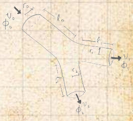
Optimal Supply Networks I: Branching

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2016

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



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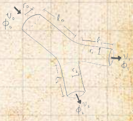
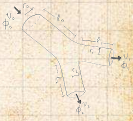


Outline

Optimal transportation

Optimal branching
- Murray's law
- Murray meets Tokunaga

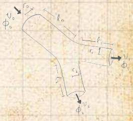
References



Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people, ...
- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
  3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems



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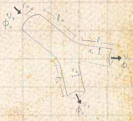

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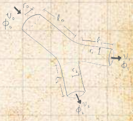

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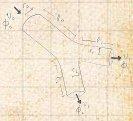

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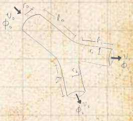

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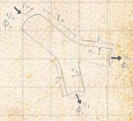

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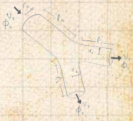


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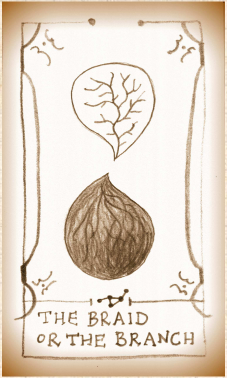
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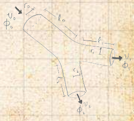


Single source optimal supply

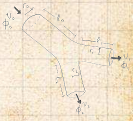
Basic question for distribution/supply networks:

- How does flow behave given cost:

$$C = \sum_j I_j^{\gamma} Z_j$$

where
I_j = current on link j
and
Z_j = link j's impedance?

- Example: $\gamma = 2$ for electrical networks.



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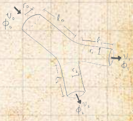

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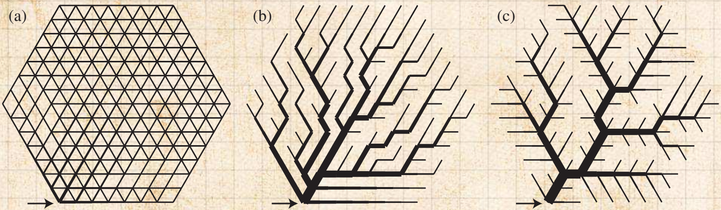
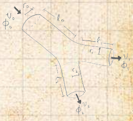


(a) $\gamma > 1$: Braided (bulk) flow

(b) $\gamma < 1$: Local minimum: Branching flow

(c) $\gamma < 1$: Global minimum: Branching flow

From Bohn and Magnasco [3]
See also Banavar et al. [1]



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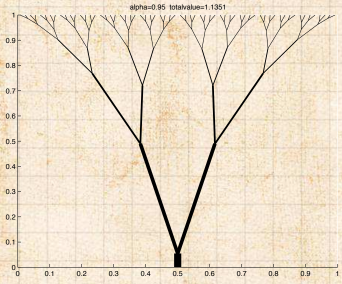
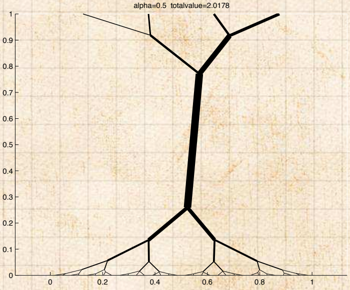
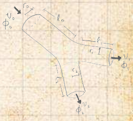
Optimal paths related to transport (Monge) problems:



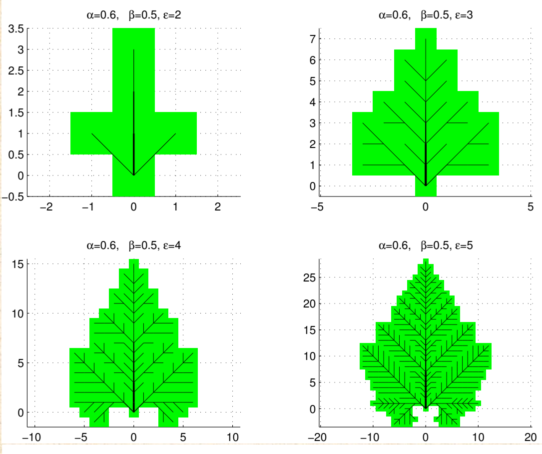


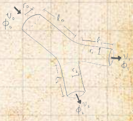
Xia (2003) [19]



Growing networks:

FIGURE 1. $\alpha = 0.6, \beta = 0.5$

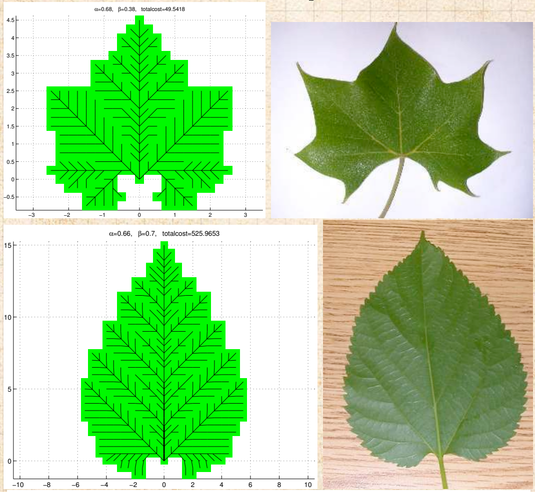




Xia (2007) [18]

Growing networks:

FIGURE 3. A maple leaf



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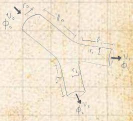


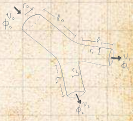
Single source optimal supply

An immensely controversial issue...

- The form of river networks and blood networks: optimal or not? [17, 2, 5, 4]

Two observations:

- Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- Real networks differ in details of scaling but reasonably agree in scaling relations.



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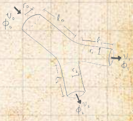

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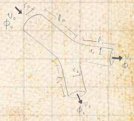


River network models

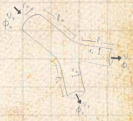
Optimality:

- Optimal channel networks [12]
- Thermodynamic analogy [13]

versus...

Randomness:

- Scheidegger's directed random networks
- Undirected random networks



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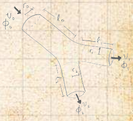


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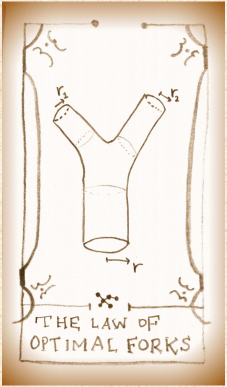
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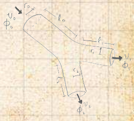

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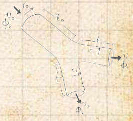
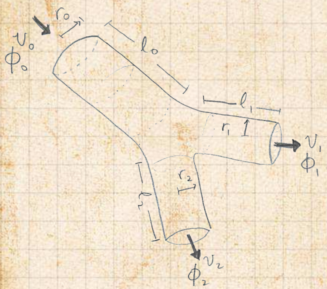


Optimization—Murray's law

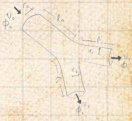


- Murray's law (1926) connects branch radii at forks: [10, 9, 11, 6, 15]

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

where r_0 = radius of main branch, and r_1 and r_2 are radii of sub-branches.

- Holds up well for outer branchings of blood networks.
- Also found to hold for trees [11, 7, 8].
- See D'Arcy Thompson's "On Growth and Form" for background inspiration [14, 15].



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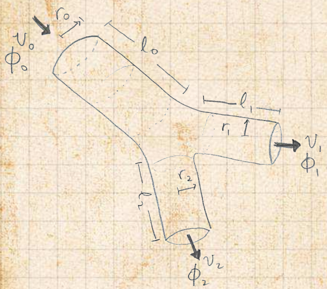

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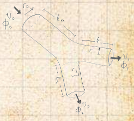

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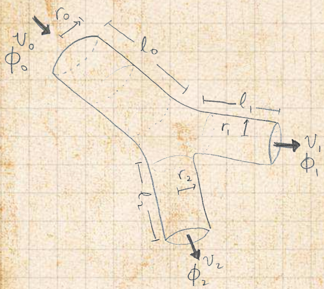

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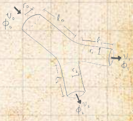

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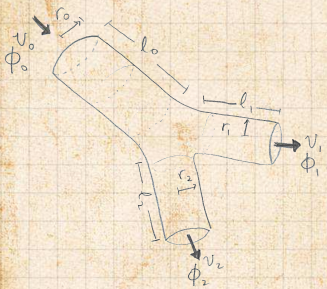

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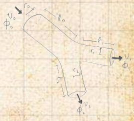
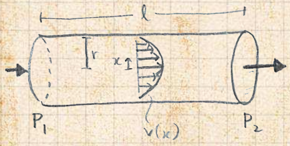


- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



- Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius r and length ℓ:

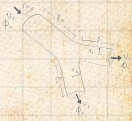
$$Z = \frac{8\eta\ell}{\pi r^4}$$

- η = dynamic viscosity (units: $ML^{-1}T^{-1}$).
- Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z$$

- Also have rate of energy expenditure in maintaining blood given metabolic constant c:

$$P_{\text{metabolic}} = c r^2 \ell$$



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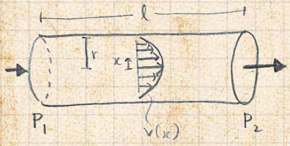

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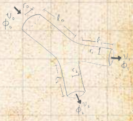

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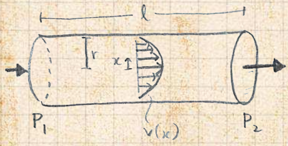

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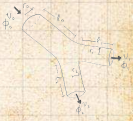

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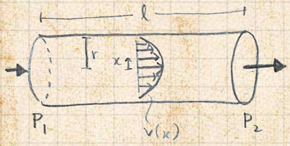

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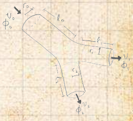
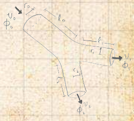


Optimization—Murray's law

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- Δp = Force per unit area
- Φ = Volume per unit time
  = cross-sectional area · velocity
- So $\Phi \Delta p$ = Force · velocity



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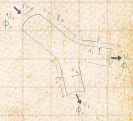

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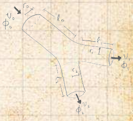

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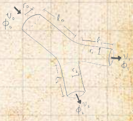

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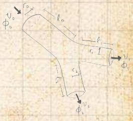

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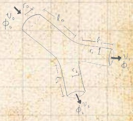

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Murray's law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

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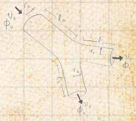

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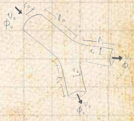

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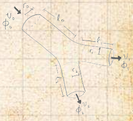

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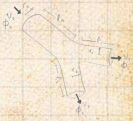

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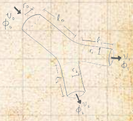

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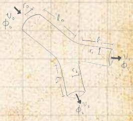

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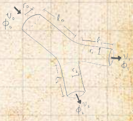
$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell\right)$$

$$= -4\Phi^2 \frac{8\eta\ell}{\pi r^5} + c2r\ell = 0$$

- Rearrange/cancel/slap

$$\Phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where k = constant.



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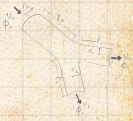

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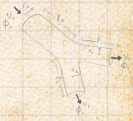

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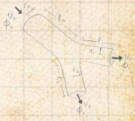

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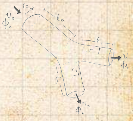

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Murray's law:

- So we now have:

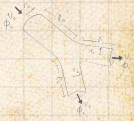
$$\Phi = kr^3$$

- Flow rates at each branching have to add up (else our organism is in serious trouble...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

  where again 0 refers to the main branch and 1 and 2 refers to the offspring branches
- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$



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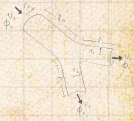

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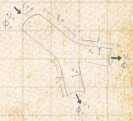
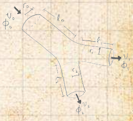


Outline



Optimization

Murray meets Tokunaga:

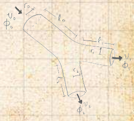
- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = k r_\omega^3$

$$r_\omega^3 = 2r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

- Find Horton ratio for vessel radius $R_r = r_\omega / r_{\omega-1}$...



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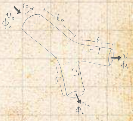

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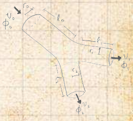

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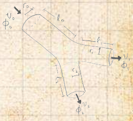

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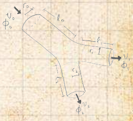

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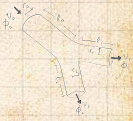

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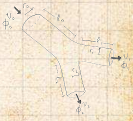

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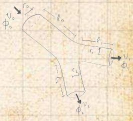

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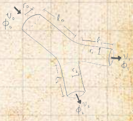

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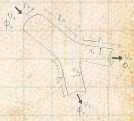

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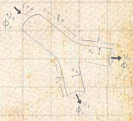

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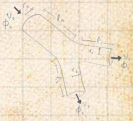

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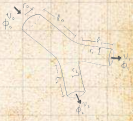

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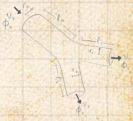

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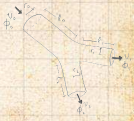

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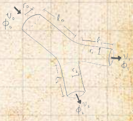

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