Optimal Supply Networks I: Branching

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2016

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

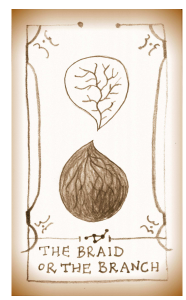
These slides are brought to you by:

Outline

Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people, ...
- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
  3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems



Single source optimal supply

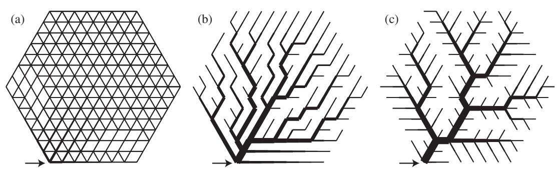
Basic question for distribution/supply networks:

- How does flow behave given cost:

$$C = \sum_j I_j^\gamma Z_j$$

  where
  I_j = current on link j
  and
  Z_j = link j's impedance?
- Example: $\gamma = 2$ for electrical networks.

Single source optimal supply

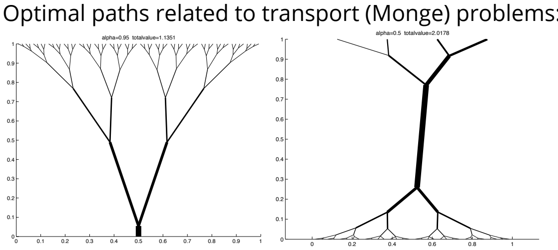


(a) $\gamma > 1$: Braided (bulk) flow
(b) $\gamma < 1$: Local minimum: Branching flow
(c) $\gamma < 1$: Global minimum: Branching flow

From Bohn and Magnasco [3]
See also Banavar et al. [1]

Single source optimal supply

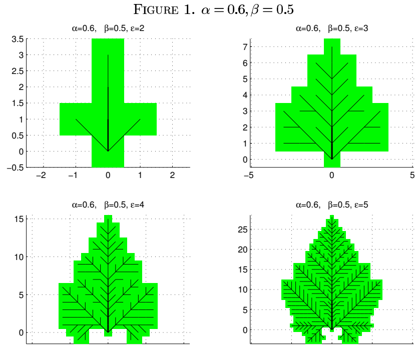
Optimal paths related to transport (Monge) problems:



Xia (2003) [19]

Growing networks:

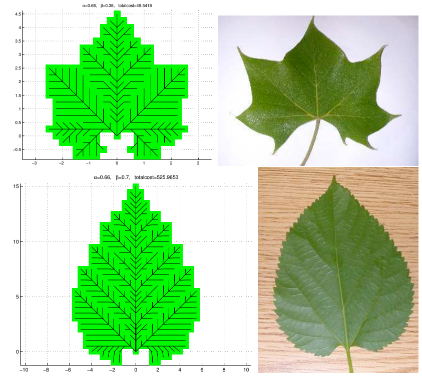
FIGURE 1. $\alpha = 0.6, \beta = 0.5$



Xia (2007) [18]

Growing networks:

FIGURE 3. A maple leaf



Xia (2007) [18]

Single source optimal supply

An immensely controversial issue...

- The form of river networks and blood networks: optimal or not? [17, 2, 5, 4]

Two observations:

- Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- Real networks differ in details of scaling but reasonably agree in scaling relations.

River network models

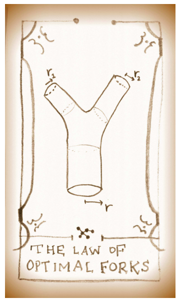
Optimality:

- Optimal channel networks [12]
- Thermodynamic analogy [13]

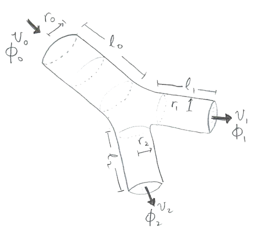
versus...

Randomness:

- Scheidegger's directed random networks
- Undirected random networks



Optimization—Murray's law



- Murray's law (1926) connects branch radii at forks:[10, 9, 11, 6, 15]

$$r_0^3 = r_1^3 + r_2^3$$

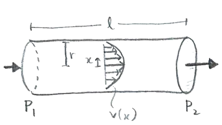
where r_0 = radius of main branch, and r_1 and r_2 are radii of sub-branches.

- Holds up well for outer branchings of blood networks.
- Also found to hold for trees[11, 7, 8].
- See D'Arcy Thompson's "On Growth and Form" for background inspiration[14, 15].

- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



- Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius r and length ℓ:

$$Z = \frac{8\eta\ell}{\pi r^4}$$

- η = dynamic viscosity (units: $ML^{-1}T^{-1}$).
- Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$

- Also have rate of energy expenditure in maintaining blood given metabolic constant c:

$$P_{\text{metabolic}} = c r^2 \ell$$

Optimization—Murray's law

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- Δp = Force per unit area
- Φ = Volume per unit time
  = cross-sectional area · velocity
- So $\Phi \Delta p$ = Force · velocity

Optimization—Murray's law

Murray's law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

- Observe power increases linearly with ℓ
- But r's effect is nonlinear:
  - increasing r makes flow easier but increases metabolic cost (as r^2)
  - decreasing r decrease metabolic cost but impedance goes up (as r^{-4})

Optimization—Murray's law

Murray's law:

- Minimize P with respect to r:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell\right)$$

$$= -4\Phi^2 \frac{8\eta\ell}{\pi r^5} + c2r\ell = 0$$

- Rearrange/cancel/slap:

$$\Phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where k = constant.

Optimization—Murray's law

Murray's law:

- So we now have:

$$\Phi = kr^3$$

- Flow rates at each branching have to add up (else our organism is in serious trouble...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

Optimization

Murray meets Tokunaga:

- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = kr_\omega^3$

$$r_\omega^3 = 2r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

- Find Horton ratio for vessel radius $R_r = r_\omega / r_{\omega-1}$...

Optimization

Murray meets Tokunaga:

- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$\boxed{R_r^3 = R_n = R_v}$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

Optimization

Murray meets Tokunaga:

- Isometry: $V_\omega \propto \ell_\omega^3$
- Gives

$$\boxed{R_\ell^3 = R_v = R_n}$$

- We need one more constraint...
- West et al (1997) [17] achieve similar results following Horton's laws.
- So does Turcotte et al. (1998) [16] using Tokunaga (sort of).

References I

[1] J. R. Banavar, F. Colaiori, A. Flammini, A. Maritan, and A. Rinaldo.
Topology of the fittest transportation network.
Phys. Rev. Lett., 84:4745–4748, 2000. pdf

[2] J. R. Banavar, A. Maritan, and A. Rinaldo.
Size and form in efficient transportation networks.
Nature, 399:130–132, 1999. pdf

[3] S. Bohn and M. O. Magnasco.
Structure, scaling, and phase transition in the optimal transport network.
Phys. Rev. Lett., 98:088702, 2007. pdf

References II

[4] P. S. Dodds.
Optimal form of branching supply and collection networks.
Phys. Rev. Lett., 104(4):048702, 2010. pdf

[5] P. S. Dodds and D. H. Rothman.
Geometry of river networks. I. Scaling, fluctuations, and deviations.
Physical Review E, 63(1):016115, 2001. pdf

[6] P. La Barbera and R. Rosso.
Reply.
Water Resources Research, 26(9):2245–2248, 1990. pdf

[7] K. A. McCulloh, J. S. Sperry, and F. R. Adler.
Water transport in plants obeys Murray's law.
Nature, 421:939–942, 2003. pdf

References III

[8] K. A. McCulloh, J. S. Sperry, and F. R. Adler.
Murray's law and the hydraulic vs mechanical functioning of wood.
Functional Ecology, 18:931–938, 2004. pdf

[9] C. D. Murray.
The physiological principle of minimum work applied to the angle of branching of arteries.
J. Gen. Physiol., 9(9):835–841, 1926. pdf

[10] C. D. Murray.
The physiological principle of minimum work. I. The vascular system and the cost of blood volume.
Proc. Natl. Acad. Sci., 12:207–214, 1926. pdf

References IV

[11] C. D. Murray.
A relationship between circumference and weight in trees and its bearing on branching angles.
J. Gen. Physiol., 10:725–729, 1927. pdf

[12] I. Rodríguez-Iturbe and A. Rinaldo.
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambrigde, UK, 1997.

[13] A. E. Scheidegger.
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.

References V

[14] D. W. Thompson.
On Growth and From.
Cambridge University Pres, Great Britain, 2nd edition, 1952.

[15] D. W. Thompson.
On Growth and Form — Abridged Edition.
Cambridge University Press, Great Britain, 1961.

[16] D. L. Turcotte, J. D. Pelletier, and W. I. Newman.
Networks with side branching in biology.
Journal of Theoretical Biology, 193:577–592, 1998. pdf

[17] G. B. West, J. H. Brown, and B. J. Enquist.
A general model for the origin of allometric scaling laws in biology.
Science, 276:122–126, 1997. pdf

References VI

[18] Q. Xia.
The formation of a tree leaf.
Submitted. pdf

[19] Q. Xia.
Optimal paths related to transport problems.
Communications in Contemporary Mathematics, 5:251–279, 2003. pdf