Optimal Supply Networks I: Branching

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CSYS/MATH 303, Spring, 2016

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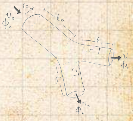

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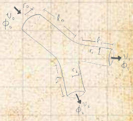
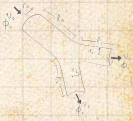


Outline

Optimal transportation

Optimal branching
- Murray's law
- Murray meets Tokunaga

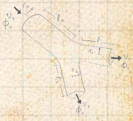
References

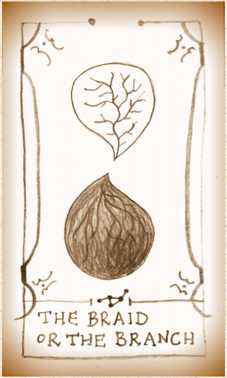


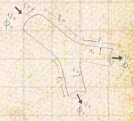
Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people, ...
- Some fundamental network problems:
    1. Distribute stuff from a single source to many sinks
    2. Distribute stuff from many sources to many sinks
    3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems







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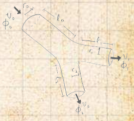
Single source optimal supply

Basic question for distribution/supply networks:

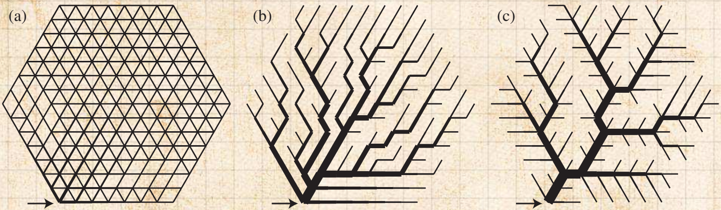
- How does flow behave given cost:

$$C = \sum_j I_j^{\gamma} Z_j$$

  where
  I_j = current on link j
  and
  Z_j = link j's impedance?
- Example: $\gamma = 2$ for electrical networks.



Single source optimal supply



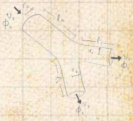
(a) $\gamma > 1$: Braided (bulk) flow

(b) $\gamma < 1$: Local minimum: Branching flow

(c) $\gamma < 1$: Global minimum: Branching flow

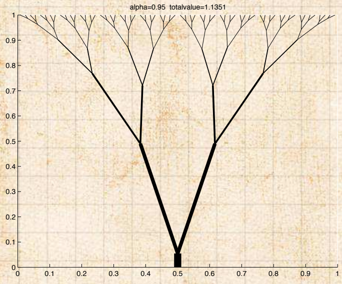
From Bohn and Magnasco [3]

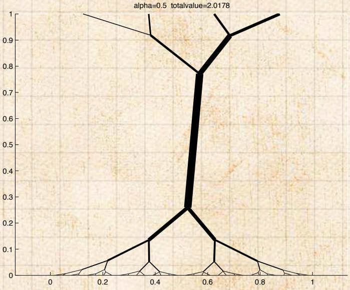
See also Banavar et al. [1]



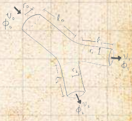
Single source optimal supply

Optimal paths related to transport (Monge) problems:



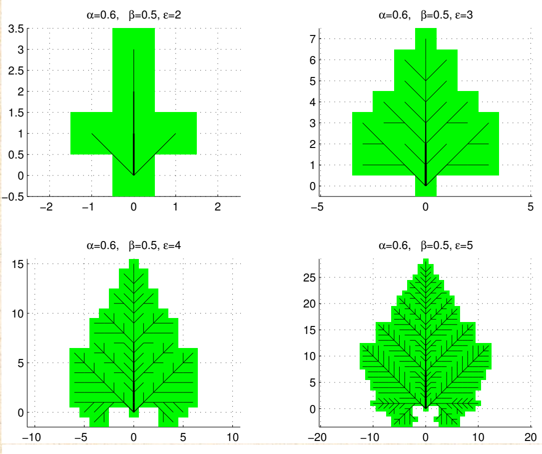


Xia (2003) [19]

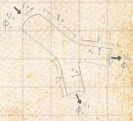


Growing networks:

FIGURE 1. $\alpha = 0.6, \beta = 0.5$

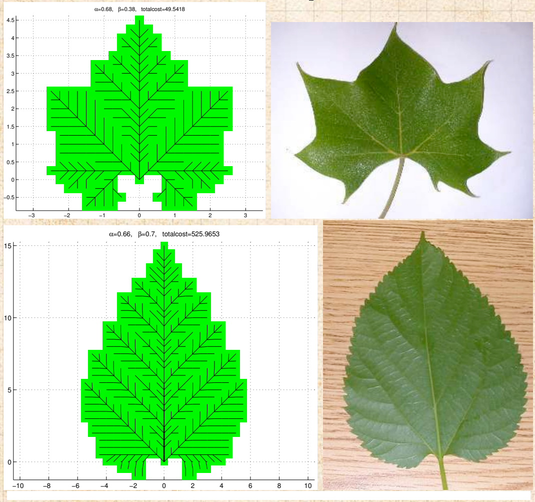


Xia (2007) [18]

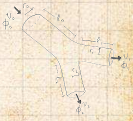


Growing networks:

FIGURE 3. A maple leaf



Xia (2007) [18]



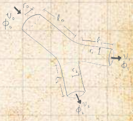
Single source optimal supply

An immensely controversial issue...

- The form of river networks and blood networks: optimal or not? [17, 2, 5, 4]

Two observations:

- Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- Real networks differ in details of scaling but reasonably agree in scaling relations.



River network models

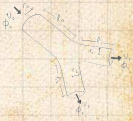
Optimality:

- Optimal channel networks [12]
- Thermodynamic analogy [13]

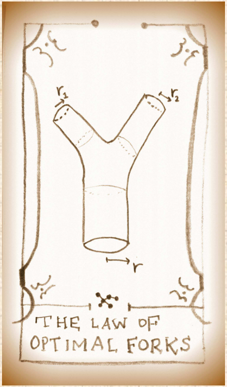
versus...

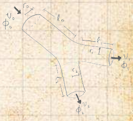
Randomness:

- Scheidegger's directed random networks
- Undirected random networks

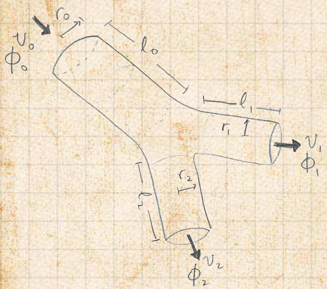



THE JOHN DORY
THE SUN
THE NETWORKS COMPLEX
"I am the Monarch of the Sea."
THE ARK OF THE RANDOM NETWORK
THE NETWORKS FREE OF SCALE
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE UNKNOWN MECHANISM
THE BRAID OR THE BRANCH


THE LAW OF OPTIMAL FORKS



Optimization—Murray's law

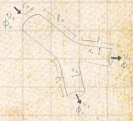


- Murray's law (1926) connects branch radii at forks: [10, 9, 11, 6, 15]

$$r_0^3 = r_1^3 + r_2^3$$

where r_0 = radius of main branch, and r_1 and r_2 are radii of sub-branches.

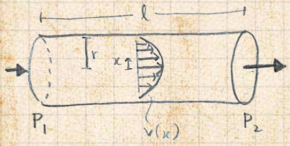
- Holds up well for outer branchings of blood networks.
- Also found to hold for trees [11, 7, 8].
- See D'Arcy Thompson's "On Growth and Form" for background inspiration [14, 15].



- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



- Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius r and length ℓ:

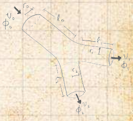
$$Z = \frac{8\eta\ell}{\pi r^4}$$

- η = dynamic viscosity (units: $ML^{-1}T^{-1}$).
- Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$

- Also have rate of energy expenditure in maintaining blood given metabolic constant c:

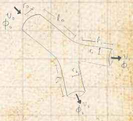
$$P_{\text{metabolic}} = c r^2 \ell$$



Optimization—Murray's law

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- Δp = Force per unit area
- Φ = Volume per unit time
  = cross-sectional area · velocity
- So $\Phi \Delta p$ = Force · velocity



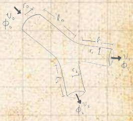
Optimization—Murray's law

Murray's law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

- Observe power increases linearly with ℓ
- But r's effect is nonlinear:
  - increasing r makes flow easier but increases metabolic cost (as r^2)
  - decreasing r decrease metabolic cost but impedance goes up (as r^{-4})



Optimization—Murray's law

Murray's law:

- Minimize P with respect to r:

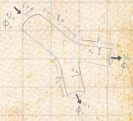
$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell\right)$$

$$= -4\Phi^2 \frac{8\eta\ell}{\pi r^5} + c2r\ell = 0$$

- Rearrange/cancel/slap:

$$\Phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where k = constant.



Optimization—Murray's law

Murray's law:

- So we now have:

$$\Phi = kr^3$$

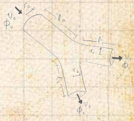
- Flow rates at each branching have to add up (else our organism is in serious trouble...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

  where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$



Optimization

Murray meets Tokunaga:

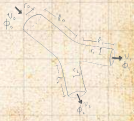
- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = k r_\omega^3$

$$r_\omega^3 = 2 r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

- Find Horton ratio for vessel radius $R_r = r_\omega / r_{\omega-1}$...



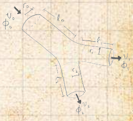
Optimization

Murray meets Tokunaga:

- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$\boxed{R_r^3 = R_n = R_v}$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?



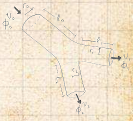
Optimization

Murray meets Tokunaga:

- Isometry: $V_\omega \propto \ell_\omega^3$
- Gives

$$\boxed{R_\ell^3 = R_v = R_n}$$

- We need one more constraint...
- West et al (1997) [17] achieve similar results following Horton's laws.
- So does Turcotte et al. (1998) [16] using Tokunaga (sort of).



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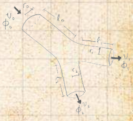

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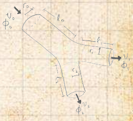

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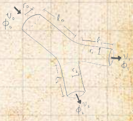

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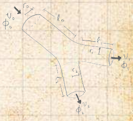

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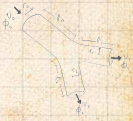

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