Mixed, correlated random networks

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2016

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Dept. of Mathematics & Statistics | Vermont Complex Systems Center
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Outline

Random directed networks:

- So far, we've largely studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have k_i and k_o incoming and outgoing edges, otherwise random.

- Network defined by joint in- and out-degree distribution: P_{k_i,k_o}
- Normalization: $\sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} P_{k_i,k_o} = 1$
- Marginal in-degree and out-degree distributions:

$$P_{k_i} = \sum_{k_o=0}^{\infty} P_{k_i,k_o} \text{ and } P_{k_o} = \sum_{k_i=0}^{\infty} P_{k_i,k_o}$$

- Required balance:

$$\langle k_i \rangle = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_i P_{k_i,k_o} = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_o P_{k_i,k_o} = \langle k_o \rangle$$

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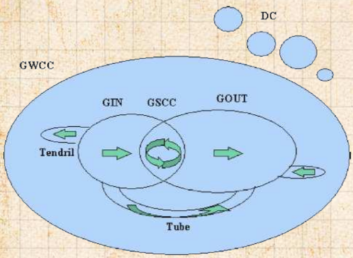

From Boguñá and Serano.[1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).

- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together.[4, 1]

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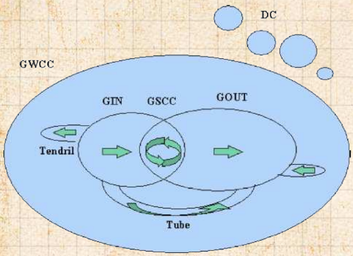

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- Directed and undirected random networks are separate families ...
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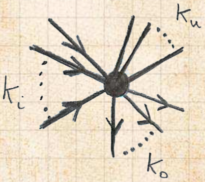

- Consider nodes with three types of edges:
  1. k_u undirected edges,
  2. k_i incoming directed edges,
  3. k_o outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\, k_u \;\; k_i \;\; k_o \,]^T.$$

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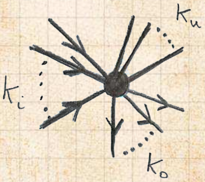

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Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\ k_{\mathrm{u}}\ \ k_{\mathrm{i}}\ \ k_{\mathrm{o}}\ ]^{\mathsf{T}}.$$

- As for directed networks, require in- and out-degree averages to match up:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{\vec{k}} = \sum_{k_{\mathrm{u}}=0}^{\infty} \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{\vec{k}} = \langle k_{\mathrm{o}} \rangle$$

- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathrm{u}}} \delta_{k_{\mathrm{i}},0} \delta_{k_{\mathrm{o}},0}.$$

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Outline

Directed random networks

Mixed random networks

- Definition
- Correlations

Mixed Random Network Contagion

- Spreading condition
- Full generalization
- Triggering probabilities

Nutshell

References

Correlations:

- Now add correlations (two point or Markovian):
  1. $P^{(u)}(\vec{k}\,|\,\vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
  2. $P^{(i)}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.
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- Now require more refined (detailed) balance.
- Conditional probabilities cannot be arbitrary.
  1. $P^{(u)}(\vec{k}\,|\,\vec{k}')$ must be related to $P^{(u)}(\vec{k}'\,|\,\vec{k})$.
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- Randomly choose an edge, and randomly choose one end.
- Say we find a degree $\vec{k}$ node at this end, and a degree $\vec{k}'$ node at the other end.
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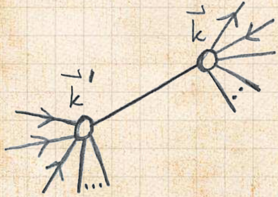

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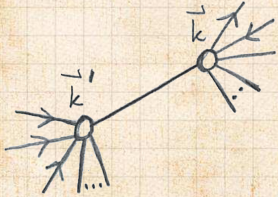

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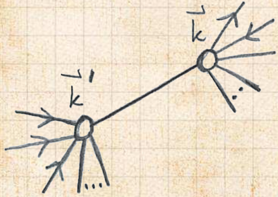

- Conditional probability connection:

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Correlations—Undirected edge balance:

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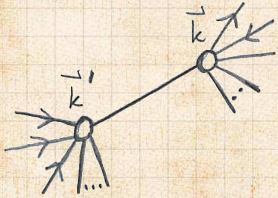

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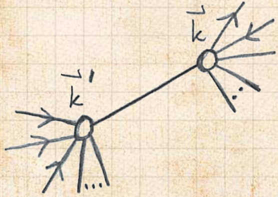

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Correlations—Directed edge balance:

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$$\frac{k_{\mathrm{o}} P(\vec{k})}{\langle k_{\mathrm{o}} \rangle} \text{ and } \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:

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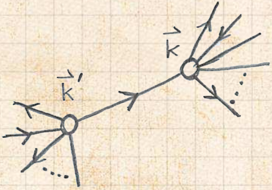

- We therefore have

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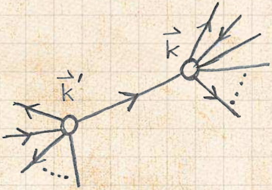

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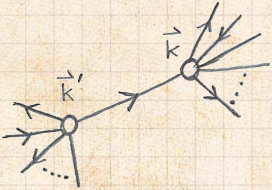

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Outline

Global spreading condition: [2]

When are cascades possible?:

- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_{\mathrm{u}}=0}^{\infty} \frac{k_{\mathrm{u}} P_{k_{\mathrm{u}}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}},1} > 1.$$

- Similar form for purely directed networks:

$$\mathbf{R} = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} \frac{k_{\mathrm{i}} P_{k_{\mathrm{i}},k_{\mathrm{o}}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \bullet B_{k_{\mathrm{i}},1} > 1.$$

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Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
- Infected edge growth equation:

$$f(d+1) = \mathbf{R} f(d).$$

- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_{\mathrm{u}},1}$ is the probability that an infected edge eventually infects a node.
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- Now have two types of edges spreading infection: directed and undirected.
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- Two separate gain equations:

- Gain ratio matrix:

$$\mathbf{R} = \sum_{\vec{k}} \left[\begin{array}{cc} \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \\ \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet k_{\mathrm{o}} & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \end{array}\right] \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1}$$

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Summary of contagion conditions for uncorrelated networks:

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III. Mixed Directed and Undirected, Uncorrelated—

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

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Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.
- Replace $P^{(i)}(\vec{k} \mid *)$ with $P^{(i)}(\vec{k} \mid \vec{k}')$ and so on.
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Summary of contagion conditions for correlated networks:

- IV. Undirected,
Correlated— $f_{k_{\mathrm{u}}}(d+1) = \sum_{k'_{\mathrm{u}}} R_{k_{\mathrm{u}} k'_{\mathrm{u}}} f_{k'_{\mathrm{u}}}(d)$

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- VI. Mixed Directed and Undirected, Correlated—

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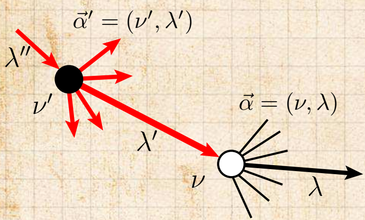
Outline

Full generalization:



$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

$$R_{\vec{\alpha}\vec{\alpha}'} = P_{\vec{\alpha}\vec{\alpha}'} \bullet k_{\vec{\alpha}\vec{\alpha}'} \bullet B_{\vec{\alpha}\vec{\alpha}'}.$$

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- Generalized contagion condition:

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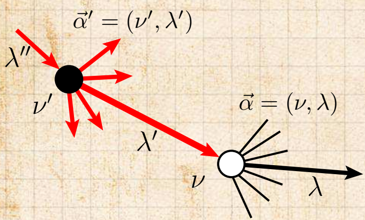

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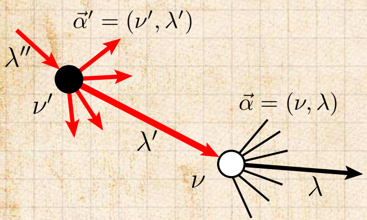

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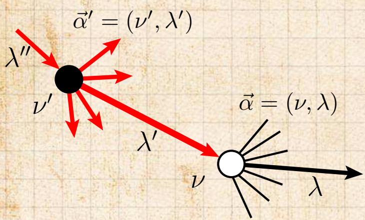

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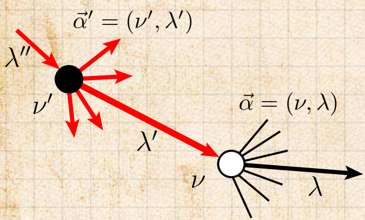

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Outline

- As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.
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Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

$$Q_{\mathrm{trig}}^{(\mathrm{u})}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q_{\mathrm{trig}}^{(\mathrm{u})}(\vec{k}'))^{k'_{\mathrm{u}}-1} (1 - Q_{\mathrm{trig}}^{(\mathrm{o})}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$Q_{\mathrm{trig}}^{(\mathrm{o})}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q_{\mathrm{trig}}^{(\mathrm{u})}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q_{\mathrm{trig}}^{(\mathrm{o})}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q_{\mathrm{trig}}^{(\mathrm{u})}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q_{\mathrm{trig}}^{(\mathrm{o})}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.

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