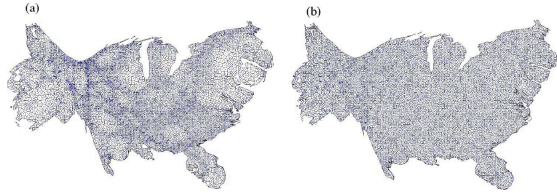


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26 of 47

Size-density law



"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



- Left: population density-equalized cartogram.
- Right: $(\text{population density})^{2/3}$ -equalized cartogram.
- Facility density is uniform for $\rho_{\text{pop}}^{2/3}$ cartogram.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



27 of 47

Size-density law

Approximations:

- For a given set of source placements $\{\vec{x}_1, \dots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells, one per source.
- Define $A(\vec{x})$ as the area of the Voronoi cell containing \vec{x} .
- As per Stephan's calculation, estimate typical distance from \vec{x} to the nearest source (say i) as

$$c_i A(\vec{x})^{1/2}$$

- where c_i is a shape factor for the i th Voronoi cell.
- Approximate c_i as a constant c .

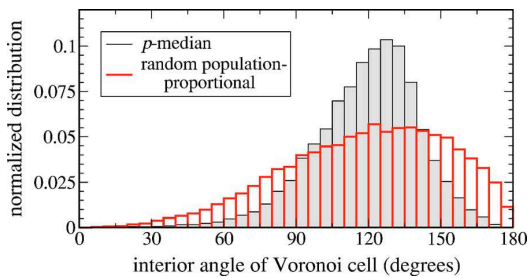
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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



31 of 47

Size-density law



From Gastner and Newman (2006) [2]

- Cartogram's Voronoi cells are somewhat hexagonal.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



28 of 47

Size-density law

Carrying on:

- The cost function is now

$$F = c \int_{\Omega} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} d\vec{x}.$$

- We also have that the constraint that Voronoi cells divide up the overall area of Ω : $\sum_{i=1}^n A(\vec{x}_i) = A_{\Omega}$.
- Sneakily turn this into an integral constraint:

$$\int_{\Omega} \frac{d\vec{x}}{A(\vec{x})} = n.$$

- Within each cell, $A(\vec{x})$ is constant.
- So ...integral over each of the n cells equals 1.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



32 of 47

Size-density law

Deriving the optimal source distribution:

- Basic idea:** Minimize the average distance from a random individual to the nearest facility. [2]
- Assume given a fixed population density ρ_{pop} defined on a spatial region Ω .
- Formally, we want to find the locations of n sources $\{\vec{x}_1, \dots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_{\Omega} \rho_{\text{pop}}(\vec{x}) \min_i \|\vec{x} - \vec{x}_i\| d\vec{x}.$$

- Also known as the p -median problem.
- Not easy ...in fact this one is an NP-hard problem. [2]
- Approximate solution originally due to Gusein-Zade [3].

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



30 of 47

Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, \dots, \vec{x}_n\}$, minimize

$$G(A) = c \int_{\Omega} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} d\vec{x} - \lambda \left(n - \int_{\Omega} [A(\vec{x})]^{-1} d\vec{x} \right)$$

- I Can Haz Calculus of Variations?
- Compute $\delta G / \delta A$, the functional derivative of the functional $G(A)$.
- This gives

$$\int_{\Omega} \left[\frac{c}{2} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{-1/2} - \lambda [A(\vec{x})]^{-2} \right] d\vec{x} = 0.$$

- Setting the integrand to be zilch, we have:

$$\rho_{\text{pop}}(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$

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Distributed Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



33 of 47

Size-density law

Now a Lagrange multiplier story:

Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\text{pop}}^{-2/3}$$

Finally, we identify $1/A(\vec{x})$ as $\rho_{\text{fac}}(\vec{x})$, an approximation of the local source density.

Substituting $\rho_{\text{fac}} = 1/A$, we have

$$\rho_{\text{fac}}(\vec{x}) = \left(\frac{c}{2\lambda} \rho_{\text{pop}}\right)^{2/3}$$

Normalizing (or solving for λ):

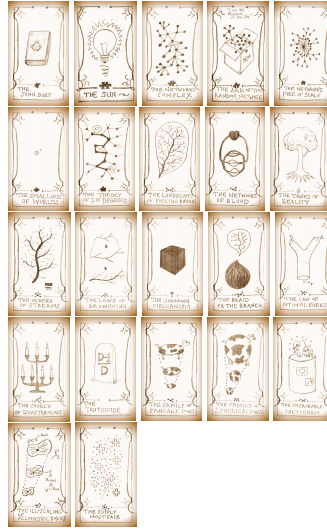
$$\rho_{\text{fac}}(\vec{x}) = n \frac{[\rho_{\text{pop}}(\vec{x})]^{2/3}}{\int_{\Omega} [\rho_{\text{pop}}(\vec{x})]^{2/3} d\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/3}$$

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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34 of 47



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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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38 of 47

Global redistribution networks

One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gastner and Newman model: cost is a function of basic maintenance and travel time:

$$C_{\text{maint}} + \gamma C_{\text{travel}}$$

Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

$$(1 - \delta)\ell_{ij} + \delta(\#\text{hops})$$

When $\delta = 1$, only number of hops matters.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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36 of 47

Public versus private facilities

Beyond minimizing distances:

- "Scaling laws between population and facility densities" by Um *et al.*, Proc. Natl. Acad. Sci., 2009. [6]
- Um *et al.* find empirically and argue theoretically that the connection between facility and population density

$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^{\alpha}$$

does not universally hold with $\alpha = 2/3$.

- Two idealized limiting classes:
 - For-profit, commercial facilities: $\alpha = 1$;
 - Pro-social, public facilities: $\alpha = 2/3$.
- Um *et al.* investigate facility locations in the United States and South Korea.

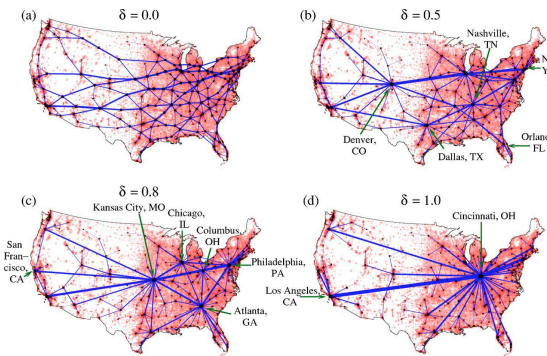
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Distributed Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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40 of 47

Global redistribution networks



From Gastner and Newman (2006) [2]

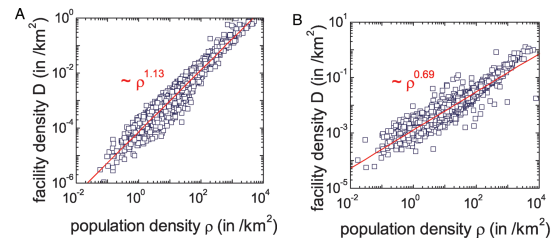
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Distributed Sources
Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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37 of 47

Public versus private facilities: evidence



- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \approx 2/3$ to $\alpha = 1$ around $\rho_{\text{pop}} \approx 100$.

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



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41 of 47

Public versus private facilities: evidence

US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87
SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition between public and private at $\alpha \approx 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.

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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



Public versus private facilities: the story

Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um *et al.* do, observing that the cost for each cell should be the same, we have:

$$\rho_{\text{fac}}(\vec{x}) = n \frac{[\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}}{\int_{\Omega} [\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)} d\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}$$

For $\beta = 0, \alpha = 1$: commercial scaling is linear.

For $\beta = 1, \alpha = 2/3$: social scaling is sublinear.

You can try this too:

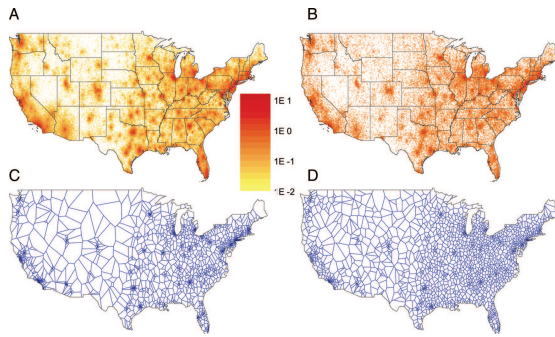
[Insert question from assignment 4](#)

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



Public versus private facilities: evidence



A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

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Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



References I

- [1] M. T. Gastner and M. E. J. Newman. Diffusion-based method for producing density-equalizing maps. [Proc. Natl. Acad. Sci., 101:7499–7504, 2004. pdf](#)
- [2] M. T. Gastner and M. E. J. Newman. Optimal design of spatial distribution networks. [Phys. Rev. E, 74:016117, 2006. pdf](#)
- [3] S. M. Gusein-Zade. Bunge's problem in central place theory and its generalizations. [Geogr. Anal., 14:246–252, 1982.](#)
- [4] G. E. Stephan. Territorial division: The least-time constraint behind the formation of subnational boundaries. [Science, 196:523–524, 1977. pdf](#)

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References



Public versus private facilities: the story

So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- Defns: For the i th facility and its Voronoi cell V_i , define
 - n_i = population of the i th cell;
 - $\langle r_i \rangle$ = the average travel distance to the i th facility.
 - A_i = area of i th cell (s_i in
- Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^\beta \text{ with } 0 \leq \beta \leq 1.$$

Limits:

- $\beta = 0$: purely commercial.
- $\beta = 1$: purely social.

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A reasonable derivation
Global redistribution
Public versus Private
References



References II

- [5] G. E. Stephan. Territorial subdivision. [Social Forces, 63:145–159, 1984. pdf](#)
- [6] J. Um, S.-W. Son, S.-I. Lee, H. Jeong, and B. J. Kim. Scaling laws between population and facility densities. [Proc. Natl. Acad. Sci., 106:14236–14240, 2009. pdf](#)

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Size-density law
Cartograms
A reasonable derivation
Global redistribution
Public versus Private
References

