Mixed, correlated random networks

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2016

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

These slides are brought to you by:

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Outline

Directed random networks

Mixed random networks

Definition
Correlations

Mixed Random Network Contagion

Spreading condition
Full generalization
Triggering probabilities

Nutshell

References

Random directed networks:

- So far, we've largely studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have k_{i} and k_{o} incoming and outgoing edges, otherwise random.

- Network defined by joint in- and out-degree distribution: $P_{k_{\mathrm{i}},k_{\mathrm{o}}}$
- Normalization: $\sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = 1$
- Marginal in-degree and out-degree distributions:

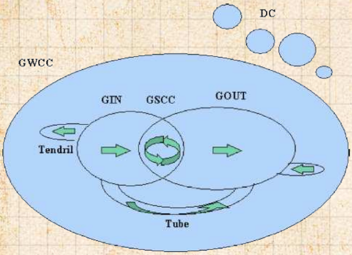
$$P_{k_{\mathrm{i}}} = \sum_{k_{\mathrm{o}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}} \text{ and } P_{k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty} P_{k_{\mathrm{i}},k_{\mathrm{o}}}$$

- Required balance:

$$\langle k_{\mathrm{i}} \rangle = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{i}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \sum_{k_{\mathrm{i}}=0}^{\infty} \sum_{k_{\mathrm{o}}=0}^{\infty} k_{\mathrm{o}} P_{k_{\mathrm{i}},k_{\mathrm{o}}} = \langle k_{\mathrm{o}} \rangle$$

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Directed network structure:



From Boguñá and Serano. [1]

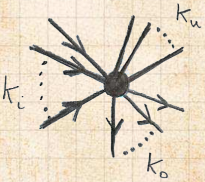
- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).

- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Important observation:

- Directed and undirected random networks are separate families ...
- ...and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.



- Consider nodes with three types of edges:
  1. k_{u} undirected edges,
  2. k_{i} incoming directed edges,
  3. k_{o} outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\, k_{\mathrm{u}} \;\; k_{\mathrm{i}} \;\; k_{\mathrm{o}} \,]^{\mathsf{T}}.$$

- Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\, k_{\mathsf{u}} \;\; k_{\mathsf{i}} \;\; k_{\mathsf{o}} \,]^{\mathsf{T}}.$$

- As for directed networks, require in- and out-degree averages to match up:

$$\langle k_{\mathsf{i}} \rangle = \sum_{k_{\mathsf{u}}=0}^{\infty} \sum_{k_{\mathsf{i}}=0}^{\infty} \sum_{k_{\mathsf{o}}=0}^{\infty} k_{\mathsf{i}} P_{\vec{k}} = \sum_{k_{\mathsf{u}}=0}^{\infty} \sum_{k_{\mathsf{i}}=0}^{\infty} \sum_{k_{\mathsf{o}}=0}^{\infty} k_{\mathsf{o}} P_{\vec{k}} = \langle k_{\mathsf{o}} \rangle$$

- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathsf{u}}} \delta_{k_{\mathsf{i}},0} \delta_{k_{\mathsf{o}},0},$$

$$\text{Directed: } P_{\vec{k}} = \delta_{k_{\mathsf{u}},0} P_{k_{\mathsf{i}},k_{\mathsf{o}}}.$$

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

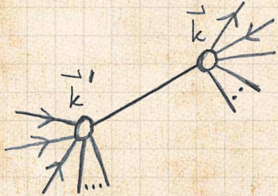
Correlations:

- Now add correlations (two point or Markovian) ☐:
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
  2. $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.
  3. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an out-directed edge relative to the destination node.
- Now require more refined (detailed) balance.
- Conditional probabilities cannot be arbitrary.
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ must be related to $P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k})$.
  2. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ and $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ must be connected.

Correlations—Undirected edge balance:

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

- Randomly choose an edge, and randomly choose one end.
- Say we find a degree $\vec{k}$ node at this end, and a degree $\vec{k}'$ node at the other end.
- Define probability this happens as $P^{(\mathrm{u})}(\vec{k},\vec{k}')$.
- Observe we must have $P^{(\mathrm{u})}(\vec{k},\vec{k}') = P^{(\mathrm{u})}(\vec{k}',\vec{k})$.



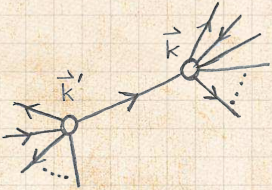
- Conditional probability connection:

$$P^{(\mathrm{u})}(\vec{k},\vec{k}') = P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')\frac{k'_{\mathrm{u}}P(\vec{k}')}{\langle k'_{\mathrm{u}}\rangle}$$

$$P^{(\mathrm{u})}(\vec{k}',\vec{k}) = P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k})\frac{k_{\mathrm{u}}P(\vec{k})}{\langle k_{\mathrm{u}}\rangle}.$$

Correlations—Directed edge balance:

- The quantities
$$\frac{k_{\mathrm{o}} P(\vec{k})}{\langle k_{\mathrm{o}} \rangle} \text{ and } \frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}$$
give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:
  1. along an outgoing edge, or
  2. against the direction of an incoming edge.



- We therefore have
$$P^{(\mathrm{dir})}(\vec{k},\vec{k}') = P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')\frac{k'_{\mathrm{o}} P(\vec{k}')}{\langle k'_{\mathrm{o}} \rangle} = P^{(\mathrm{o})}(\vec{k}'\,|\,\vec{k})\frac{k_{\mathrm{i}} P(\vec{k})}{\langle k_{\mathrm{i}} \rangle}.$$
- Note that $P^{(\mathrm{dir})}(\vec{k},\vec{k}')$ and $P^{(\mathrm{dir})}(\vec{k}',\vec{k})$ are in general not related if $\vec{k} \neq \vec{k}'$.

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Global spreading condition: [2]

When are cascades possible?:

- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_u=0}^{\infty} \frac{k_u P_{k_u}}{\langle k_u \rangle} \bullet (k_u - 1) \bullet B_{k_u,1} > 1.$$

- Similar form for purely directed networks:

$$\mathbf{R} = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} \frac{k_i P_{k_i,k_o}}{\langle k_i \rangle} \bullet k_o \bullet B_{k_i,1} > 1.$$

- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Global spreading condition:

Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
- Infected edge growth equation:

$$f(d+1) = \mathbf{R} f(d).$$

- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_{\mathrm{u}},1}$ is the probability that an infected edge eventually infects a node.
- Also allows for recovery of nodes (SIR).

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Global spreading condition:

Mixed, uncorrelated random netwoks:

- Now have two types of edges spreading infection: directed and undirected.
- Gain ratio now more complicated:
  1. Infected directed edges can lead to infected directed or undirected edges.
  2. Infected undirected edges can lead to infected directed or undirected edges.
- Define $f^{(u)}(d)$ and $f^{(o)}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance d from seed.

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

- Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(u)}(d+1) \\ f^{(o)}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(u)}(d) \\ f^{(o)}(d) \end{bmatrix}$$

- Two separate gain equations:

$$f^{(u)}(d+1) = \sum_{\vec{k}} \left[\frac{k_u P_{\vec{k}}}{\langle k_u \rangle} \bullet (k_u - 1) \bullet B_{k_u+k_i,1} f^{(u)}(d) + \frac{k_i P_{\vec{k}}}{\langle k_i \rangle} \bullet k_u \bullet B_{k_u+k_i,1} f^{(o)}(\right.$$

$$f^{(o)}(d+1) = \sum_{\vec{k}} \left[\frac{k_u P_{\vec{k}}}{\langle k_u \rangle} \bullet k_o B_{k_u+k_i,1} f^{(u)}(d) + \frac{k_i P_{\vec{k}}}{\langle k_i \rangle} \bullet k_o \bullet B_{k_u+k_i,1} f^{(o)}(d) \right]$$

- Gain ratio matrix:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_u P_{\vec{k}}}{\langle k_u \rangle} \bullet (k_u - 1) & \frac{k_i P_{\vec{k}}}{\langle k_i \rangle} \bullet k_u \\ \frac{k_u P_{\vec{k}}}{\langle k_u \rangle} \bullet k_o & \frac{k_i P_{\vec{k}}}{\langle k_i \rangle} \bullet k_o \end{bmatrix} \bullet B_{k_u+k_i,1}$$

- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

Global spreading condition:

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle}$ and $P^{(\mathrm{i})}(\vec{k}\,|\,*) = \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).
- Also write $B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{k_{\mathrm{u}} k_{\mathrm{i}},*}$$

Summary of contagion conditions for uncorrelated networks:

- I. Undirected, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_{\mathsf{u}}} P^{(\mathsf{u})}(k_{\mathsf{u}} \,|\, *) \bullet (k_{\mathsf{u}} - 1) \bullet B_{k_{\mathsf{u}},*}$$

- II. Directed, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_{\mathsf{i}},k_{\mathsf{o}}} P^{(\mathsf{i})}(k_{\mathsf{i}}, k_{\mathsf{o}} \,|\, *) \bullet k_{\mathsf{o}} \bullet B_{k_{\mathsf{i}},*}$$

- III. Mixed Directed and Undirected, Uncorrelated—

$$\begin{bmatrix} f^{(\mathsf{u})}(d+1) \\ f^{(\mathsf{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathsf{u})}(d) \\ f^{(\mathsf{o})}(d) \end{bmatrix}$$

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathsf{u})}(\vec{k} \,|\, *) \bullet (k_{\mathsf{u}} - 1) & P^{(\mathsf{i})}(\vec{k} \,|\, *) \bullet k_{\mathsf{u}} \\ P^{(\mathsf{u})}(\vec{k} \,|\, *) \bullet k_{\mathsf{o}} & P^{(\mathsf{i})}(\vec{k} \,|\, *) \bullet k_{\mathsf{o}} \end{bmatrix} \bullet B_{k_{\mathsf{u}} k_{\mathsf{i}},*}$$

Correlated version:

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.
- Replace $P^{(\mathrm{i})}(\vec{k} \,|\, *)$ with $P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}')$ and so on.
- Edge types are now more diverse beyond directed and undirected as originating node type matters.
- Sums are now over $\vec{k}'$.

Summary of contagion conditions for correlated networks:

- IV. Undirected,
  Correlated— $f_{k_{\mathrm{u}}}(d+1) = \sum_{k'_{\mathrm{u}}} R_{k_{\mathrm{u}} k'_{\mathrm{u}}} f_{k'_{\mathrm{u}}}(d)$

$$R_{k_{\mathrm{u}} k'_{\mathrm{u}}} = P^{(\mathrm{u})}(k_{\mathrm{u}} \,|\, k'_{\mathrm{u}}) \bullet (k_{\mathrm{u}} - 1) \bullet B_{k_{\mathrm{u}} k'_{\mathrm{u}}}$$

- V. Directed,
  Correlated— $f_{k_{\mathrm{i}} k_{\mathrm{o}}}(d+1) = \sum_{k'_{\mathrm{i}}, k'_{\mathrm{o}}} R_{k_{\mathrm{i}} k_{\mathrm{o}} k'_{\mathrm{i}} k'_{\mathrm{o}}} f_{k'_{\mathrm{i}} k'_{\mathrm{o}}}(d)$

$$R_{k_{\mathrm{i}} k_{\mathrm{o}} k'_{\mathrm{i}} k'_{\mathrm{o}}} = P^{(\mathrm{i})}(k_{\mathrm{i}}, k_{\mathrm{o}} \,|\, k'_{\mathrm{i}}, k'_{\mathrm{o}}) \bullet k_{\mathrm{o}} \bullet B_{k_{\mathrm{i}} k_{\mathrm{o}} k'_{\mathrm{i}} k'_{\mathrm{o}}}$$

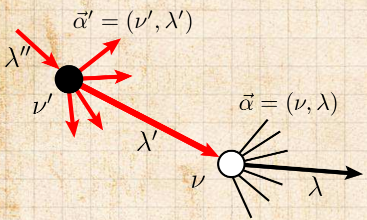
- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}}(d+1) \\ f^{(\mathrm{o})}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{k'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}'}(d) \\ f^{(\mathrm{o})}_{\vec{k}'}(d) \end{bmatrix}$$

$$\mathbf{R}_{\vec{k}\vec{k}'} = \begin{bmatrix} P^{(\mathrm{u})}(\vec{k} \,|\, \vec{k}') \bullet (k_{\mathrm{u}} - 1) & P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}') \bullet k_{\mathrm{u}} \\ P^{(\mathrm{u})}(\vec{k} \,|\, \vec{k}') \bullet k_{\mathrm{o}} & P^{(\mathrm{i})}(\vec{k} \,|\, \vec{k}') \bullet k_{\mathrm{o}} \end{bmatrix} \bullet B_{\vec{k}\vec{k}'}$$

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Full generalization:



$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

$$R_{\vec{\alpha}\vec{\alpha}'} = P_{\vec{\alpha}\vec{\alpha}'} \bullet k_{\vec{\alpha}\vec{\alpha}'} \bullet B_{\vec{\alpha}\vec{\alpha}'}.$$

- $P_{\vec{\alpha}\vec{\alpha}'}$ = conditional probability that a type λ' edge emanating from a type ν' node leads to a type ν node.
- $k_{\vec{\alpha}\vec{\alpha}'}$ = potential number of newly infected edges of type λ emanating from nodes of type ν.
- $B_{\vec{\alpha}\vec{\alpha}'}$ = probability that a type ν node is eventually infected by a single infected type λ' link arriving from a neighboring node of type ν'.
- Generalized contagion condition:

$$\max |\mu| : \mu \in \sigma(\mathbf{R}) > 1$$

- As we saw earlier, the triggering probability for simple contagion on random networks can be determined with a straightforward physical argument.
- Two good things:

$$Q_{\text{trig}} = \sum_{k=0}^{\infty} \frac{kP_k}{\langle k \rangle} \bullet B_{k1} \bullet \left[1 - \left(1 - Q_{\text{trig}} \right)^{k-1} \right],$$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_{k} P_k \bullet \left[1 - (1 - Q_{\text{trig}})^{k} \right].$$

- Equivalent to result found via the eldritch route of generating functions.
- Generating functions arguably make some kinds of calculations easier (but perhaps we don't care about component sizes that much).
- On the other hand, a plainspoken physical argument helps us generalize to correlated networks more easily.

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—
$Q_{\text{trig}} = \sum_{k'_{\text{u}}} P^{(\text{u})}(k'_{\text{u}} \,|\, \cdot) B_{k'_{\text{u}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}-1} \right]$

$$P_{\text{trig}} = S_{\text{trig}} = \sum_{k'_{\text{u}}} P(k'_{\text{u}}) \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{u}}} \right]$$

- II. Directed, Uncorrelated—
$Q_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P^{(\text{u})}(k'_{\text{i}}, k'_{\text{o}} | \cdot) B_{k'_{\text{i}}1} \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{o}}} \right]$

$$S_{\text{trig}} = \sum_{k'_{\text{i}}, k'_{\text{o}}} P(k'_{\text{i}}, k'_{\text{o}}) \left[1 - (1 - Q_{\text{trig}})^{k'_{\text{o}}} \right]$$

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q^{(\mathrm{u})}_{\mathrm{trig}} = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}-1} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}} \right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}} = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \cdot) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}} \right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}})^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}})^{k'_{\mathrm{o}}} \right]$$

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated— $Q_{\textrm{trig}}(k_{\textrm{u}}) = \sum_{k'_{\textrm{u}}} P^{(\textrm{u})}(k'_{\textrm{u}} \,|\, k_{\textrm{u}}) B_{k'_{\textrm{u}}1} \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{u}}))^{k'_{\textrm{u}}-1}\right]$

$$S_{\textrm{trig}} = \sum_{k'_{\textrm{u}}} P(k'_{\textrm{u}}) \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{u}}))^{k'_{\textrm{u}}}\right]$$

- V. Directed, Correlated— $Q_{\textrm{trig}}(k_{\textrm{i}}, k_{\textrm{o}}) = \sum_{k'_{\textrm{i}}, k'_{\textrm{o}}} P^{(\textrm{u})}(k'_{\textrm{i}}, k'_{\textrm{o}} | \, k_{\textrm{i}}, k_{\textrm{o}}) B_{k'_{\textrm{i}}1} \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{i}}, k'_{\textrm{o}}))^{k'_{\textrm{o}}}\right]$

$$S_{\textrm{trig}} = \sum_{k'_{\textrm{i}}, k'_{\textrm{o}}} P(k'_{\textrm{i}}, k'_{\textrm{o}}) \left[1 - (1 - Q_{\textrm{trig}}(k'_{\textrm{i}}, k'_{\textrm{o}}))^{k'_{\textrm{o}}}\right]$$

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

$$Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}-1} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}) = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \vec{k}) B_{\vec{k}'1} \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(\mathrm{u})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{u}}} (1 - Q^{(\mathrm{o})}_{\mathrm{trig}}(\vec{k}'))^{k'_{\mathrm{o}}}\right]$$

Nutshell:

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.

References I

[1] M. Boguñá and M. Ángeles Serrano.
Generalized percolation in random directed networks.
Phys. Rev. E, 72:016106, 2005. pdf

[2] P. S. Dodds, K. D. Harris, and J. L. Payne.
Direct, phyiscally motivated derivation of the contagion condition for spreading processes on generalized random networks.
Phys. Rev. E, 83:056122, 2011. pdf

[3] K. D. Harris, J. L. Payne, and P. S. Dodds.
Direct, physically-motivated derivation of triggering probabilities for contagion processes acting on correlated random networks.
http://arxiv.org/abs/1108.5398, 2014.

References II

[4] M. E. J. Newman, S. H. Strogatz, and D. J. Watts.
Random graphs with arbitrary degree distributions and their applications.
Phys. Rev. E, 64:026118, 2001. pdf

Directed random networks
Mixed random networks
Definition
Correlations
Mixed Random Network Contagion
Spreading condition
Full generalization
Triggering probabilities
Nutshell
References