Random Networks Nutshell

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2016

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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Outline

Pure random networks
Definitions
How to build theoretically
Some visual examples
Clustering
Degree distributions
Generalized Random Networks
Configuration model
How to build in practice
Motifs
Random friends are strange
Largest component
References

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THE JOHN DORY
THE SUN
THE NETWORKS COMPLEX
THE ARK OF THE RANDOM NETWORK
THE NETWORKS FREE OF SCALE
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE UNKNOWN MECHANISM
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
THE TRUTHICIDE
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
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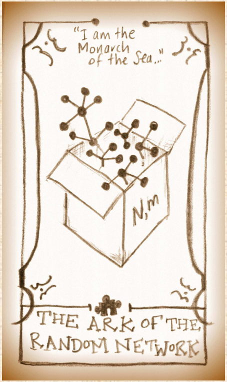
"I am the Monarch of the Sea."
N,m
THE ARK OF THE RANDOM NETWORK


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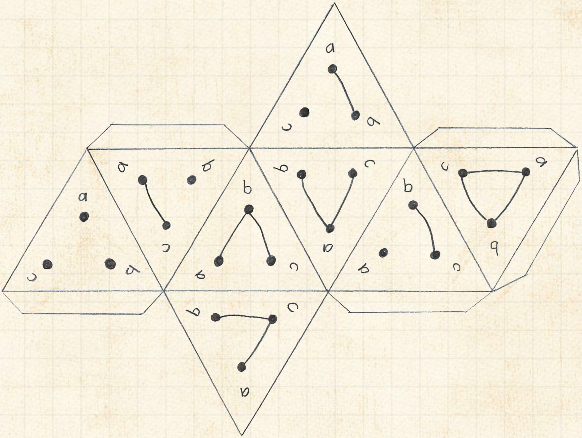
Random network generator for $N = 3$:



- Get your own exciting generator here.
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

Outline

Random networks

Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = one randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.

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Random networks—basic features:

- Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

- Limit of $m = 0$: empty graph.
- Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- Number of possible networks with N labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N^2}.$$

- Given m edges, there are $\binom{\binom{N}{2}}{m}$ different possible networks.
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- Real world: links are usually costly so real networks are almost always sparse.

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Outline

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Random networks

How to build standard random networks:

- Given N and m.
- Two probablistic methods (we'll see a third later on)

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p.
    - Useful for theoretical work.
2. Take N nodes and add exactly m links by selecting edges without replacement.
    - Algorithm: Randomly choose a pair of nodes i and j, $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
    - Best for adding relatively small numbers of links (most cases).
    - 1 and 2 are effectively equivalent for large N.

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A few more things:

- For method 1, # links is probablistic:

$$\langle m \rangle = p\binom{N}{2} = p\frac{1}{2}N(N-1)$$

- So the expected or average degree is

$$\langle k \rangle = \frac{2\,\langle m \rangle}{N}$$

$$= \frac{2}{N}p\frac{1}{2}N(N-1) = \frac{2}{N}p\frac{1}{2}N(N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.

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Random networks: examples

Next slides:

Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.

Random networks: examples

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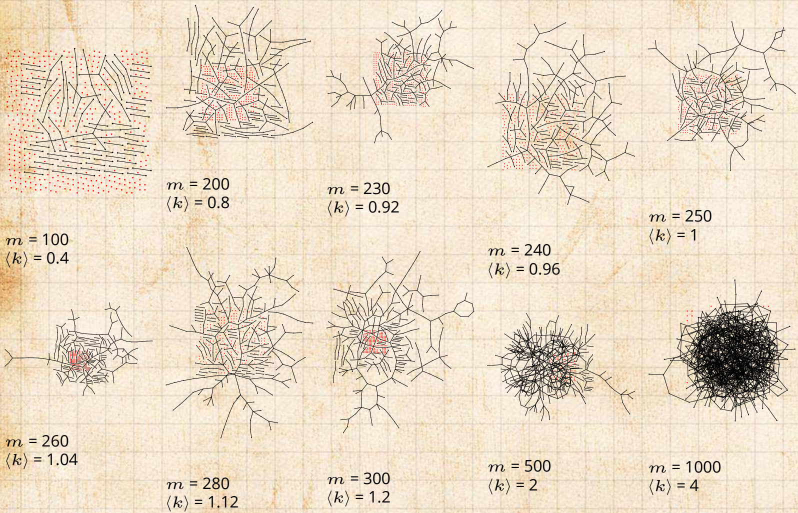
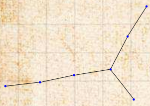
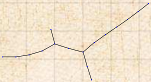
Random networks: examples for N=500



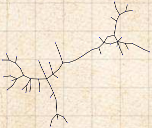
Random networks: largest components



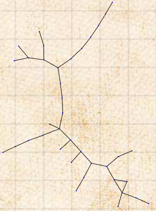
m = 100
$\langle k \rangle$ = 0.4



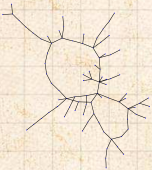
m = 200
$\langle k \rangle$ = 0.8



m = 230
$\langle k \rangle$ = 0.92



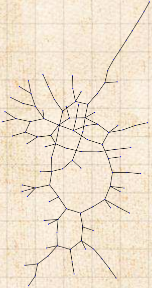
m = 240
$\langle k \rangle$ = 0.96



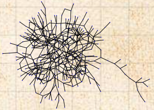
m = 250
$\langle k \rangle$ = 1

m = 260
$\langle k \rangle$ = 1.04

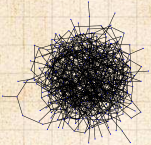
m = 280
$\langle k \rangle$ = 1.12



m = 300
$\langle k \rangle$ = 1.2



m = 500
$\langle k \rangle$ = 2



m = 1000
$\langle k \rangle$ = 4

Random networks: examples for N=500

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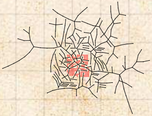

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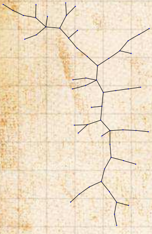

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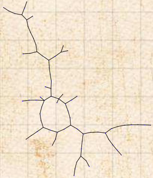

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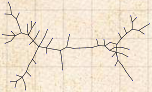

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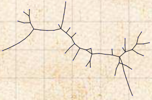

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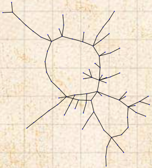

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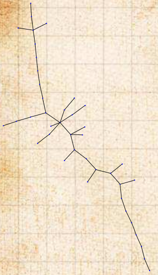

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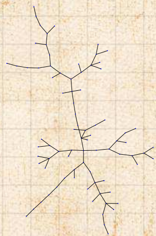

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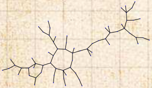

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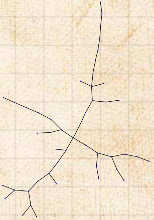

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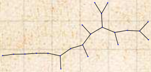

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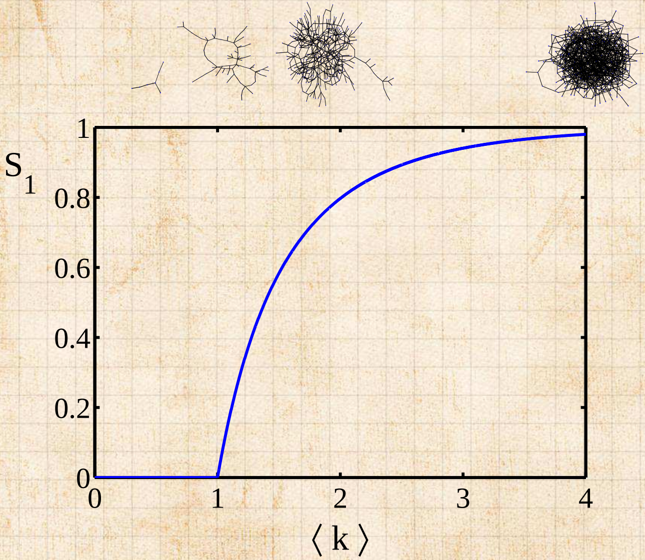

Outline

Clustering in random networks:

- For construction method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient: [5]

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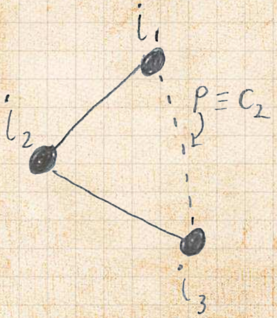

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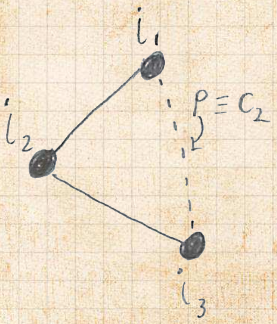

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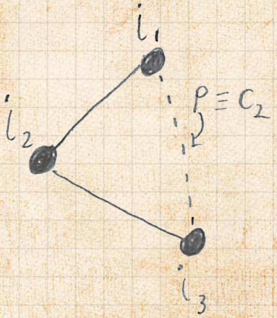

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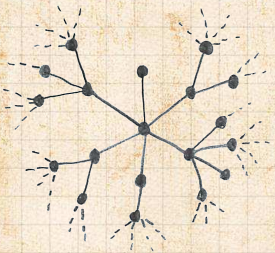

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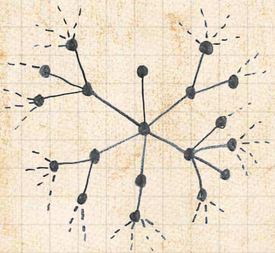

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Outline

Pure random networks
- Definitions
- How to build theoretically
- Some visual examples
- Clustering
- Degree distributions

Generalized Random Networks
- Configuration model
- How to build in practice
- Motifs
- Random friends are strange
- Largest component

References

Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are '$N-1$ choose k' ways the node can be connected to k of the other $N-1$ nodes.
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- Therefore have a binomial distribution:

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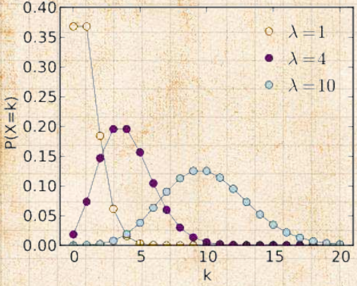
Poisson basics:

$$P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.: phone calls/minute, horse-kick deaths.
- 'Law of small numbers'

Poisson basics:

- The variance of degree distributions for random networks turns out to be very important.
- Using calculation similar to one for finding $\langle k \rangle$ we find the second moment to be:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

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General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution P_k.
- Also known as the configuration model.
- Can generalize construction method from ER random networks.
- Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- But we'll be more interested in
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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

- $N = 1000$.
- $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- Set $P_0 = 0$ (no isolated nodes).
- Vary exponent γ between 2.10 and 2.91.
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.

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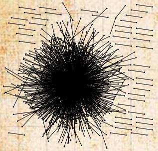
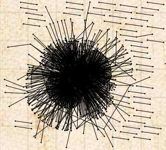
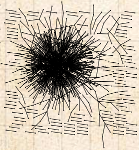


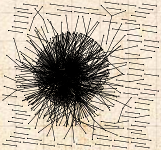
γ = 2.1
$\langle k \rangle$ = 3.448



γ = 2.19
$\langle k \rangle$ = 2.986



γ = 2.28
$\langle k \rangle$ = 2.306



γ = 2.37
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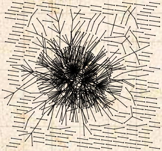
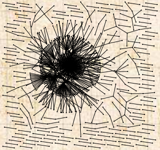


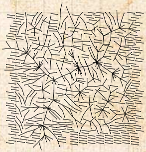
γ = 2.46
$\langle k \rangle$ = 1.856

γ = 2.55
$\langle k \rangle$ = 1.712

γ = 2.64
$\langle k \rangle$ = 1.6



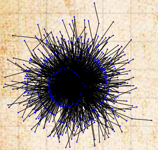
γ = 2.73
$\langle k \rangle$ = 1.862



γ = 2.82
$\langle k \rangle$ = 1.386

γ = 2.91
$\langle k \rangle$ = 1.49

Random networks: largest components



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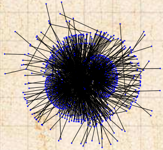

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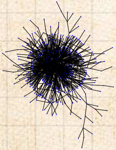

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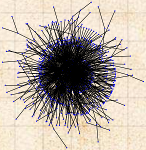

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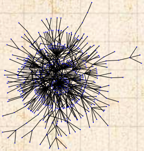

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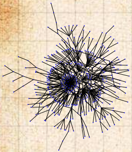

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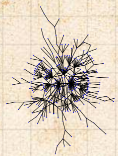

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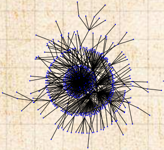

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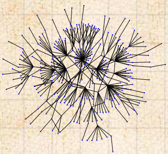

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Outline

Models

Generalized random networks:

- Arbitrary degree distribution P_k.
- Create (unconnected) nodes with degrees sampled from P_k.
- Wire nodes together randomly.
- Create ensemble to test deviations from randomness.

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Building random networks: Stubs

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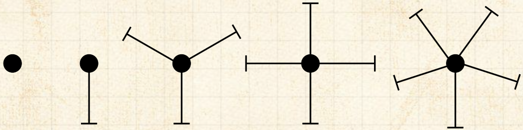
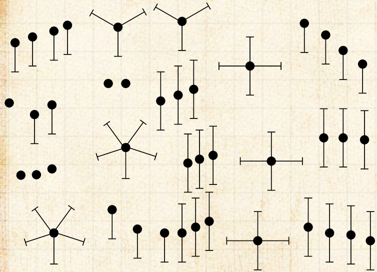
- Idea: start with a soup of unconnected nodes with stubs (half-edges):





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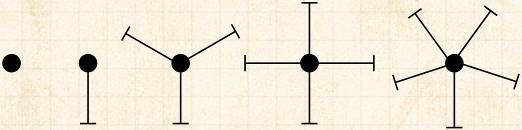

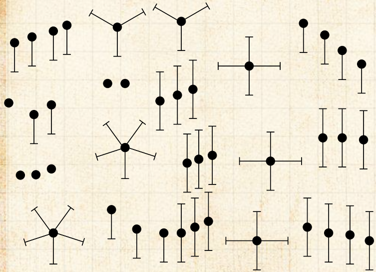

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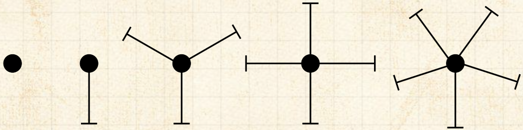

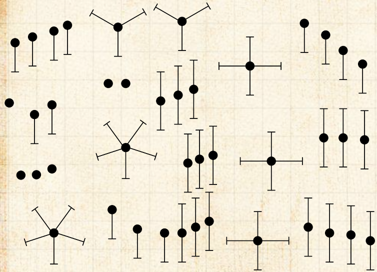

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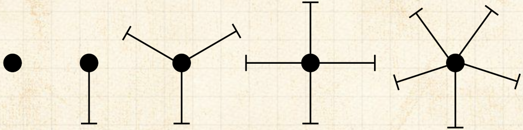

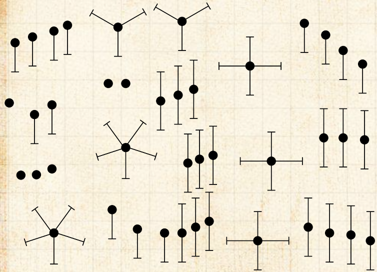

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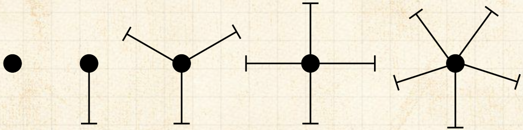

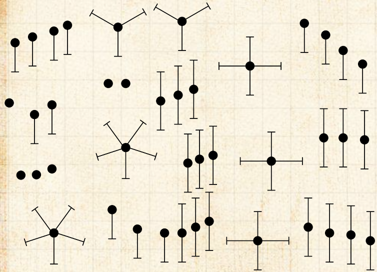

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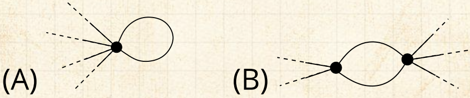
Building random networks: First rewiring

Phase 2:

- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.



- Being careful: we can't change the degree of any node, so we can't simply move links around.
- Simplest solution: randomly rewire two edges at a time.

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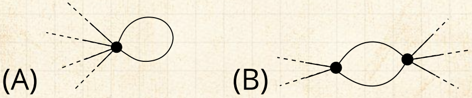

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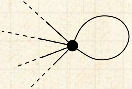
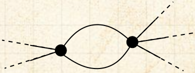
(A) 

(B) 

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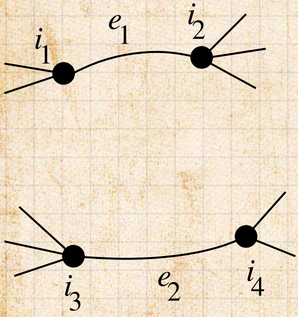
General random rewiring algorithm



- Randomly choose two edges. (Or choose problem edge and a random edge)
- Check to make sure edges are disjoint.
- Rewire one end of each edge.
- Node degrees do not change.
- Works if e_1 is a self-loop or repeated edge.
- Same as finding on/off/on/off 4-cycles, and rotating them.

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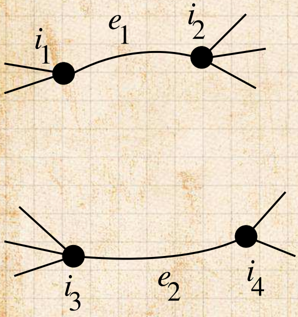

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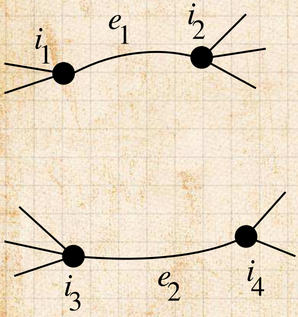

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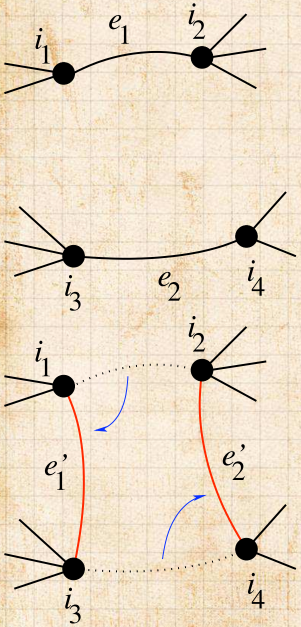

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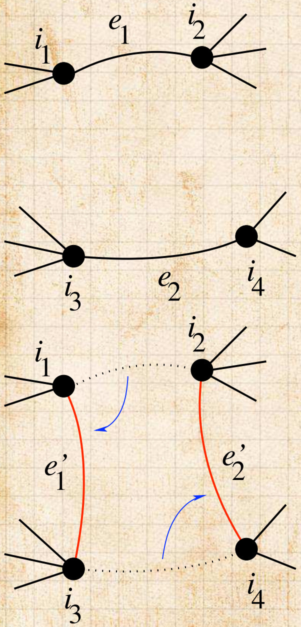

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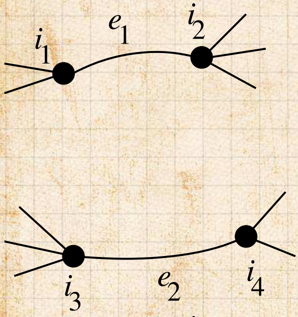

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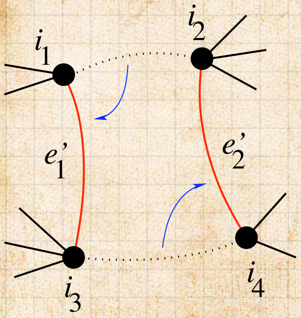

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Phase 2:

- Use rewiring algorithm to remove all self and repeat loops.

Phase 3:

- Randomize network wiring by applying rewiring algorithm liberally.
- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [3].

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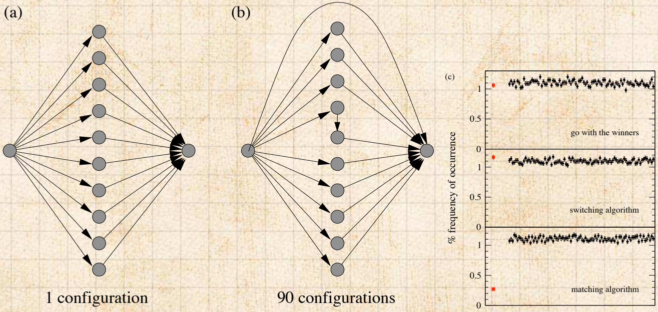

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- What if we have P_k instead of N_k?
- Must now create nodes before start of the construction algorithm.
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Outline

Pure random networks
- Definitions
- How to build theoretically
- Some visual examples
- Clustering
- Degree distributions

Generalized Random Networks
- Configuration model
- How to build in practice
- Motifs
- Random friends are strange
- Largest component

References

Network motifs

- Idea of motifs [6] introduced by Shen-Orr, Alon et al. in 2002.
- Looked at gene expression within full context of transcriptional regulation networks.
- Specific example of Escherichia coli.
- Directed network with 577 interactions (edges) and 424 operons (nodes).
- Used network randomization to produce ensemble of alternate networks with same degree frequency N_k.
- Looked for certain subnetworks (motifs) that appeared more or less often than expected

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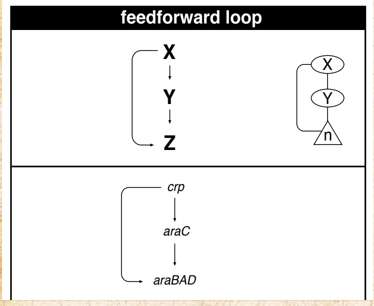

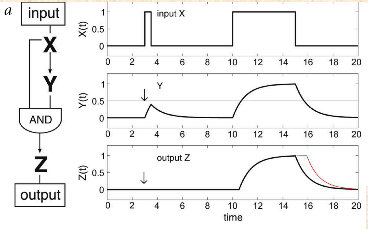

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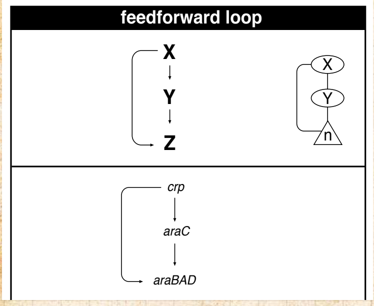

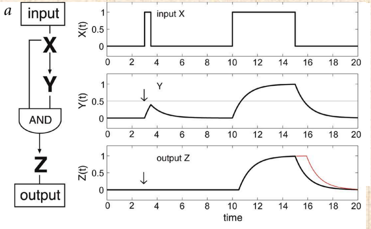

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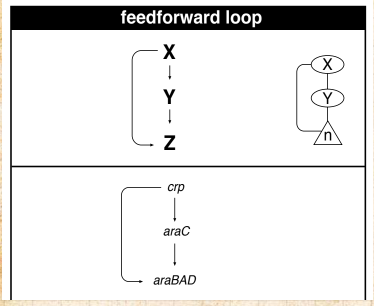

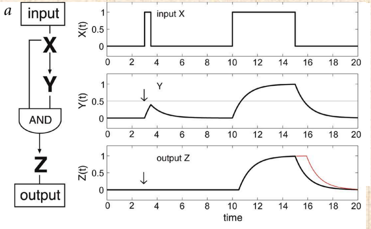

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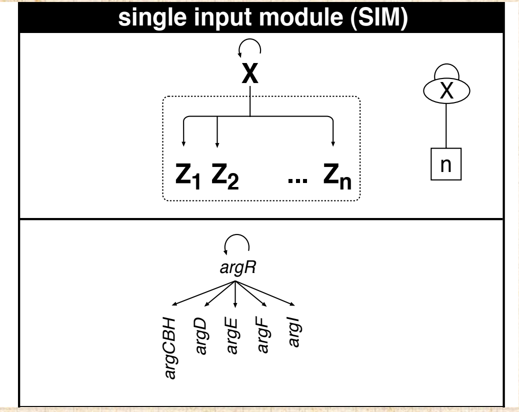
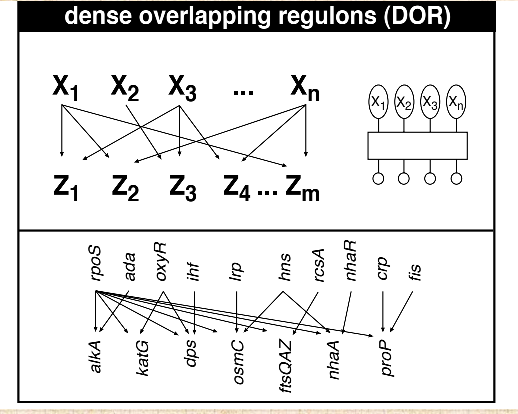


- Master switch.

Network motifs



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Outline

Pure random networks
- Definitions
- How to build theoretically
- Some visual examples
- Clustering
- Degree distributions

Generalized Random Networks
- Configuration model
- How to build in practice
- Motifs
- Random friends are strange
- Largest component

References

The edge-degree distribution:

- The degree distribution P_k is fundamental for our description of many complex networks
- Again: P_k is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define Q_k to be the probability the node at a random end of a randomly chosen edge has degree k.
- Now choosing nodes based on their degree (i.e., size):

$$Q_k \propto k P_k$$

- Normalized form:

$$Q_k = \frac{k P_k}{\sum_{k'=0}^{\infty} k' P_{k'}} = \frac{k P_k}{\langle k \rangle}.$$

- Big deal: Rich-get-richer mechanism is built into this selection process.

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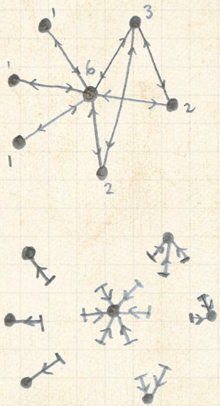

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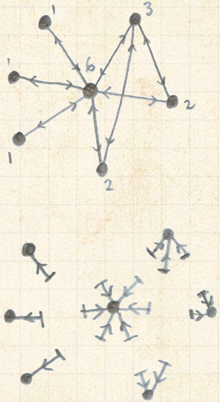

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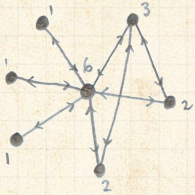

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- Useful variant on Q_k:

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Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} \left(\langle k^2 \rangle - \langle k \rangle \right) = \langle k^2 \rangle - \langle k \rangle.$$

- Key: Average depends on the 1st and 2nd moments of P_k and not just the 1st moment.
- Three peculiarities:
  1. We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
  2. If P_k has a large second moment, then $\langle k_2 \rangle$ will be big (e.g., in the case of a power-law distribution)
  3. Your friends really are different from you... [2, 4]
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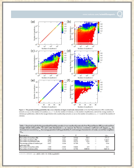

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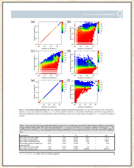

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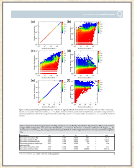

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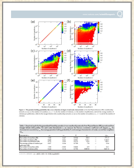

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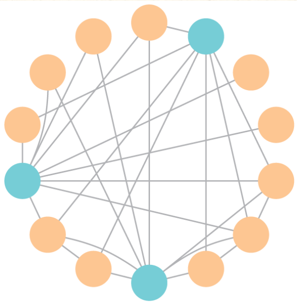

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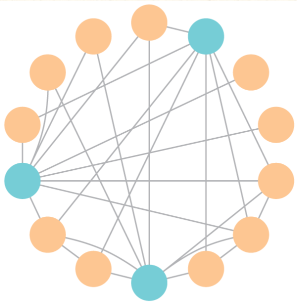

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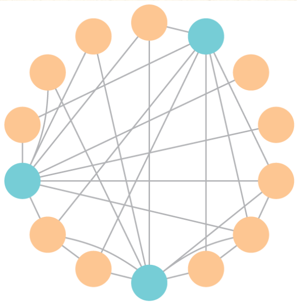

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- Again: thinking in edge space changes everything.

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Two reasons why this matters

(Big) Reason #2:

- $\langle k \rangle_R$ is key to understanding how well random networks are connected together.
- e.g., we'd like to know what's the size of the largest component within a network.
- As $N \to \infty$, does our network have a giant component?
- Defn: Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
- Defn: Giant component = component that comprises a non-zero fraction of a network as $N \to \infty$.
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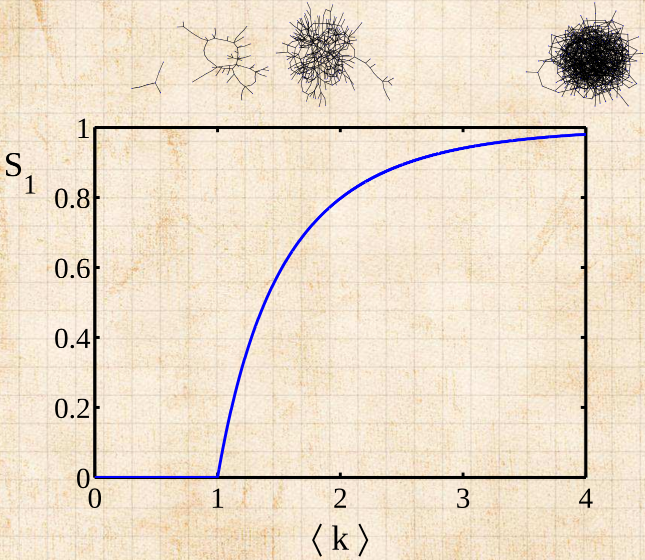
Outline

Pure random networks
Definitions
How to build theoretically
Some visual examples
Clustering
Degree distributions
Generalized Random Networks
Configuration model
How to build in practice
Motifs
Random friends are strange
Largest component
References

Giant component



Structure of random networks

Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.
- Equivalently, expect exponential growth in node number as we move out from a random node.
- All of this is the same as requiring $\langle k \rangle_R > 1$.
- Giant component condition (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

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- Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
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- Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
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Random networks with skewed P_k:

- e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$, $k \geq 1$, then

$$\langle k^2 \rangle = c \sum_{k=1}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=1}^{\infty} x^{2-\gamma} \mathrm{d}x$$

$$\propto x^{3-\gamma} \Big|_{x=1}^{\infty} = \infty \quad (\gg \langle k \rangle).$$

- So giant component always exists for these kinds of networks.
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And how big is the largest component?

- Define S_1 as the size of the largest component.
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- Let's find S_1 with a back-of-the-envelope argument.
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- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- First, we can write $\langle k \rangle$ in terms of S_1:

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- As $\langle k \rangle \to 0$, $S_1 \to 0$.
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- Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- Only solvable for $S_1 > 0$ when $\langle k \rangle > 1$.
- Really a transcritical bifurcation. [?]

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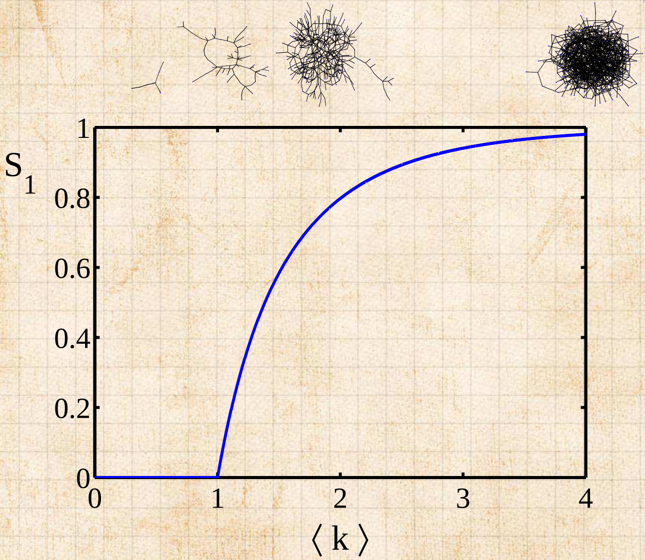

Giant component

Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
- We need a separate probability δ' for the chance that an edge leads to the giant (infinite) component.
- We can sort many things out with sensible probabilistic arguments...
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