Branching Networks II

Complex Networks | @networksvox
CSYS/MATH 303, Spring, 2016

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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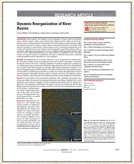
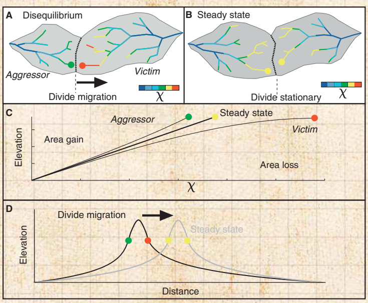
Outline

Piracy on the high χ's:



"Dynamic Reorganization of River Basins"
Willett et al.,
Science Magazine, **343**, 1248765, 2014. [21]



$$\frac{\partial z(x,t)}{\partial t} = U - K A^m \left| \frac{\partial z(x,t)}{\partial x} \right|^n$$

$$z(x) = z_{\mathrm{b}} + \left(\frac{U}{K A_0^m} \right)^{1/n} \chi$$

$$\chi = \int_{x_{\mathrm{b}}}^{x} \left(\frac{A_0}{A(x')} \right)^{m/n} \mathrm{d}x'$$

Piracy on the high χ's:

More: How river networks move across a landscape
(Science Daily)

Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- In terms of network achitecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's laws have four parameters and Tokunaga has two parameters.
- R_n, R_a, R_ℓ, and R_s versus T_1 and R_T. One simple redundancy: $R_\ell = R_s$.
  Insert question from assignment 1
- To make a connection, clearest approach is to start with Tokunaga's law...
- Known result: Tokunaga $\rightarrow$ Horton [18, 19, 20, 9, 21]

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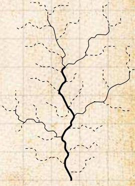

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Let us make them happy

We need one more ingredient:

Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:
  Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.
- In terms of basin characteristics:

$$\rho_{dd} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega}{a_\Omega}$$

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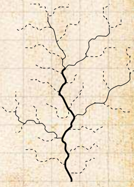

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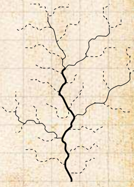

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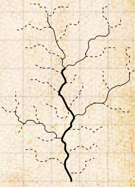

More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law: $n_\omega / n_{\omega+1} = R_n$.
- Estimate n_ω, the number of streams of order ω in terms of other $n_{\omega'}$, $\omega' > \omega$.
- Observe that each stream of order ω terminates by either:

1. Running into another stream of order ω and generating a stream of order $\omega + 1$...
    - $2n_{\omega+1}$ streams of order ω do this
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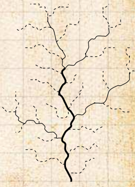

The University of Vermont

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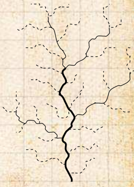

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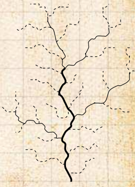

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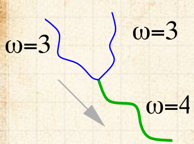

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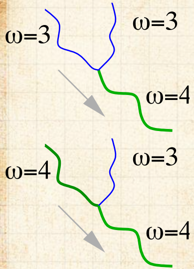

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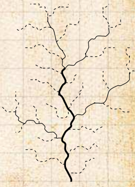

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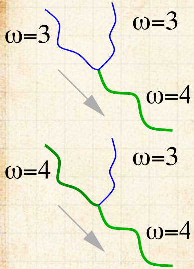

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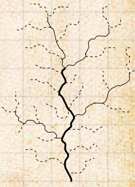

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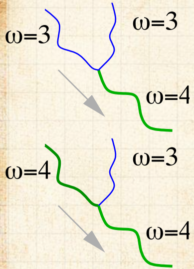

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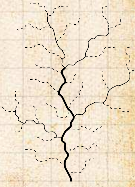

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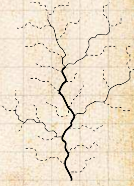
Putting things together:

- $$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega} n_{\omega'}}_{\text{absorption}}$$
- Use Tokunaga's law and manipulate expression to find Horton's law for stream numbers follows and hence obtain R_n.
- Insert question from assignment 1
- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

(The larger value is the one we want.)



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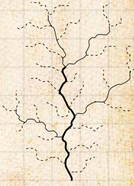

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Finding other Horton ratios

Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd}.
- Assume side streams are roughly separated by distance $1/\rho_{\mathsf{dd}}$.
- For an order ω stream segment, expected length is

$$\bar{s}_\omega \simeq \rho_{\mathsf{dd}}^{-1}\left(1 + \sum_{k=1}^{\omega-1} T_k\right)$$

- Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_\omega \simeq \rho_{\mathsf{dd}}^{-1}\left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1}\right) \propto R_T^{\omega}$$

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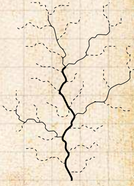

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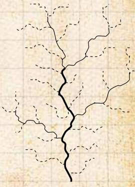

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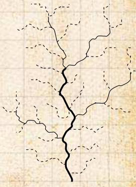

Horton and Tokunaga are happy

Altogether then:

- $$\Rightarrow \bar{s}_{\omega}/\bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$

- Recall $R_\ell = R_s$ so

$$R_\ell = R_s = R_T$$

- And from before:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

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Some observations:

- R_n and R_ℓ depend on T_1 and R_T.
- Seems that R_a must as well...
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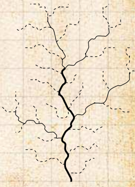

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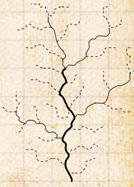

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The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.
- $$R_T = R_\ell,$$
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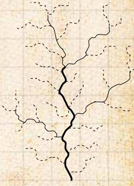

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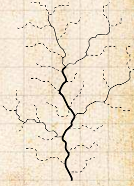

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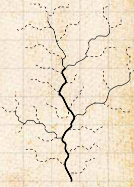

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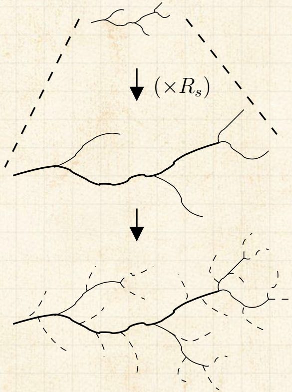

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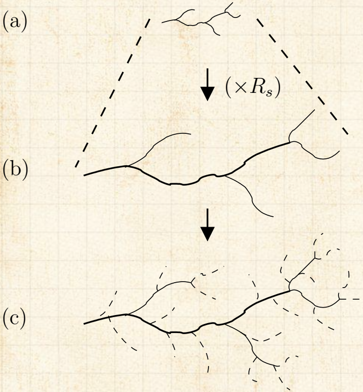

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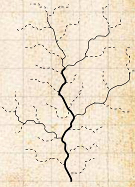

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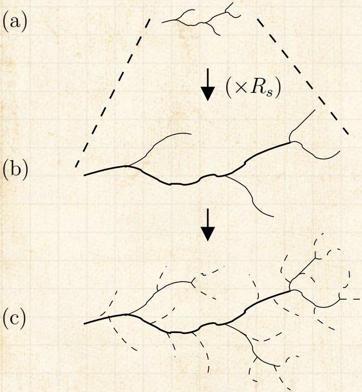

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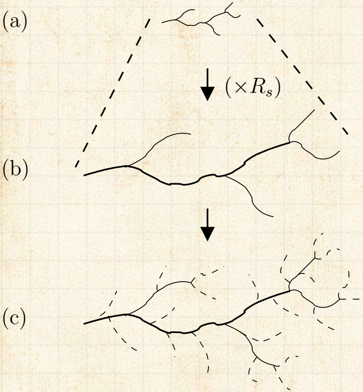

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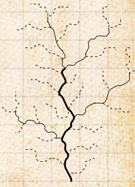

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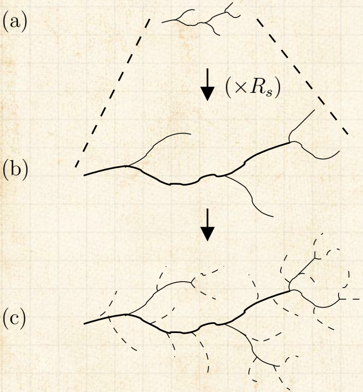

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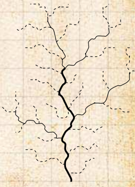

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...and in detail:

- Must retain same drainage density.
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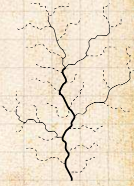

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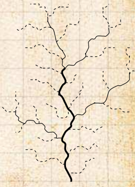

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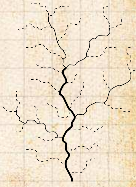

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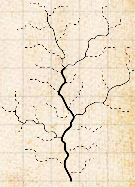

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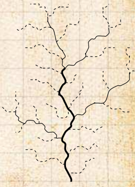

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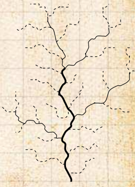

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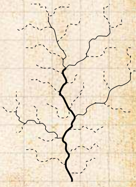
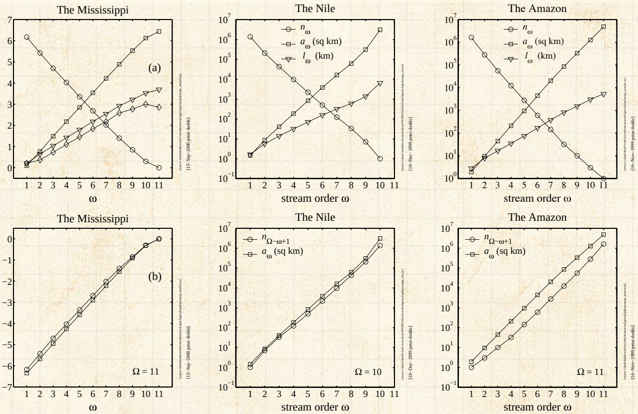
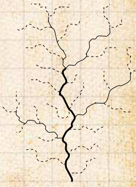


Horton's laws of area and number:



- In bottom plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$...



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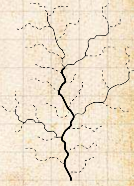
Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	5.27	5.26	2.48	2.30	1.00
$[2,5]$	4.86	4.96	2.42	2.31	1.02
$[2,7]$	4.77	4.88	2.40	2.31	1.02
$[3,4]$	4.72	4.91	2.41	2.34	1.04
$[3,6]$	4.70	4.83	2.40	2.35	1.03
$[3,8]$	4.60	4.79	2.38	2.34	1.04
$[4,6]$	4.69	4.81	2.40	2.36	1.02
$[4,8]$	4.57	4.77	2.38	2.34	1.05
$[5,7]$	4.68	4.83	2.36	2.29	1.03
$[6,7]$	4.63	4.76	2.30	2.16	1.03
$[7,8]$	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024



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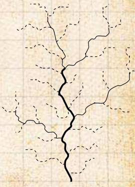
$[2,5]$	4.55	4.58	2.32	2.12	1.01
$[2,7]$	4.42	4.53	2.24	2.10	1.02
$[3,5]$	4.45	4.52	2.26	2.14	1.01
$[3,7]$	4.35	4.49	2.20	2.10	1.03
$[4,6]$	4.38	4.54	2.22	2.18	1.03
$[5,6]$	4.38	4.62	2.22	2.21	1.06
$[6,7]$	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019

Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

- $a_\Omega \propto$ sum of all stream segment lengths in a order Ω basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{\mathrm{dd}}$$



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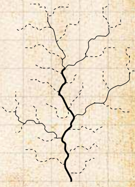

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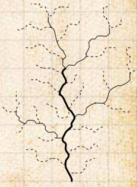

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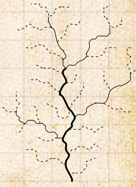

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Reducing Horton's laws:

Continued ...

- $$a_\Omega \propto \frac{R_n^\Omega}{R_s}\bar{s}_1 \sum_{\omega=1}^{\Omega}\left(\frac{R_s}{R_n}\right)^\omega$$

$$= \frac{R_n^\Omega}{R_s}\bar{s}_1 \frac{R_s}{R_n}\frac{1-(R_s/R_n)^\Omega}{1-(R_s/R_n)}$$

$$\sim R_n^{\Omega-1}\bar{s}_1 \frac{1}{1-(R_s/R_n)} \text{ as } \Omega \nearrow$$

- So, a_Ω is growing like R_n^Ω and therefore:

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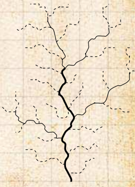
Not quite:

- ... But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranching.
- Insert question from assignment 2



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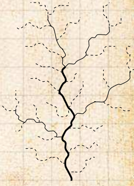

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Intriguing division of area:

- Observe: Combined area of basins of order ω independent of ω.
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

- Reason:

$$n_\omega \propto (R_n)^{-\omega}$$

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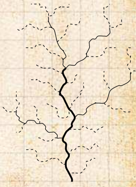

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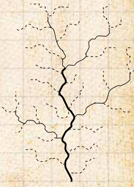

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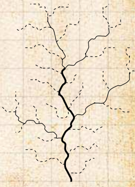

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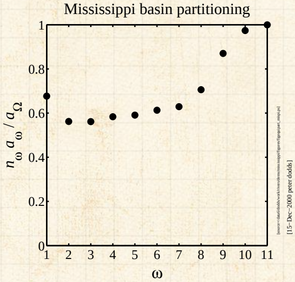
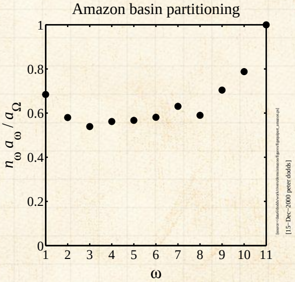
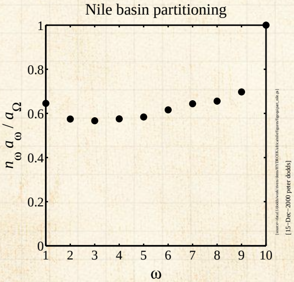
Some examples:







Neural reboot (NR):

Fwoompff

Scaling laws

The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture: $T_k = T_1 R_T^{k-1}$.
- We have connected Tokunaga's and Horton's laws
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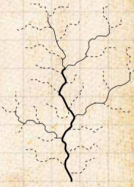

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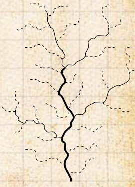

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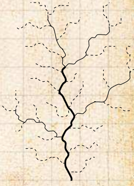

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- Ignore stream ordering for the moment
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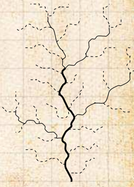

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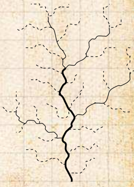

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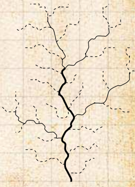

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  - City sizes (Zipf's law)
  - Word frequency (Zipf's law)[22]
  - Wealth (maybe not—at least heavy-tailed)
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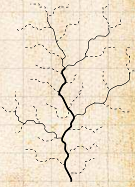

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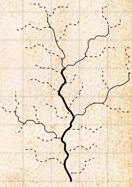

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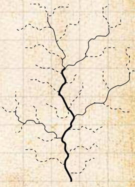

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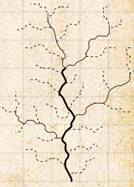

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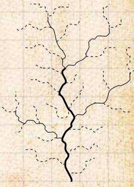

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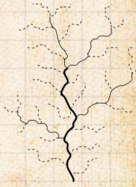

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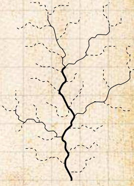

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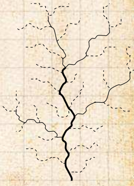

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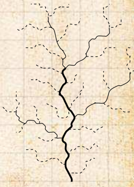

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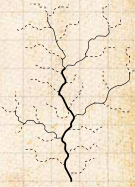

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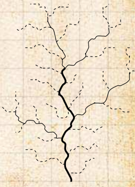

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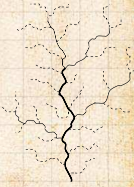

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Finding γ:

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
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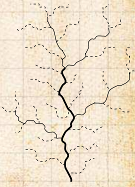

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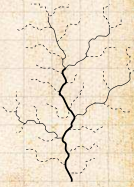

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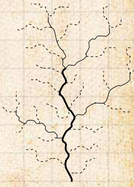

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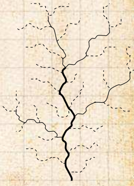

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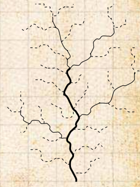

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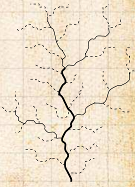

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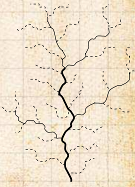

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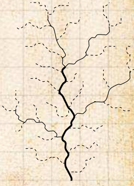

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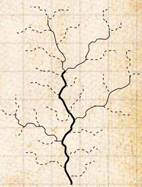

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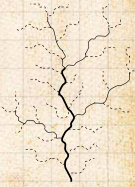

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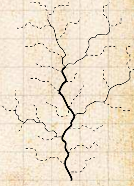

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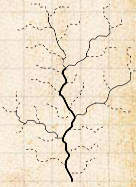

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Scaling laws

Finding γ:

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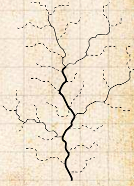

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$$P_>(\bar{\ell}_\omega) \propto \left(\frac{R_n}{R_s}\right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

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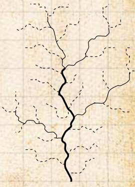

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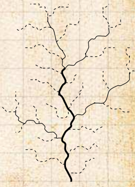

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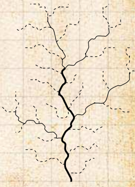

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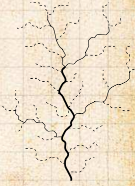

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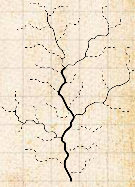

Scaling laws

Finding γ:

- Therefore:

$$P_>(\bar{\ell}_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}\right)^{-\ln(R_n/R_s)/\ln(R_s)}$$

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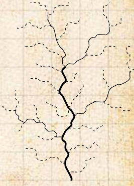

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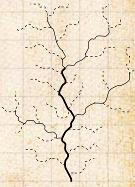

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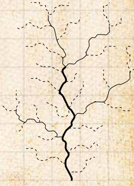

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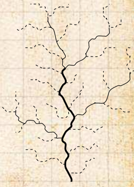

Scaling laws

Finding γ:

- And so we have:

$$\gamma = \ln R_n / \ln R_s$$

- Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

Insert question from assignment 2

- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law

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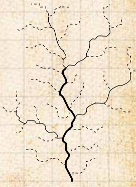

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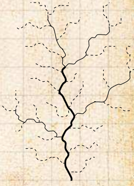

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Hack's law: [6]

- $$\ell \propto a^h$$
- Typically observed that $0.5 \lesssim h \lesssim 0.7$.
- Use Horton laws to connect h to Horton ratios:

$$\bar{\ell}_\omega \propto R_s^\omega \text{ and } \bar{a}_\omega \propto R_n^\omega$$

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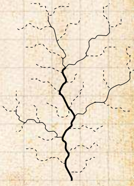

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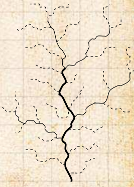

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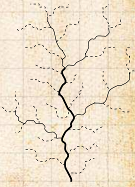

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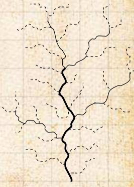

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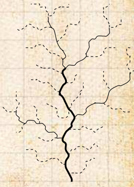
Connecting exponents

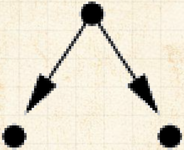
Only 3 parameters are independent:
e.g., take d, R_n, and R_s

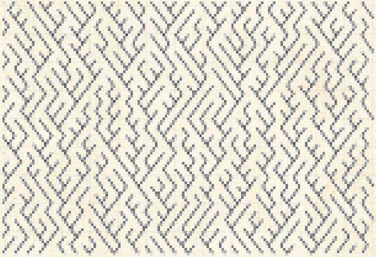
relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_\omega / n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \log R_s / \log R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$



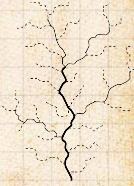
Scheidegger's model

Directed random networks [11, 12]





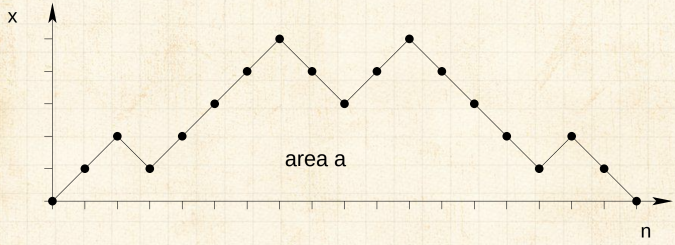
- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]
- Useful and interesting test case



A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks



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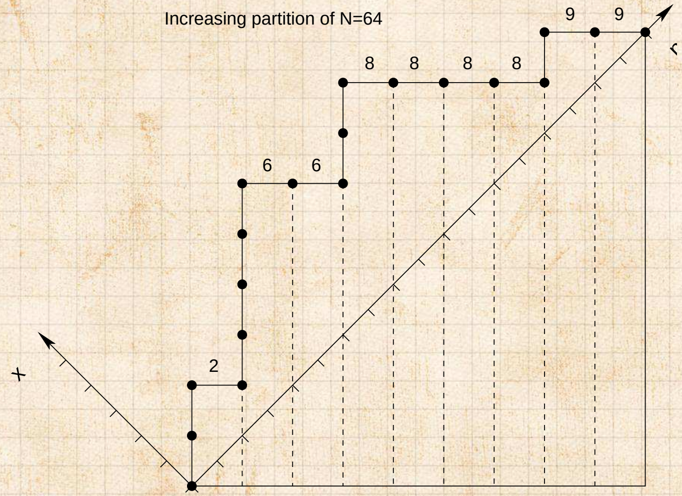
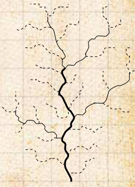
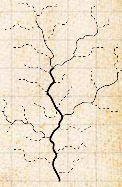




Scheidegger's model

Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

- $$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$
  and so $P(\ell) \propto \ell^{-3/2}$.
- Typical area for a walk of length n is $\propto n^{3/2}$:
  $$\ell \propto a^{2/3}.$$
- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.
- Note $\tau = 2 - h$ and $\gamma = 1/h$.
- R_n and R_ℓ have not been derived analytically.



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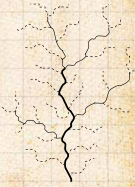

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- $$P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

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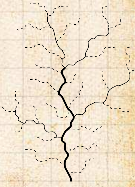
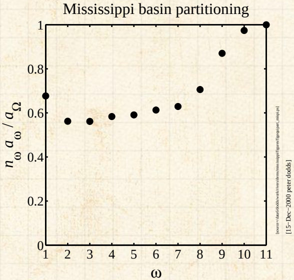
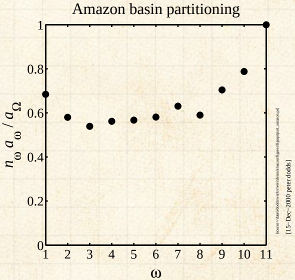
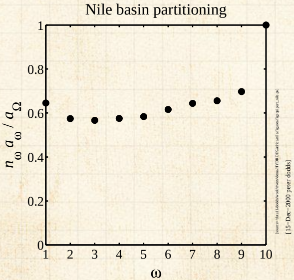


Equipartitioning reexamined:

Recall this story:







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- What about
$$P(a) \sim a^{-\tau} \qquad ?$$
- Since $\tau > 1$, suggests no equipartitioning:
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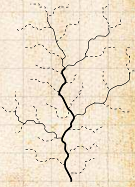
- $P(a)$ overcounts basins within basins...
- while stream ordering separates basins...



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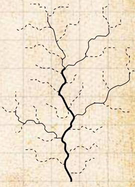

Neural reboot (NR):

Feline elevation

Fluctuations

Moving beyond the mean:

- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_{\omega}/\bar{s}_{\omega-1} = R_s$$

- Natural generalization to consider relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness...

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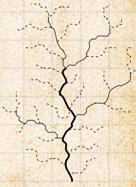

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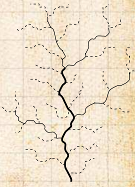

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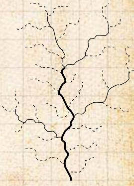
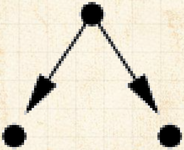
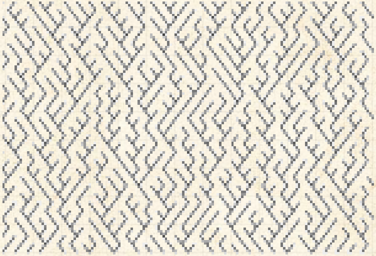
A toy model—Scheidegger's model



Directed random networks [11, 12]





- $$P(\searrow) = P(\swarrow) = 1/2$$
- Flow is directed downwards

Generalizing Horton's laws

- $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$
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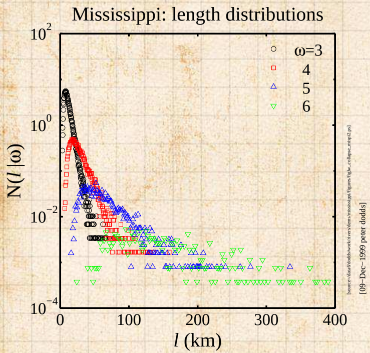

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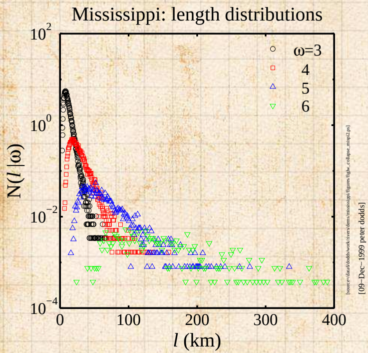

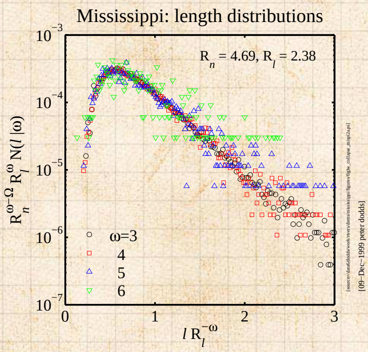

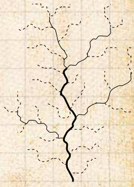

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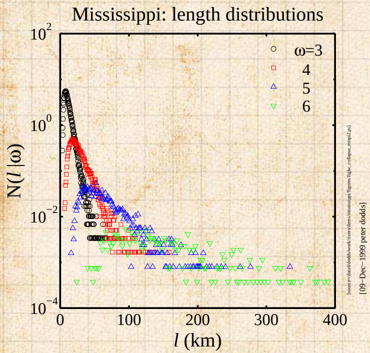

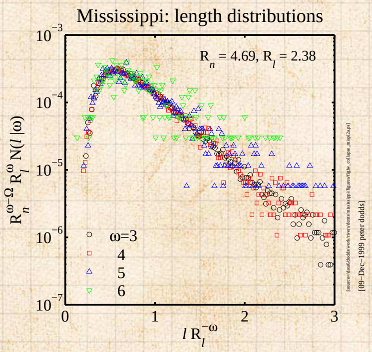

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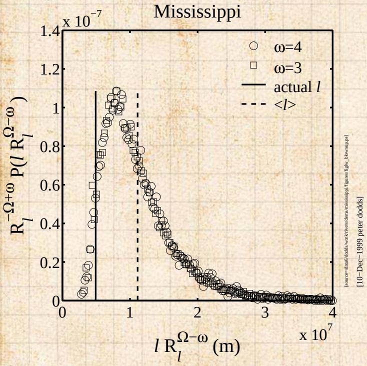
Generalizing Horton's laws

- How well does overall basin fit internal pattern?



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- Predicted Mean length = 11100 km
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- Actual length/Mean length = 44 %
- Okay.

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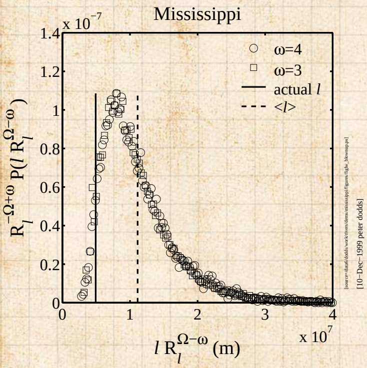

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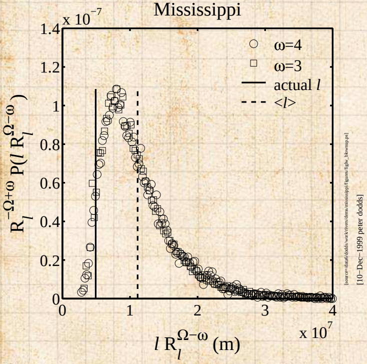

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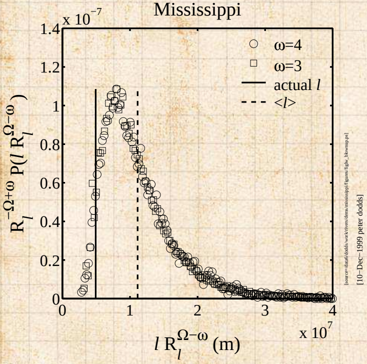

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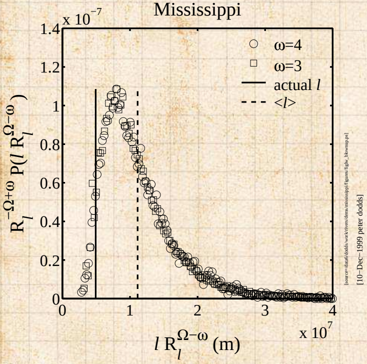

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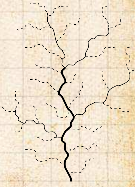

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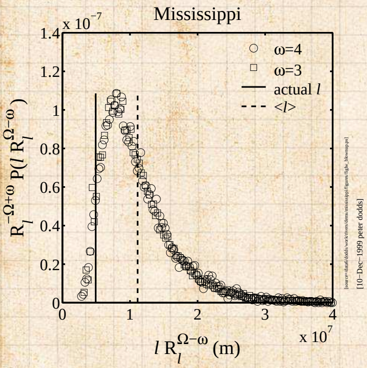

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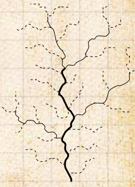
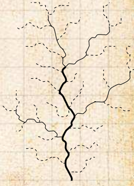


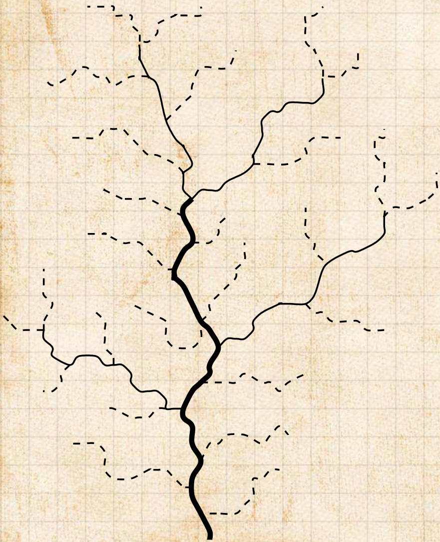
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

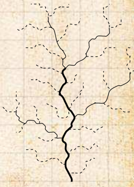
basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell_\Omega/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
	a_Ω	$\bar{a}_\Omega$	σ_a	$a_\Omega/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86



Combining stream segments distributions:



- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$
- $P(\ell_\omega)$ is a convolution of distributions for the s_μ



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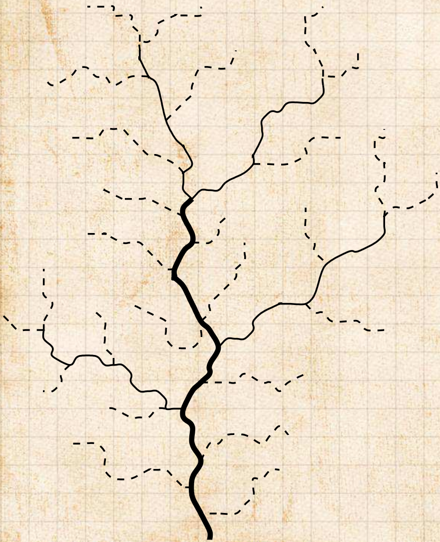

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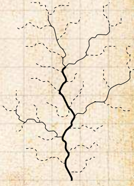
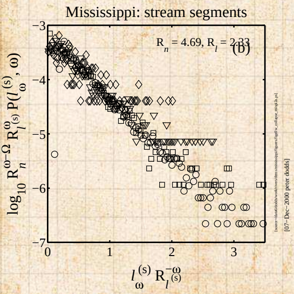


Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

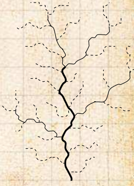
$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F(s/R_\ell^\omega)$$

$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.



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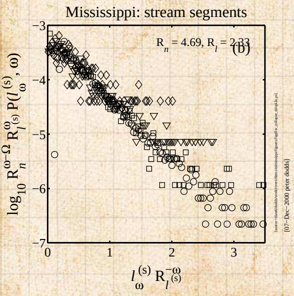

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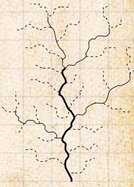
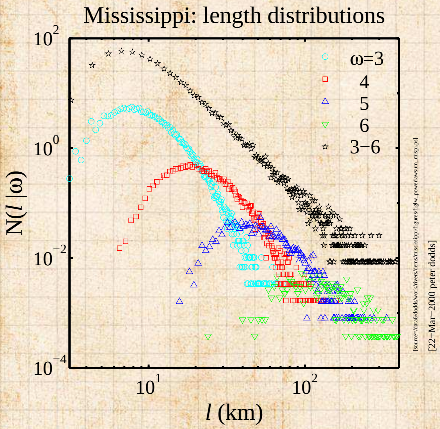
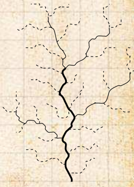


Generalizing Horton's laws

- Next level up: Main stream length distributions must combine to give overall distribution for stream length



- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting...



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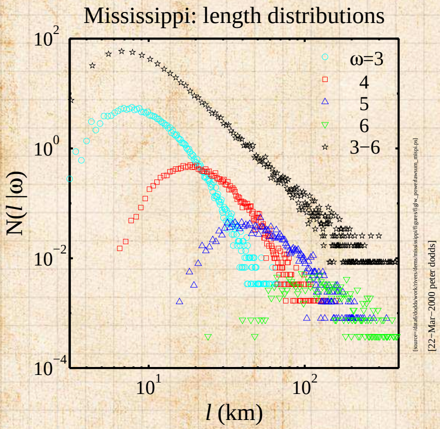

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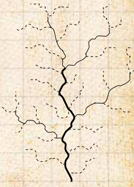
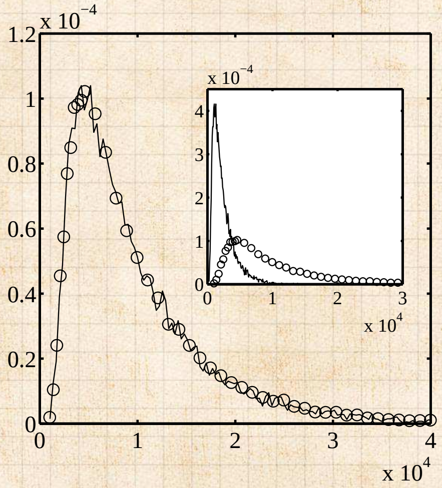


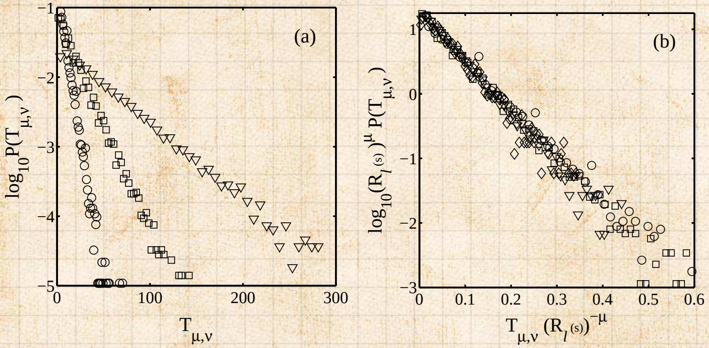
Generalizing Horton's laws

- Number and area distributions for the Scheidegger model [3]
- $P(n_{1,6})$ versus $P(a_6)$ for a randomly selected $\omega = 6$ basin.

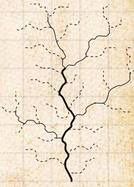


Generalizing Tokunaga's law

Scheidegger:



- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s



Generalizing Tokunaga's law

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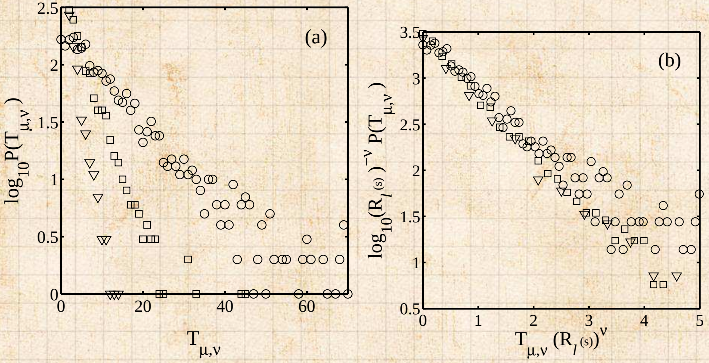
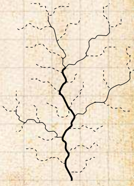


- Same data collapse for Mississippi...



Generalizing Tokunaga's law

So

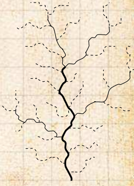
$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu} / (R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

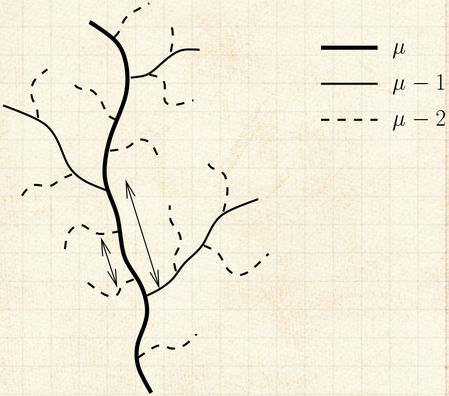
- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

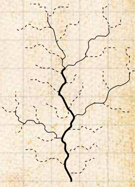


Generalizing Tokunaga's law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments





Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1}\xi_s$$

- Probability decays exponentially with stream order
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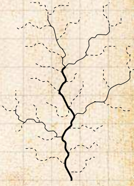

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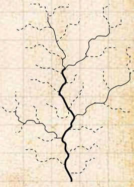

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- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

where
  - $p_\nu =$ probability of absorbing an order ν side stream
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- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.

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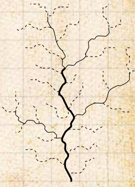

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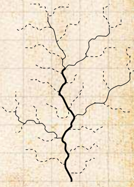

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- Now deal with this thing:

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- Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.
- Obtain

$$P(x, y) = N x^{-1/2} \left[F(y/x)\right]^x$$

where

$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

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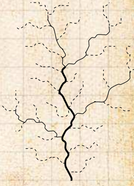

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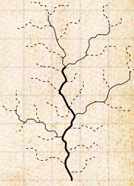
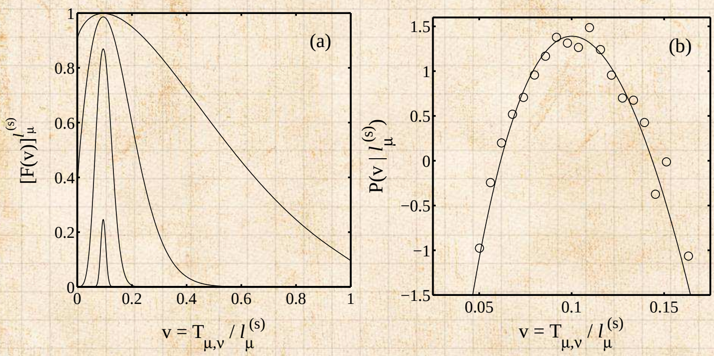
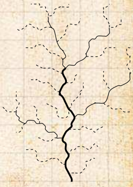


Generalizing Tokunaga's law

- Checking form of $P(s_{\mu}, T_{\mu,\nu})$ works:

Scheidegger:





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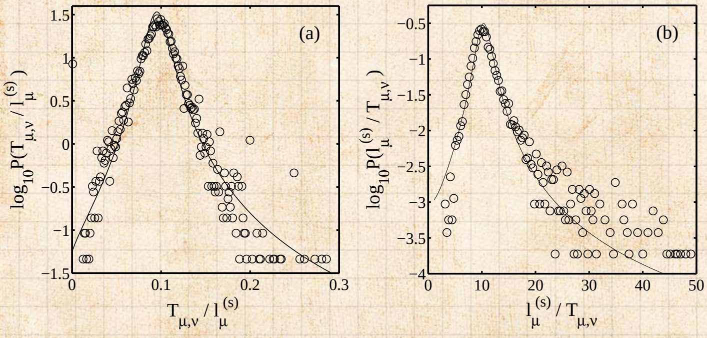

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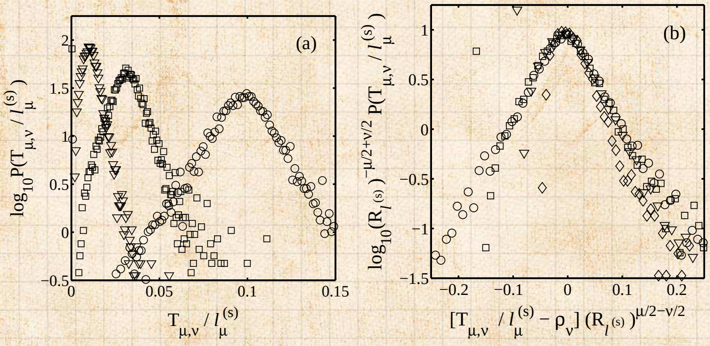

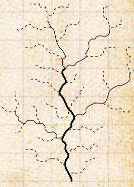

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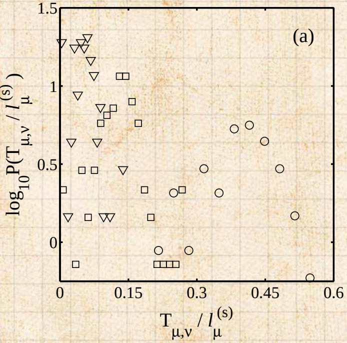

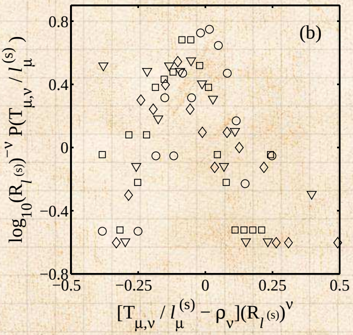

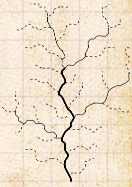

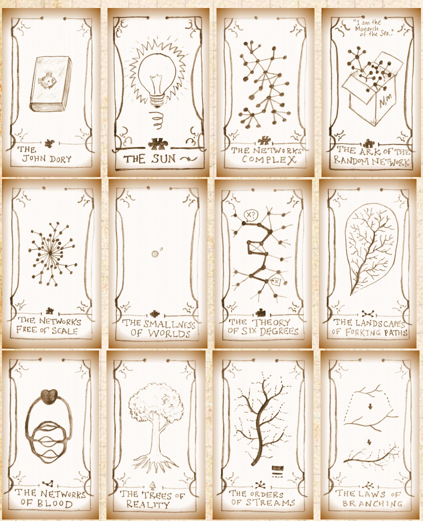
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"I am the Monarch of the Sea."
N,m
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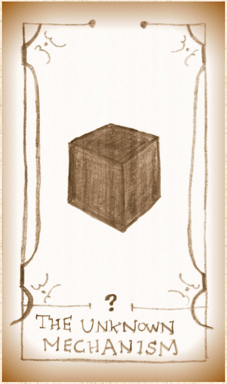
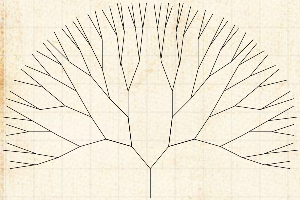

?
THE UNKNOWN MECHANISM

Models

Random subnetworks on a Bethe lattice [13]



- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of “Statistical inevitability” of river network statistics [7]
- But Bethe lattices unconnected with surfaces.
- In fact, Bethe lattices ≃ infinite dimensional spaces (oops).
- So let’s move on...

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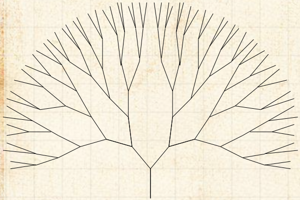

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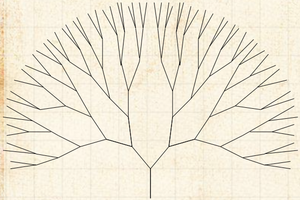

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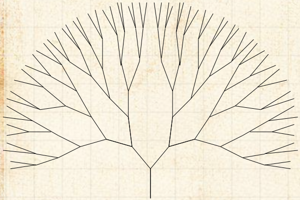

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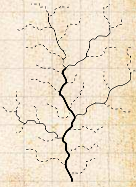

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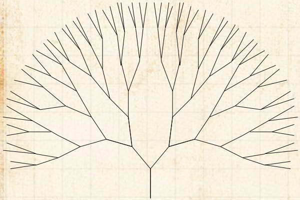

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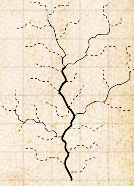

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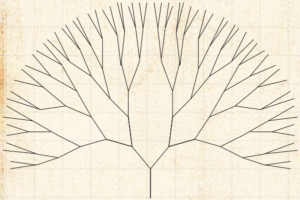

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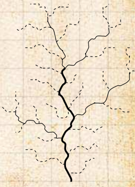

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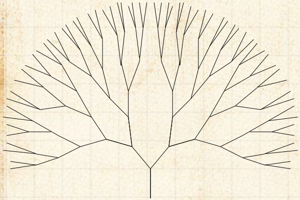

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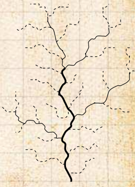
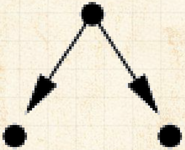
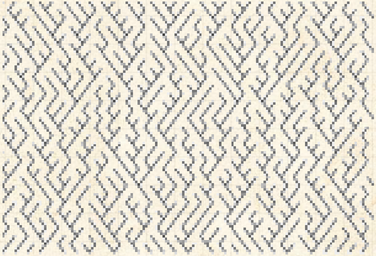
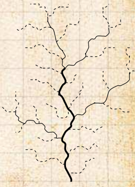


Scheidegger's model

Directed random networks [11, 12]







- $$P(\searrow) = P(\swarrow) = 1/2$$
- Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]

Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [10]

- Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\varepsilon}$ is minimized, where

$$\dot{\varepsilon} \propto \int \mathrm{d}\vec{r}\,(\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^{\gamma}$$

- Landscapes obtained numerically give exponents near that of real networks.
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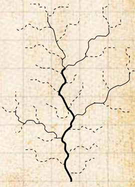

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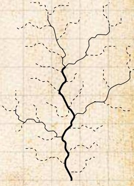
Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$h \Rightarrow \ell \propto a^h$ (Hack's law).

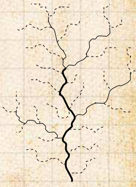
$d \Rightarrow \ell \propto L_{\parallel}^{d}$ (stream self-affinity).



Nutshell

Branching networks II Key Points:

- Horton's laws and Tokunaga law all fit together.
- For 2-d networks, these laws are 'planform' laws and ignore slope.
- Abundant scaling relations can be derived.
- Can take R_n, R_ℓ, and d as three independent parameters necessary to describe all 2-d branching networks.
- For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
- Laws can be extended nicely to laws of distributions.
- Numerous models of branching network evolution exist: nothing rock solid yet.



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- Laws can be extended nicely to laws of distributions.
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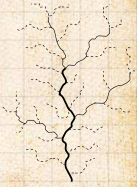

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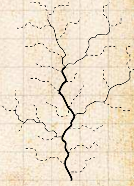

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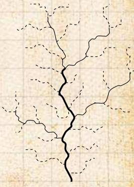

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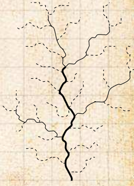

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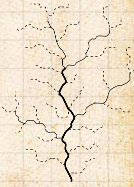

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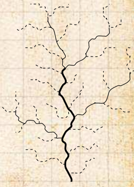

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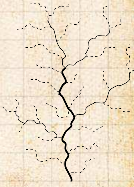

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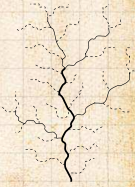

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