Principles of Complex Systems | @pocsvox CSYS/MATH 300, Fall, 2015 | #FallPoCS2015

# Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center Vermont Advanced Computing Core | University of Vermont









UNIVERSITY

VERMON













Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story

Model details

Analysis

more plan

mechanism

Robustness Krapivisky & Redner's

nodel

eneralized model

alusis

nalysis

iblinear attachm

Superlinear attachme

Nutshell

References







# These slides are brought to you by:



### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

A more plausible

Robustness

Krapivisky & Redner's

Universality?

Nutshell







# Outline

# Scale-free networks

Main story

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

Nutshell

# References

# PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story

Model details

Analysis

more plau

mechanism

obustness

Krapivisky & Redner's model

Generalized mode

Analysis

Universali

Sublinear attach

uperlinear attachme

Nutshell

Nutsriell









### Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

Analysis

Universality?

Sublinear attachment kernels

kernels

Nutshell











### Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

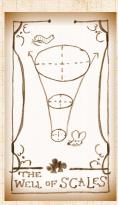
Nutshell











# Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

model

Generalized model

Analysis

Universality?
Sublinear attachment

kernels Superlinear attachmen

kernels Nutshell









# Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible

mechanism

Robustness

Krapivisky & Redner's

Analysis

Universality?

Sublinear attachment kernels

kernels

Nutshell











### Scale-free networks

## Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

model

eneralized model

Analysis
Universality?

Sublinear attachment kernels

Superlinear attachment

kernels Nutshell











### Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

RODUSTITESS

Krapivisky & Redner's model

Seneralized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachmen kernels

Nutshell











### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

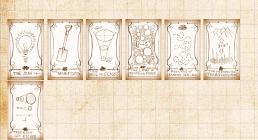
Superlinear attachmen kernels

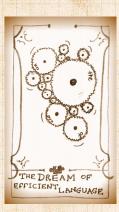
Nutshell











# Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

nodel

Generalized model

Analysis

Universality?

Sublinear attachment kernels

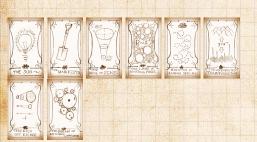
Superlinear attachmen kernels

Nutshell











# Scale-free networks

## Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

Analysis

Universality?

Sublinear attachment kernels

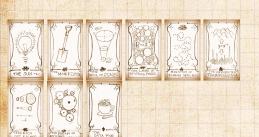
Superlinear attachment kernels

Nutshell











### Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

model

Generalized model

Analysis
Universality?

Sublinear attachment kernels

Superlinear attachment kernels

kernels Nutshell











# Scale-free networks

### Scale-free networks

Main story Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

Congralized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachmen kernels

Nutshell











### Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment

kernels Nutshell











### Scale-free networks

### Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

model

Generalized model

Analysis
Universality?

Sublinear attachment kernels

Superlinear attachment kernels

Nutshell











### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism

Robustness

Krapivisky & Redner's

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

kernels Nutshell

References



What's the Story?









# Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism

Robustness

Krapivisky & Redner's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

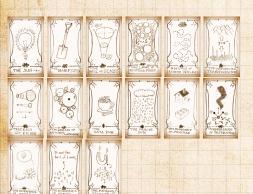
Superlinear attachment kernels

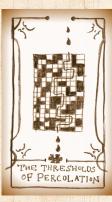
Nutshell











### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism Robustness

Krapivisky & Redner's

Analysis

Universality?

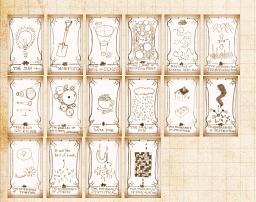
Sublinear attachment kernels

Superlinear attachment kernels

Nutshell









# Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

eneralized model

Analysis

Universality?

Sublinear attachment kernels

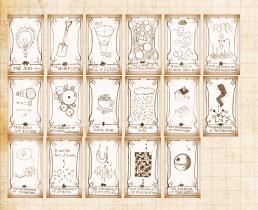
Superlinear attachment kernels

Nutshell











### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

Congralized model

Analysis

Universality?

Sublinear attachment kernels

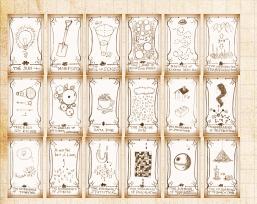
Superlinear attachment kernels

Nutshell ...











# PoCS | @pocsvox Scale-free

### Scale-free networks

networks

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

nodel

Generalized model

Analysis
Universality?

Sublinear attachment kernels

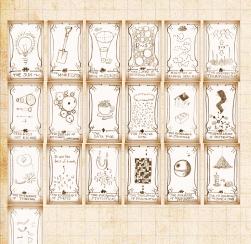
Superlinear attachment kernels

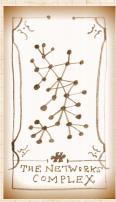
kernels Nutshell











### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism Robustness

Krapivisky & Redner's

nodel

Analysis

Universality?

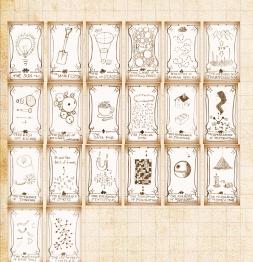
Sublinear attachment kernels

Superlinear attachment kernels

Nutshell









### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism

Robustness

Krapivisky & Redner's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

Nutshell









### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism

Robustness

Krapivisky & Redner's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

Nutshell











### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism Robustness

Krapivisky & Redner's

nodel

Generalized model

Analysis

Universality? Sublinear attachment

kernels

Superlinear attachment kernels

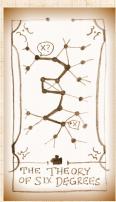
Nutshell











### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism Robustness

Krapivisky & Redner's

nodel

Generalized model

Analysis

Universality?
Sublinear attachment

kernels Superlinear attachment

kernels Nutshell









### Scale-free networks

# Scale-free networks

Model details

Analysis

A more plausible

mechanism

Robustness

Krapivisky & Redner's

operalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

Nutshell





# Outline

# Scale-free networks Main story

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details Analysis

Robustness

Krapivisky & Redner's

Universality?





Networks with power-law degree distributions have become known as scale-free networks.

PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

 $P_k \sim k^{-\gamma}$  for 'large' k

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks"
- Somewhat misleading nomenclature...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story Model details

Analysis

more plan

nechanism

Robustness

Krapivisky & Redner's model

seneralized model

Analysis

Universality?

kernels

vernels Nutshell





- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large'  $k$ 

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large'  $k$ 

One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2] Times cited: ~ 20, 734 (as of September 23, 2014)

Somewhat misleading nomenclature...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story Model details

Analysis

A more plausil mechanism

Robustness
Krapivisky & Redner's

model

eneralized model

Analysis Universality?

Sublinear attachment kernels

kernels Nutshell





- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large'  $k$ 

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2] Times cited: ~ 20, 734 (as of September 23, 2014)
- Somewhat misleading nomenclature...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story Model details

Analysis

A more plausib mechanism

Krapivisky & Redner's model

Generalized model

Analysis

Universality? Sublinear attachm

kernels





- ▶ Scale-free networks are not fractal in any sense.

# PoCS | @pocsvox

### Scale-free networks

#### Scale-free networks

#### Main story Model details

# Analysis

Robustness

Krapivisky & Redner's

Universality?







- ▶ Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not

# PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

more plausible

hustness

Krapivisky & Redner's

Generalized model

nalysis

Universality? Sublinear attachmen

Superlinear attachment kernels





- ▶ Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)
- ▶ Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not.

PoCS | @pocsvox
Scale-free

Scale-free networks

networks

Main story Model details

Analysis

A more plausible

Robustness

Krapivisky & Redner's model

Generalized model

Iniversality?

ublinear attachme

Superlinear attachmen kernels





- ▶ Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

A more plausib mechanism

Robustness

Krapivisky & Redner's model

seneralized model

Analysis

Universality?

ublinear attachmen ernels

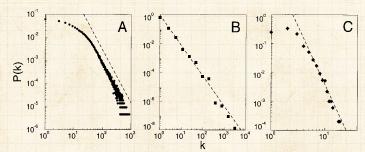
Superlinear attachme kernels





## Some real data (we are feeling brave):

## From Barabási and Albert's original paper [2]:



**Fig. 1.** The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with N=212,250 vertices and average connectivity  $\langle k \rangle=28.78$ . **(B)** WWW, N=325,729,  $\langle k \rangle=5.46$  **(6)**. **(C)** Power grid data, N=4941,  $\langle k \rangle=2.67$ . The dashed lines have slopes (A)  $\gamma_{\rm actor}=2.3$ , (B)  $\gamma_{\rm www}=2.1$  and (C)  $\gamma_{\rm power}=4$ .

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's

model

Generalized model

Universality?

Superlinear attachmen kernels

Nutshell





## Random networks: largest components









 $\gamma = 2.5$  $\langle k \rangle = 1.8$ 

 $\gamma = 2.5$ 

 $\langle k \rangle = 1.6$ 

 $\gamma = 2.5$  $\langle k \rangle = 2.05333$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.66667$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.92$ 









 $\gamma = 2.5$  $\langle k \rangle = 1.50667$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.62667$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.8$ 

## PoCS | @pocsvox

#### Scale-free networks

#### Scale-free networks

### Main story

#### Analysis

A more plausible

Robustness

Krapivisky & Redner's

Analysis

Universality? Sublinear attachment

Nutshell







## The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

#### PoCS | @pocsvox

#### Scale-free networks

#### Scale-free networks

## Main story

Analysis

more plausible

Robustness

Krapivisky & Redner's

model Redner S

eneralized model

Analysis
Universality?

ublinear attachment ernels

Superlinear attachmen kernels Nutshell







## The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

## A big deal for scale-free networks:

- $\blacktriangleright$  How does the exponent  $\gamma$  depend on the mechanism?

#### PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's







## The big deal:

We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

## A big deal for scale-free networks:

- $\blacktriangleright$  How does the exponent  $\gamma$  depend on the mechanism?
- Do the mechanism details matter?

## Scale-free

networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's





## Outline

## Scale-free networks

## Model details

### PoCS | @pocsvox

#### Scale-free networks

Scale-free networks

Main story

Model details Analysis

Robustness

Krapivisky & Redner's

Universality?







- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA
- $\triangleright$  Step 1: start with  $m_0$  disconnected nodes.
  - Step 2

- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

# PoCS | @pocsvox Scale-free

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

A more plausibl

Robustness

Krapivisky & Redner's

nodel

Generalized model

Universality?

Sublinear attachme

Superlinear attachme

vernels Nutshell







- Barabási-Albert model = BA model.
- Key ingredients:
   Growth and Preferential Attachment (PA).
- $\triangleright$  Step 1: start with  $m_0$  disconnected nodes.
  - Ste

- In essence, we have a rich gets richer scheme.
- Yes, we've seen this all before in Simon's model.

# PoCS | @pocsvox Scale-free

## networks

## Scale-free networks

Main story

#### Model details

Analysis

more plausible

iechanism

Robustness

Krapivisky & Redner's model

Generalized model

Universality?

Sublinear att

kernels

kernels Nutshell







- Barabási-Albert model = BA model.
- Key ingredients:
   Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.

- In essence, we have a rich gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

# PoCS | @pocsvox Scale-free

Scale-free networks

#### Scale-free networks

Main story

#### Model details Analysis

more plausible

Robustness

Krapivisky & Redner's

nodel

Generalized model

Universality?

Sublinear attachme

kernels







- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.
- ► Step 2:
  - 1. Growth—a new node appears at each time step
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to ith node is  $\propto k_i$ .
- ▶ In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

# PoCS | @pocsvox Scale-free

Scale-free networks

Scale-free networks

Main story

#### Model details Analysis

more plausible

mechanism Robustness

Krapivisky & Redner's

Generalized model

Analysis

Universality? Sublinear attachmer

Superlinear attachmen kernels

vernels Nutshell





# Scale-free networks

- ▶ Barabási-Albert model = BA model.
- Key ingredients:
   Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.
- ► Step 2:
  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to ith node is  $\infty k_i$ .
- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

## Scale-free networks

Main story
Model details

#### Analysis

more plausible

echanism

Krapivisky & Redner's

Generalized model

Analysis

Iniversality?

Superlinear attachmen kernels





Scale-free networks

- Barabási-Albert model = BA model.
- Key ingredients:
   Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.
- ► Step 2:
  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to ith node is  $\propto k_i$ .
- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

## Scale-free networks

Main story
Model details

#### Analysis

more plausible nechanism

Robustness Krapivisky & Redner's

nodel

Analysis

Universality? Sublinear attachmer

Superlinear attachmen kernels

Nutshell





Scale-free networks

- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.
- ► Step 2:
  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to *i*th node is  $\propto k_i$ .
- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

## Scale-free networks

Main story
Model details

#### Analysis

more plausible nechanism

Krapivisky & Redner's

Generalized model

Universality?

Sublinear attachment kernels

ernels Jutshell





- Barabási-Albert model = BA model.
- Key ingredients:
   Growth and Preferential Attachment (PA).
- ▶ Step 1: start with  $m_0$  disconnected nodes.
- ► Step 2:
  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to *i*th node is  $\propto k_i$ .
- ▶ In essence, we have a rich-gets-richer scheme.

Yes, we've seen this all before in Simon's model.

## Scale-free networks

Main story
Model details

#### Analysis

more plausible nechanism

Krapivisky & Redner's

Generalized model

Universality?

Superlinear attachmer kernels

kernels Nutshell





Key ingredients:
 Growth and Preferential Attachment (PA).

- ▶ Step 1: start with  $m_0$  disconnected nodes.
- ► Step 2:
  - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to *i*th node is  $\propto k_i$ .
- ▶ In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.

#### Scale-free networks

Main story

#### Model details

more plausible nechanism

Krapivisky & Redner's

eneralized model

Universality?

Superlinear attachme kernels

kernels Nutshell





## Outline

## Scale-free networks

**Analysis** 

## PoCS | @pocsvox

#### Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







- ▶ Definition: A<sub>k</sub> is the attachment kernel for a node with degree k.

$$A_k = k$$

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Nutshell





- **Definition:**  $A_k$  is the attachment kernel for a node with degree k.
- ▶ For the original model:

$$A_k = k$$

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Krapivisky & Redner's

Universality?

Nutshell





- **Definition:**  $A_k$  is the attachment kernel for a node with degree k.
- ▶ For the original model:

$$A_k = k$$

- **Definition:**  $P_{\text{attach}}(k,t)$  is the attachment probability.

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's

Universality?

Nutshell





- **Definition:**  $A_k$  is the attachment kernel for a node with degree k.
- ▶ For the original model:

$$A_k = k$$

- **Definition:**  $P_{\text{attach}}(k,t)$  is the attachment probability.
- ▶ For the original model:

$$P_{\mathrm{attach}}(\mathsf{node}\ i,t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\mathrm{max}}(t)} k_j(t)}$$

PoCS | @pocsvox Scale-free

Scale-free networks

networks

Main story Model details

Analysis

Krapivisky & Redner's

Nutshell







- **Definition:**  $A_k$  is the attachment kernel for a node with degree k.
- ▶ For the original model:

$$A_k = k$$

- **Definition:**  $P_{\text{attach}}(k,t)$  is the attachment probability.
- ▶ For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{N_{\text{max}}(t)} k_j(t)}$$

where  $N(t) = m_0 + t$  is # nodes at time t

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Krapivisky & Redner's

Nutshell





▶ For the original model:

$$A_k = k$$

- **Definition:**  $P_{\text{attach}}(k,t)$  is the attachment probability.
- ▶ For the original model:

$$P_{\mathrm{attach}}(\mathsf{node}\ i,t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\max}(t)} k N_k(t)}$$

where  $N(t) = m_0 + t$  is # nodes at time t

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's

Nutshell







▶ For the original model:

$$A_k = k$$

- **Definition:**  $P_{\text{attach}}(k,t)$  is the attachment probability.
- ▶ For the original model:

$$P_{\text{attach}}(\text{node } i,t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\max}(t)} kN_k(t)}$$

where  $N(t) = m_0 + t$  is # nodes at time t and  $N_k(t)$  is # degree k nodes at time t.

#### Scale-free networks

Main story Model details

#### Analysis

Krapivisky & Redner's

Nutshell







 $\blacktriangleright$  When (N+1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

## PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story Model details

#### Analysis

#### Robustness Krapivisky & Redner's

## Universality?

Nutshell







When (N+1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- $\blacktriangleright$  Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i,N}$

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

viouei det

Analysis

more plausibl echanism

Robustness Krapivisky & Redner's

nodel

Seneralized model

Universality?

ernels

kernels
Nutshell





When (N+1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_{j}(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ► Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i}$ .

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story

Model details

Analysis

more plausible nechanism

Krapivisky & Redner's model

Generalized model

Universality?

ernels uperlinear attachmo ernels

vernels Nutshell





When (N+1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_{j}(t)}.$$

- Assumes probability of being connected to is small.
- ▶ Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ :

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story
Model details

Analysis

A more plausibl mechanism

Robustness

Krapivisky & Redner's

Generalized model

nalysis

Sublinear attachment kernels

kernels Nutshell





When (N+1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_{j}(t)}.$$

- Assumes probability of being connected to is small.
- ▶ Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ :

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)}$$

where  $t = N(t) - m_0$ .

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story

Model details

Analysis A more plat

more plausible nechanism

Krapivisky & Redner's model

Generalized model

niversality?

kernels

kernels Nutshell







$$\sum_{i=1}^{N(t)} k_j(t) = 2tm$$

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$$

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

#### PoCS | @pocsvox Scale-free

Scale-free networks

networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Analysis

Universality? Sublinear attachment

Nutshell







$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_j(t)}$$

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

## PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

#### Analysis

Robustness

Krapivisky & Redner's

Analysis

Universality?

Nutshell







$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

▶ The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_{i}(t)}{\sum_{j=1}^{N(t)}k_{j}(t)} = m\frac{k_{i}(t)}{2mt} - \frac{1}{2t}k_{i}(t)$$

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

#### PoCS | @pocsvox Scale-free

# networks

#### Scale-free networks

Main story Model details

Analysis

Robustness Krapivisky & Redner's

Universality?

Nutshell







$$\div \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

▶ The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_{i}(t)}{\sum_{j=1}^{N(t)}k_{j}(t)} = m\frac{k_{i}(t)}{2mt} - \frac{1}{2t}k_{i}(t)$$

Rearrange and solve

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

Next find a

# PoCS | @pocsvox Scale-free networks

#### Scale-free networks

Main story

#### Analysis

more plausible

#### Robustness

Krapivisky & Redner's

#### Generalized model

Analysis
Universality?

#### Sublinear attachn

Superlinear attachmen

#### kernels Nutshell





$$\div \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

▶ The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

#### PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness Krapivisky & Redner's

Universality?

Nutshell





$$\div \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

▶ The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{}$$

► Next find e;

# PoCS | @pocsvox Scale-free

Scale-free networks

networks

Main story

Model details

Analysis

a laly sis

more plausible echanism

Robustness

Krapivisky & Redner's model

Generalized model

Analysis

Universality?

kernels Superlinear attachmer

kernels Nutshell







$$\div \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

▶ The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i\,t^{1/2}.}$$

## PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Model details

#### Analysis

Robustness

Krapivisky & Redner's

Universality?







$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

▶ The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i\,t^{1/2}.}$$

ightharpoonup Next find  $c_i$  ...

# PoCS | @pocsvox Scale-free

networks

#### Scale-free networks

Main story

## Analysis

more plausible

## mechanism

Robustness Krapivisky & Redner's

### nodel

Generalized model

## Universality?

kernels

Superlinear attachmen kernels





Know ith node appears at time

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

$$k_i(t) = m \left( rac{t}{t_{i, ext{start}}} 
ight)^{1/2} ext{ for } t \geq t_{i, ext{start}}$$

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story Model details

#### Analysis

Robustness

Krapivisky & Redner's

Universality?







Know ith node appears at time

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

lacktriangle So for  $i>m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i, \text{start}}}\right)^{1/2} \text{ for } t \geq t_{i, \text{start}}.$$

- All node degrees grow as
- ➤ First-mover advantage: Early nodes do
- Clearly, a

## PoCS | @pocsvox

## Scale-free networks

## Scale-free networks

Main story

#### Analysis

more plausible

### Robustness

Krapivisky & Redner's model

### neralized model

lysis

## Universality?

Superlinear attachment kernels Nutshell







Know ith node appears at time

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

 $\blacktriangleright$  So for  $i > m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i, \, \mathrm{start}}}\right)^{1/2} \, \, \mathrm{for} \, \, t \geq t_{i, \, \mathrm{start}}.$$

- $\blacktriangleright$  All node degrees grow as  $t^{1/2}$  but later nodes have

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story Model details

#### Analysis

#### Robustness Krapivisky & Redner's

Nutshell







Know ith node appears at time

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

 $\blacktriangleright$  So for  $i > m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i, \, \mathrm{start}}}\right)^{1/2} \, \, \mathrm{for} \, \, t \geq t_{i, \, \mathrm{start}}.$$

- ightharpoonup All node degrees grow as  $t^{1/2}$  but later nodes have larger  $t_{i,\text{start}}$  which flattens out growth curve.

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's

Nutshell





Know ith node appears at time

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

▶ So for  $i > m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i, \text{start}}}\right)^{1/2} \text{ for } t \geq t_{i, \text{start}}.$$

- All node degrees grow as  $t^{1/2}$  but later nodes have larger  $t_{i,\text{start}}$  which flattens out growth curve.
- ► First-mover advantage: Early nodes do best.
- ► Clearly, a

PoCS | @pocsvox
Scale-free

Scale-free networks

Scale-free networks

Main story
Model details

Analysis

Allalysis

more plausib nechanism

Robustness

Krapivisky & Redner's

odel

ysis

niversality?

kernels Superlinear attachme

kernels Nutshell





Know ith node appears at time

$$t_{i, \mathrm{start}} = \left\{ \begin{array}{ll} i - m_0 & \mathrm{for} \ i > m_0 \\ 0 & \mathrm{for} \ i \leq m_0 \end{array} \right.$$

ightharpoonup So for  $i>m_0$  (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i, \text{start}}}\right)^{1/2} \text{ for } t \geq t_{i, \text{start}}.$$

- ▶ All node degrees grow as  $t^{1/2}$  but later nodes have larger  $t_{i,\text{start}}$  which flattens out growth curve.
- ► First-mover advantage: Early nodes do best.
- ► Clearly, a Ponzi scheme .

PoCS | @pocsvox
Scale-free

Scale-free networks

Scale-free networks

Main story Model details

Analysis

Alidiyala

more plausible nechanism

Krapivisky & Redner's

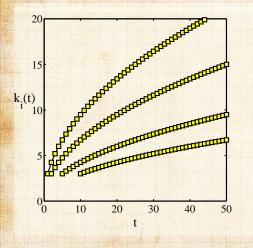
neralized model

Universality? Sublinear attachmen

Superlinear at kernels Nutshell







 $\rightarrow m=3$ 

 $ightharpoonup t_{i,start} =$ 1, 2, 5, and 10.

## PoCS | @pocsvox

## Scale-free networks

#### Scale-free networks

Main story Model details

### Analysis

Krapivisky & Redner's

Universality?

Nutshell







- ▶ So what's the degree distribution at time t?

$$\mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}t_{i, \mathrm{start}} \simeq \frac{\mathrm{d}t_{i, \mathrm{start}}}{t}$$

$$k_i(t) = m \left(\frac{t}{t_{i, \text{start}}}\right)^{1/2} + t_{i, \text{start}} = \frac{m^2 t}{k_i(t)^2}$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's

Universality?

Nutshell







- ▶ So what's the degree distribution at time *t*?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i, \mathsf{start}}) \mathsf{d}t_{i, \mathsf{start}} \simeq \frac{\mathsf{d}t_{i, \mathsf{start}}}{t}$$

Also use

 $k_i(t) = m \left(\frac{t}{t_i}\right)^{1/2} - t_i$ , start

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Analysis

rulaly313

more plausible echanism

Robustness

Krapivisky & Redner's model

Seneralized model

Universality?

kernels

Superlinear attachment kernels Nutshell





- ▶ So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}t_{i, \mathrm{start}} \simeq \frac{\mathrm{d}t_{i, \mathrm{start}}}{t}$$

Also use

$$k_i(t) = m \left(\frac{t}{t_{i, \mathrm{start}}}\right)^{1/2} {\Rightarrow} t_{i, \mathrm{start}} = \frac{m^2 t}{k_i(t)^2}.$$

$$\frac{\mathrm{d}t_{i,\mathrm{start}}}{\mathrm{d}k_{i}} = -2 \frac{m^{2}t}{k_{i}(t)^{3}}$$

## PoCS | @pocsvox

Scale-free networks

### Scale-free networks

Main story Model details

#### Analysis

Krapivisky & Redner's

Universality?







- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}t_{i, \mathrm{start}} \simeq \frac{\mathrm{d}t_{i, \mathrm{start}}}{t}$$

Also use

$$k_i(t) = m \left(\frac{t}{t_{i, \mathrm{start}}}\right)^{1/2} \Rightarrow t_{i, \mathrm{start}} = \frac{m^2 t}{k_i(t)^2}.$$

Transform variables—Jacobian:

$$\frac{\mathrm{d}t_{i,\mathrm{start}}}{\mathrm{d}k_i} = -2\frac{m^2t}{k_i(t)^3}.$$

PoCS | @pocsvox
Scale-free

Scale-free

networks

networks

Model details

Analysis

nore plausible chanism

Robustness

Krapivisky & Redner's model

Generalized model

Universality? Sublinear attachmer

Superlinear attachmen kernels





 $\mathbf{Pr}(k_i) dk_i = \mathbf{Pr}(t_{i,\text{start}}) dt_{i,\text{start}}$ 

$$\mathbf{Pr}(t_{i,\mathsf{start}})\mathsf{d}k_i \left| egin{array}{c} \mathsf{d}t_{i,\mathsf{start}} \ \mathsf{d}k_i \end{array} \right|$$

$$\frac{1}{t} \mathrm{d}k_i \, 2 \frac{m^2 t}{k_i(t)^3}$$

$$=2\frac{m^2}{k_i(t)^3}\mathsf{d}k_i$$

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story

Model details

#### Analysis

Robustness

Krapivisky & Redner's

Analysis

Universality?

Nutshell







$$\mathbf{Pr}(k_i) \mathrm{d}k_i = \mathbf{Pr}(t_{i,\mathrm{start}}) \mathrm{d}t_{i,\mathrm{start}}$$

$$= \mathbf{Pr}(t_{i, \text{start}}) \mathsf{d} k_i \left| \frac{\mathsf{d} t_{i, \text{start}}}{\mathsf{d} k_i} \right|$$

$$=\frac{1}{t} dk_i 2 \frac{m^2 t}{k_i(t)^3}$$

$$=2\frac{m^2}{k_s(t)^3}\mathsf{d}k_i$$

 $dk_i$ 

## PoCS | @pocsvox

## Scale-free networks

## Scale-free networks

Main story

Model details

## Analysis

more plausible echanism

Robustness

Krapivisky & Redner's model

Generalized model

Universality?

kernels Superlinear attachmen

kernels Nutshell







$$\mathbf{Pr}(k_i) \mathrm{d}k_i = \mathbf{Pr}(t_{i,\mathrm{start}}) \mathrm{d}t_{i,\mathrm{start}}$$

$$= \mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}k_i \left| \frac{\mathrm{d}t_{i, \mathrm{start}}}{\mathrm{d}k_i} \right|$$

$$=\frac{1}{t}\mathrm{d}k_i\,2\frac{m^2t}{k_i(t)^3}$$

$$=2\frac{m^2}{k_i(t)^3}\mathsf{d}k_i$$

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story

Model details

### Analysis

Krapivisky & Redner's

Universality?

Nutshell







$$\begin{split} \mathbf{Pr}(k_i) \mathrm{d}k_i &= \mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}t_{i, \mathrm{start}} \\ &= \mathbf{Pr}(t_{i, \mathrm{start}}) \mathrm{d}k_i \left| \frac{\mathrm{d}t_{i, \mathrm{start}}}{\mathrm{d}k_i} \right| \end{split}$$

$$1_{\mathsf{d}k} \, 2^{m^2t}$$

$$=\frac{1}{t}\mathrm{d}k_i\,2\frac{m^2t}{k_i(t)^3}$$

$$=2\frac{m^2}{k_i(t)^3}\mathsf{d} k_i$$

## PoCS | @pocsvox

## Scale-free networks

## Scale-free networks

Main story Model details

## Analysis

Krapivisky & Redner's

Universality?

Nutshell









$$\begin{split} \mathbf{Pr}(k_i) \mathrm{d}k_i &= \mathbf{Pr}(t_{i,\text{start}}) \mathrm{d}t_{i,\text{start}} \\ &= \mathbf{Pr}(t_{i,\text{start}}) \mathrm{d}k_i \left| \frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i} \right| \end{split}$$

$$1 \cdot m^2 t$$

$$=\frac{1}{t}\mathrm{d}k_i\,2\frac{m^2t}{k_i(t)^3}$$

$$=2\frac{m^2}{k_i(t)^3}\mathsf{d}k_i$$

$$\propto k_i^{-3} dk_i$$
.

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story

## Model details

Analysis
A more plausible

#### echanism obustness -

Krapivisky & Redner's model

## Generalized model

Universality? Sublinear attachment

#### kernels Superlinear attachme

kernels Nutshell





- ▶ We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story Model details

## Analysis

Robustness

Krapivisky & Redner's

Universality?







- ▶ We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story Model details

#### Analysis

Robustness

Krapivisky & Redner's

Universality?







- ▶ We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- ▶ Range true more generally for events with size distributions that have power-law tails.

## PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's

Universality?





- We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- ▶ Range true more generally for events with size distributions that have power-law tails.
- $ightharpoonup 2 < \gamma < 3$ : finite mean and 'infinite' variance
- In practice,  $\gamma < 3$  means variance is governed by upper cutoff.
- > 3: finite mean and variance

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Analysis

Analysis

more plausible echanism

Krapivisky & Redner's

Generalized model

Analysis
Universality?

Sublinear attachment kernels

uperlinear attachmer ernels Jurshell





- We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- ► Range true more generally for events with size distributions that have power-law tails.
- $ightharpoonup 2 < \gamma < 3$ : finite mean and 'infinite' variance
- In practice,  $\gamma < 3$  means variance is governed by upper cutoff.
- > 3: finite mean and variance

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

woder de

Analysis

more plausib nechanism

Robustness

Krapivisky & Redner's model

Generalized model

Universality?

ernels uperlinear attachmer

ernels lutshell





- We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- ► Range true more generally for events with size distributions that have power-law tails.
- $ightharpoonup 2 < \gamma < 3$ : finite mean and 'infinite' variance
- In practice,  $\gamma < 3$  means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$ : finite mean and variance

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story
Model details

viodei deta

Analysis

mechanism

Krapivisky & Redner's

Generalized model

Analysis

Sublinear attachme

uperlinear attachmer ernels





- We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- ► Range true more generally for events with size distributions that have power-law tails.
- $ightharpoonup 2 < \gamma < 3$ : finite mean and 'infinite' variance (wild)
- In practice,  $\gamma < 3$  means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$ : finite mean and variance

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story
Model details

lodel deta

Analysis

A more plausi mechanism

Krapivisky & Redner's

Generalized model

Analysis

Sublinear attachmen

uperlinear attachr ernels lutshell





- We thus have a very specific prediction of  $\mathbf{Pr}(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- ▶ Typical for real networks:  $2 < \gamma < 3$ .
- ▶ Range true more generally for events with size distributions that have power-law tails.
- $ightharpoonup 2 < \gamma < 3$ : finite mean and 'infinite' variance (wild)
- In practice,  $\gamma < 3$  means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$ : finite mean and variance (mild)

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Model details

Analysis

A more plausi mechanism

Krapivisky & Redner's

Generalized model

Universality?

kernels Superlinear attachme

urnels utshell





## Back to that real data:

## From Barabási and Albert's original paper [2]:

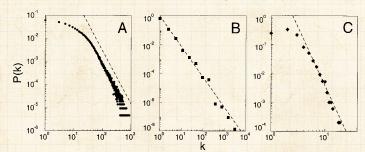


Fig. 1. The distribution function of connectivities for various large networks. (A) Actor collaboration graph with N=212,250 vertices and average connectivity  $\langle k \rangle=28.78$ . (B) WWW, N=325,729,  $\langle k \rangle=5.46$  (c) Power grid data, N=4941,  $\langle k \rangle=2.67$ . The dashed lines have slopes (A)  $\gamma_{actor} = 2.3$ , (B)  $\gamma_{www} = 2.1$  and (C)  $\gamma_{power} = 4$ .

PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story Model details

#### Analysis

Krapivisky & Redner's

Universality?

Nutshell







# Examples

 $\gamma \simeq 2.1$  for in-degree Web  $\gamma \simeq 2.45$  for out-degree Web Movie actors  $\gamma \simeq 2.3$ Words (synonyms)  $\gamma \simeq 2.8$ 

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story

Model details

## Analysis

Krapivisky & Redner's

Universality?

Nutshell







# Examples

 $\gamma \simeq 2.1$  for in-degree Web  $\gamma \simeq 2.45$  for out-degree Web  $\gamma \simeq 2.3$ Movie actors Words (synonyms)  $\gamma \simeq 2.8$ 

The Internets is a different business...

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story

#### Model details Analysis

Krapivisky & Redner's

Universality?

Nutshell







- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.

PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's

Universality?







- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- ▶ Important Q.: Are there distinct universality classes for these networks?

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's





- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- $\triangleright$  Q.: How does changing the model affect  $\gamma$ ?
- Q: Do we need preferential attachment and growth?
- O: Do model details matter?

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Analysis

Analysis

more plat

nechanism

Robustness

Krapivisky & Redner's

odel

eneralized model

nalysis

Universality?

Sublinear attachment kernels

kernels







- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- ► Important Q.: Are there distinct universality classes for these networks?
- ▶ Q.: How does changing the model affect  $\gamma$ ?
- Q.: Do we need preferential attachment and growth?

O: Do model details matter?

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story
Model details

Analysis

Arialysis

more plausib nechanism

obustness

Krapivisky & Redner's model

Seneralized model

Analysis

Sublinear attachment kernels

kernels





- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- ► Important Q.: Are there distinct universality classes for these networks?
- ▶ Q.: How does changing the model affect  $\gamma$ ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter? Maybe ...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story

Analysis

analysis

A more plausil mechanism

Robustness

Krapivisky & Redner's

Model

Generalized model

peneralized model

Universality?

Sublinear attachment kernels Superlinear attachmen

kernels Nutshell





- Vary attachment kernel.
- Vary mechanisms:
  - 1. Add edge deletion
  - 2. Add node deletion
  - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- ► Important Q.: Are there distinct universality classes for these networks?
- ▶ Q.: How does changing the model affect  $\gamma$ ?
- Q.: Do we need preferential attachment and growth?
- ▶ Q.: Do model details matter? Maybe ...

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story
Model details

Analysis

anaiysis

A more plausit mechanism

Krapivisky & Redner's

Generalized model

nalysis

Universality?

kernels

kernels Nutshell





## Outline

## Scale-free networks

## A more plausible mechanism

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

A more plausible mechanism

Krapivisky & Redner's

Universality?







- ▶ Let's look at preferential attachment (PA) a little more closely.

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Model details Analysis

A more plausible mechanism

Krapivisky & Redner's

Universality?







- ► Let's look at preferential attachment (PA) a little more closely.
- ▶ PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality
- We need to know what everyone's degree is.
- ▶ PA is :: an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story
Model details

Analysis

A more plausible mechanism

Krapivisky & Redner's

Generalized model

Analysis Universality?

Sublinear attachment kernels

ernels lutshell





- ► Let's look at preferential attachment (PA) a little more closely.
- ▶ PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.
- We need to know what everyone's degree is
- ► PA is :: an outrageous assumption of node capability.
- ➤ But a very simple mechanism saves the day.

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story
Model details

Analysis

A more plausible mechanism

Robustness Krapivisky & Redner's

model ....

Analysis

Sublinear attachment kernels

uperlinear attachr ernels utshell





- ► Let's look at preferential attachment (PA) a little more closely.
- ▶ PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...
- ➤ PA is : an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story
Model details

Analysis

A more plausible mechanism

Robustness Krapivisky & Redner's

model Redners

Generalized model

Universality?

kernels

ernels lutshell





- ► Let's look at preferential attachment (PA) a little more closely.
- ▶ PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- ► PA is : an outrageous assumption of node capability.
- ➤ But a very simple mechanism saves the day...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story Model details

Analysis A more plausible

A more plausible mechanism Robustness

Krapivisky & Redner's model

Generalized model

Analysis Universality?

Sublinear attachment kernels Superlinear attachme

kernels Nutshell





- ► Let's look at preferential attachment (PA) a little more closely.
- ▶ PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- ► PA is :: an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story
Model details

Analysis

A more plausible mechanism

Robustness Krapivisky & Redner's

Generalized model

Analysis

Sublinear attachment kernels

kernels Nutshell





# Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.

PoCS | @pocsvox Scale-free networks

Scale-free

networks Main story

Model details Analysis

A more plausible mechanism

Krapivisky & Redner's

Universality?







# Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

A more plausible mechanism

Krapivisky & Redner's







# Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.

Scale-free networks

Scale-free networks

Main story Model details

Analysis

A more plausible mechanism

Krapivisky & Redner's







# Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

➤ So rich gets sicher scheme can now be seen to work in a natural way.

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Model details

Analysis

A more plausible mechanism

Robustness Krapivisky & Redner's

model

alvsis

Iniversality?

kernels Superlinear attachme

ernels lutshell





# Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

So rich-gets-richer scheme can now be seen to work in a natural way. PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details Analysis

A more plausible mechanism

Krapivisky & Redner's model

Generalized model

Universality? Sublinear attachmen

Superlinear attachmer kernels





### Outline

### Scale-free networks

### Robustness

### PoCS | @pocsvox

#### Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

#### Robustness

Krapivisky & Redner's

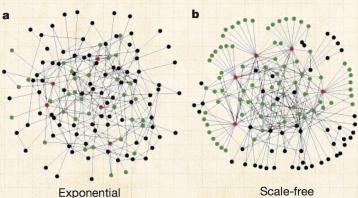
Universality?







- ▶ Albert et al., Nature, 2000: "Error and attack tolerance of complex networks" [1]
- Standard random networks (Erdős-Rényi) versus Scale-free networks:



Scale-free

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

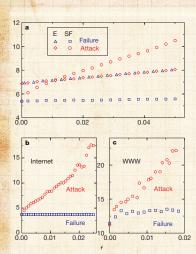
Robustness

Krapivisky & Redner's









 Plots of network diameter as a function of fraction of nodes removed

- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

PoCS | @pocsvox Scale-free

Scale-free networks

Scale-free networks

Main story Model details Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's model

eneralized model nalysis

Universality? Sublinear attachment kernels

kernels







- Scale-free networks are thus robust to random failures yet fragile to targeted ones.

### PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:

Need to explore cost of various targeting schemes.

### PoCS | @pocsvox

Scale-free networks

### Scale-free networks

Main story

Model details Analysis

more plausible

#### Robustness

Krapivisky & Redner's model

Analysis

Universality?

kernels Superlinear attachmen

kernels Nutshell







### Scale-free networks

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- Main story ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.

#### Scale-free networks

Model details Analysis

#### Robustness

Krapivisky & Redner's

Universality?









Scale-free networks

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:

Scale-free networks

Main story Model details Analysis

more plausible

Robustness

obustness

Krapivisky & Redner's model

Analysis

Universality? Sublinear attachm kernels

Superlinear attac kernels

References



Need to explore cost of various targeting schemes



- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:

Main story Model details Analysis

Krapivisky & Redner's







Main story Model details

Analysis

- ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  - 1. Physically larger nodes that may be harder to 'target'

References

Krapivisky & Redner's







- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  - 1. Physically larger nodes that may be harder to 'target'
  - 2. or subnetworks of smaller, normal-sized nodes.

Main story Model details

Krapivisky & Redner's







- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  - Physically larger nodes that may be harder to 'target'
  - 2. or subnetworks of smaller, normal-sized nodes.
- ▶ Need to explore cost of various targeting schemes.

Main story
Model details
Analysis

more plausible echanism

#### Robustness

Krapivisky & Redner's model

Analysis

Universality? Sublinear attach

Superlinear





#### PoCS | @pocsvox Scale-free networks

### Not a robust paper:



"The "Robust yet Fragile" nature of the Internet" Doyle et al., Proc. Natl. Acad. Sci., 2005, 14497–14502, 2005. [3]

- ▶ HOT networks versus scale-free networks
- Same degree distributions, different arrangements.
- ▶ Doyle et al. take a look at the actual Internet.
- Excellent project material.

### Scale-free networks

Model details

Analysis

nore plausible echanism

#### Robustness

Krapivisky & Redner's model

Analysis

Sublinear attachmen kernels

Superlinear attach kernels





### Outline

### Scale-free networks

Krapivisky & Redner's model

PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







### Outline

### Scale-free networks

### Generalized model

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

#### Generalized model

Analysis Universality?









### Fooling with the mechanism:

▶ 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness Krapivisky & Redner's

#### Generalized model

Universality?







### Fooling with the mechanism:

▶ 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

$$\mathbf{Pr}(\text{attach to node } i) \propto A_k = k_i^{\nu}$$

where  $A_{\nu}$  is the attachment kernel and  $\nu > 0$ .

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Krapivisky & Redner's

#### Generalized model







### Fooling with the mechanism:

▶ 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

$$\mathbf{Pr}(\mathsf{attach}\ \mathsf{to}\ \mathsf{node}\ i) \propto A_k = k_i^{\nu}$$

where  $A_k$  is the attachment kernel and  $\nu > 0$ .

- ► KR also looked at changing the details of the attachment kernel.
- KR model will be fully studied in CoNKS

# PoCS | @pocsvox Scale-free

sala fran

networks

Scale-free networks

Main story
Model details

Analysis

nechanism

Robustness Krapivisky & Redner's

#### Generalized model

nalysis

Universality? Sublinear attachment

Superlinear attachmer kernels





### Fooling with the mechanism:

▶ 2001: Krapivsky & Redner (KR) [4] explored the general attachment kernel:

$$\mathbf{Pr}(\mathsf{attach}\ \mathsf{to}\ \mathsf{node}\ i) \propto A_k = k_i^{\nu}$$

where  $A_k$  is the attachment kernel and  $\nu > 0$ .

- ► KR also looked at changing the details of the attachment kernel.
- ▶ KR model will be fully studied in CoNKS.

# PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story

Model details

Analysis

nechanism

Krapivisky & Redner's

#### Generalized model

Universality?

kernels Superlinear attachmer

kernels Nutshell





- ▶ We'll follow KR's approach using rate equations .

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

#### Generalized model

Universality?

Nutshell







- ▶ We'll follow KR's approach using rate equations .
- ► Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

#### Generalized model

Universality?







- ▶ We'll follow KR's approach using rate equations .
- ► Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

- 1. One node with one link is added per unit time.

PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

#### Generalized model

Universality?







- ▶ We'll follow KR's approach using rate equations .
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

- 1. One node with one link is added per unit time.
- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 3. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 4. A is the correct normalization (coming up)
- 5. Seed with some initial network
- 6. Detail:  $A_0 = 0$

PoCS | @pocsvox

Scale-free networks

### Scale-free networks

Main story

Model details

Analysis

echanism

Robustness

Krapivisky & Redner's model

#### Generalized model

Analysis
Universality?

ublinear attachment

Superlinear attachmen kernels







- ▶ We'll follow KR's approach using rate equations .
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

- 1. One node with one link is added per unit time.
- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 3. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 4. A is the correct normalization (coming up
- 5. Seed with some initial network
- 6. Detail:  $A_0 = 0$

PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Analysis

ialysis

echanism

lobustness

Krapivisky & Redner's model

#### Generalized model

niversality?

ublinear attachment

Superlinear attachi kernels





- ▶ We'll follow KR's approach using rate equations .
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

- 1. One node with one link is added per unit time.
- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 3. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 4. *A* is the correct normalization (coming up).

PoCS | @pocsvox
Scale-free

Scale-free networks

### Scale-free networks

Main story

Model details

Analysis

echanism

Robustness Krapivisky & Redner's

model Redriers

#### Generalized model

niversality?

Sublinear attachment

Superlinear attachr kernels







- ▶ We'll follow KR's approach using rate equations .
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

- 1. One node with one link is added per unit time.
- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 3. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 4. *A* is the correct normalization (coming up).
- 5. Seed with some initial network

(e.g., a connected pair)

6. Detail:  $A_0 = 0$ 

PoCS | @pocsvox
Scale-free

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

echanism

hustness

Krapivisky & Redner's

Generalized model

Analysis

Universality?

Superlinear attachmen kernels

kernels Nutshell







- ▶ We'll follow KR's approach using rate equations .
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

- 1. One node with one link is added per unit time.
- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 3. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 4. *A* is the correct normalization (coming up).
- 5. Seed with some initial network (e.g., a connected pair)

6. Detail:  $A_0 = 0$ 

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story

Model details

Analysis

echanism

bustness

Krapivisky & Redner's model

### Generalized model

niversality

ublinear attachment ernels

Superlinear attachi kernels







- ▶ We'll follow KR's approach using rate equations .
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

where  $N_k$  is the number of nodes of degree k.

- 1. One node with one link is added per unit time.
- 2. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 3. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 4. *A* is the correct normalization (coming up).
- 5. Seed with some initial network (e.g., a connected pair)
- 6. Detail:  $A_0 = 0$

PoCS | @pocsvox
Scale-free

Scale-free networks

#### Scale-free networks

Main story
Model details

Analysis

more plans

ecnanism

Krapivisky & Redner's

model Redner S

### Generalized model

Iniversality?

kernels Superlinear attachmen

kernels Nutshell







### Outline

### Scale-free networks

Main story

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisiy & Redners mode

### **Analysis**

**Universality** 

Sublinear attachment kernels Superlinear attachment kernels Nutshell

References

### PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story

Model details

Analysis

more plausible

Robustness

Krapivisky & Redner's

model

Generalized mou

### Analysis Universality?

Sublinear attachment

Superlinear attachmen

Nutshell







In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\mathsf{attach}\ \mathsf{to}\ \mathsf{node}\ i) = \frac{A_k}{A(t)}$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

#### Analysis Universality?







In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

#### Analysis Universality?







In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

#### Analysis Universality?







In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .

### PoCS | @pocsvox

#### Scale-free networks

#### Scale-free networks

Main story

Model details Analysis

Robustness Krapivisky & Redner's

### Analysis

Nutshell







In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .
- ightharpoonup For  $A_k = k$ , we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

### Analysis

Nutshell







In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .
- For  $A_k = k$ , we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t)$$

since one edge is being added per unit time

 Detail: we are ignoring initial seed network's edges.

### PoCS | @pocsvox

Scale-free networks

### Scale-free networks

Main story

Model details Analysis

Arialysis

nechanism

Robustness

Krapivisky & Redner's model

eneralized model

### Analysis

ublinear attachme

Superlinear attachmen kernels Nutshell





In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .
- ightharpoonup For  $A_k = k$ , we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

### PoCS | @pocsvox

#### Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness Krapivisky & Redner's

### Analysis

Nutshell







In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .
- For  $A_k = k$ , we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

## PoCS | @pocsvox

Scale-free networks

## Scale-free networks

Main story Model details

Analysis

more plausi

mechanism

Krapivisky & Redner's

odel

Analysis

#### Universali

ublinear attachmen

Superlinear attachmen kernels Nutshell





In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where 
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .
- ightharpoonup For  $A_k = k$ , we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

 Detail: we are ignoring initial seed network's edges.

# PoCS | @pocsvox Scale-free

Scale-free networks

## Scale-free networks

Main story Model details

Analysis

mechanism

obustness

Krapivisky & Redner's model

eneralized model

### Analysis

blinear attachment

Superlinear attachmer kernels Nutshell





So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

$$n_k = \frac{1}{2!} [(k-1)n_{k+1}/-kn_k/] + \delta_k$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

#### Analysis Universality?

Nutshell









So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution:

$$=\frac{1}{2}[(k-1)n_{k+1}/-kn_k/]+\delta_{k1}$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details Analysis

Robustness

Krapivisky & Redner's

Analysis Universality?

Nutshell







So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .

$$=\frac{1}{2!}[(k-1)n_{k+1}/-kn_k/]+\delta_{k1}$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Analysis Universality?

Nutshell







So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .
- We replace  $dN_k/dt$  with  $dn_k t/dt = n_k$ .
- ▶ We arrive at a difference equation:

 $=\frac{1}{2l}[(k-1)n_{k+1}/-kn_k/]+\delta_{k1}$ 

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

echanism

Robustness

Krapivisky & Redner's model

eneralized model

### Analysis

ublinear attachmen ernels

Superlinear attachment kernels Nutshell







So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_kN_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .
- We replace  $dN_k/dt$  with  $dn_k t/dt = n_k$ .
- ▶ We arrive at a difference equation:

$$n_{\pmb{k}} = \frac{1}{2 \textcolor{red}{t}} \left[ (k-1) n_{\pmb{k}-1} \textcolor{red}{t} - k n_{\pmb{k}} \textcolor{red}{t} \right] + \delta_{\pmb{k} \pmb{1}}$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

### Analysis

Nutshell







### Outline

### Scale-free networks

Main story

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redners mode

Generalized model

Analysis

### Universality?

Sublinear attachment kernels Superlinear attachment kernels Nutshell

References

### PoCS | @pocsvox

#### Scale-free networks

### Scale-free networks

Main story

Model details

Analysis

more plausible

Robustness

Krapivisky & Redner's

model

Analysis

### Universality?

kernels kernels

Superlinear attachmen kernels

Nutshell







As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel  $A_k$ ?
- Again, we're asking if the result  $\gamma = 3$
- $\blacktriangleright$  KR's natural modification:  $A_k = k^{\nu}$  with  $\nu \neq$
- But we'll first explore a more subtle modification of A<sub>k</sub> made by Krapivsky/Redner
- Keep A<sub>k</sub> linear in k but tweak details
- ▶ Idea: Relax from  $A_k = k$  to  $A_k \sim k$  as  $k \to \infty$ .

### PoCS | @pocsvox

#### Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

undiyoto

echanism

Robustness

Krapivisky & Redner's

eneralized model

Analysis

Universality?

### Sublinear attachm

kernels

kernels

Nutshell





As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel  $A_h$ ?

PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel  $A_k$ ?
- ▶ Again, we're asking if the result  $\gamma = 3$  universal  $\checkmark$ ?
- $\blacktriangleright$  KR's natural modification:  $A_k = k^{\nu}$  with  $\nu$
- lacktriangle But we'll first explore a more subtle modification of  $A_k$  made by Krapivsky/Redner
- Keep A<sub>k</sub> linear in k but tweak details
- ▶ Idea: Relax from  $A_k = k$  to  $A_k \sim k$  as  $k \to \infty$ .

Main story

Model details

Analysis

more plausil

Robustness

Krapivisky & Redner's

odel

inalysis

Universality?

Sublinear attachm

Superlinear attachmo kernels





$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel  $A_h$ ?
- Again, we're asking if the result  $\gamma = 3$  universal  $\square$ ?
- ▶ KR's natural modification:  $A_k = k^{\nu}$  with  $\nu \neq 1$ .

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?





$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel  $A_h$ ?
- Again, we're asking if the result  $\gamma = 3$  universal  $\square$ ?
- ▶ KR's natural modification:  $A_k = k^{\nu}$  with  $\nu \neq 1$ .
- ▶ But we'll first explore a more subtle modification of  $A_k$  made by Krapivsky/Redner [4]

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel  $A_h$ ?
- Again, we're asking if the result  $\gamma = 3$  universal  $\square$ ?
- ▶ KR's natural modification:  $A_k = k^{\nu}$  with  $\nu \neq 1$ .
- ▶ But we'll first explore a more subtle modification of  $A_k$  made by Krapivsky/Redner [4]
- $\blacktriangleright$  Keep  $A_k$  linear in k but tweak details.

Main story Model details

Analysis

Krapivisky & Redner's

Universality?







$$N_k(t) = n_k(t)t \propto k^{-3}t$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel  $A_k$ ?
- ▶ Again, we're asking if the result  $\gamma = 3$  universal  $\checkmark$ ?
- ▶ KR's natural modification:  $A_k = k^{\nu}$  with  $\nu \neq 1$ .
- ▶ But we'll first explore a more subtle modification of  $A_k$  made by Krapivsky/Redner [4]
- ▶ Keep  $A_k$  linear in k but tweak details.
- ▶ Idea: Relax from  $A_k = k$  to  $A_k \sim k$  as  $k \to \infty$ .

Main story Model details

Analysis

A more plan

nechanism

Krapivisky & Redner's

oneralized model

seneralized model

Universality?

Sublinear attach

Superlinear attachr kernels

Nutshell





Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness Krapivisky & Redner's

Universality?









Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of  $A_k$ .

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of  $A_k$ .

- $\blacktriangleright$  We assume that  $A = \mu t$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

▶ We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of  $A_k$ .

- $\blacktriangleright$  We assume that  $A = \mu t$
- We'll find  $\mu$  later and make sure that our assumption is consistent.
- As before, also assume  $N_k(t) = n_k t$ .

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

more plausible

1.

Krapivisky & Redner's

del

neralized model

Analysis
Universality?

Cublinger

kernels

Superlinear attachment kernels







Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of  $A_k$ .

- $\blacktriangleright$  We assume that  $A = \mu t$
- $\blacktriangleright$  We'll find  $\mu$  later and make sure that our assumption is consistent.
- As before, also assume  $N_k(t) = n_k t$ .

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Krapivisky & Redner's

Universality?







ightharpoonup For  $A_k = k$  we had

$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

$$[n_k] = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_k n_k] + \delta_k$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

### Universality?

Nutshell







For  $A_k = k$  we had

$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

▶ This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k+1}n_{k-1} + \mu\delta_k$$

Again two cases:

### PoCS | @pocsvox

Scale-free networks

### Scale-free networks

networks Main story

Model details

Analysis

more plausible

Robustness

Krapivisky & Redner's

odel

Analysis

Universality?

Sublinear attachmen

Superlinear attachment

kernels Nutshell





ightharpoonup For  $A_k = k$  we had

$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

### Universality?







ightharpoonup For  $A_k = k$  we had

$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

Again two cases:

$$\frac{k}{k} = 1 : n_1 = \frac{\mu}{\mu + A_1};$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?









ightharpoonup For  $A_k = k$  we had

$$n_k = \frac{1}{2} \left[ (k-1) n_{k-1} - k n_k \right] + \delta_{k1}$$

This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu) n_k = A_{k-1} n_{k-1} + \mu \delta_{k1}$$

Again two cases:

$$\frac{k=1}{n_1} : n_1 = \frac{\mu}{\mu + A_1}; \qquad \frac{k}{n_k} > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}.$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?







- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto$$

### PoCS | @pocsvox

### Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Nutshell







- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
- ▶ For large k, we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu - 1}$$

# PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Nutshell







- ▶ Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
- $\blacktriangleright$  For large k, we find:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \propto k^{-\mu - 1}$$

 $\blacktriangleright$  Since  $\mu$  depends on  $A_k$ , details matter...

# PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Nutshell







- Now we need to find  $\mu$ .

# PoCS | @pocsvox

### Scale-free networks

#### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

# Universality?







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$

# PoCS | @pocsvox

### Scale-free networks

### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

### Universality?







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ightharpoonup Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$

# PoCS | @pocsvox

### Scale-free networks

#### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

### Universality?

Nutshell







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ightharpoonup Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

Scale-free networks

#### Scale-free networks

Main story

Model details Analysis

Robustness

Krapivisky & Redner's

Universality?

Nutshell







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

Scale-free networks

#### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

# Universality?







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for  $n_k$ :

$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{\cancel{A_k}} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \cancel{A_k}$$

# PoCS | @pocsvox

Scale-free networks

### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

### Universality?







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ightharpoonup Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{\mathcal{A}_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \mathcal{A}_k$$

# PoCS | @pocsvox

### Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

### Universality?







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{\mathcal{A}_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \mathcal{A}_k$$

- $\triangleright$  Closed form expression for  $\mu$ .

Scale-free networks

### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

# Universality?







- Now we need to find  $\mu$ .
- ▶ Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{\mathcal{A}_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \mathcal{A}_k$$

- $\triangleright$  Closed form expression for  $\mu$ .
- $\blacktriangleright$  We can solve for  $\mu$  in some cases.

Scale-free

networks

#### Scale-free networks

Main story

Model details Analysis

Robustness

Krapivisky & Redner's

### Universality?







- Now we need to find  $\mu$ .
- Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{\mathcal{A}_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \mathcal{A}_k$$

- $\triangleright$  Closed form expression for  $\mu$ .
- $\blacktriangleright$  We can solve for  $\mu$  in some cases.
- $\blacktriangleright$  Our assumption that  $A = \mu t$  looks to be not too horrible.

Scale-free

networks

Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?









- ▶ Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}$$

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

# PoCS | @pocsvox

Scale-free networks

### Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

## Universality?







- ▶ Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- ▶ Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ .
- Closed form expression for µ

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}$$

Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

▶ Craziness

# PoCS | @pocsvox

Scale-free networks

# Scale-free networks

networks Main story

Model details

Analysis

more plausi

mechanism

Robustness

Krapivisky & Redner's

Generalized mode

Analysis

### Universality?

Sublinear attachme

## kernels

Superlinear attachment kernels

Nutshell







- ▶ Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for k > 2.
- ▶ Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ .
- $\triangleright$  Closed form expression for  $\mu$ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

# #mathisfun

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}$$

# PoCS | @pocsvox

Scale-free networks

### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

# Universality?









- ▶ Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- $\blacktriangleright$  Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ .
- ▶ Closed form expression for  $\mu$ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

# #mathisfun

 $\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}$ 

# PoCS | @pocsvox

### Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?





- ▶ Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ .
- ▶ Closed form expression for  $\mu$ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

# #mathisfun

•

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

▶ Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

Craziness

# PoCS | @pocsvox

Scale-free networks

# Scale-free networks

Main story

Model details

Analysis

more plan

mechanism

Robustness

Krapivisky & Redner's model

Generalized mode

nalysis

### Universality?

ernels

Superlinear attachmen kernels





- ▶ Consider tunable  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- ▶ Again, we can find  $\gamma = \mu + 1$  by finding  $\mu$ .
- ▶ Closed form expression for  $\mu$ :

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

# #mathisfun

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

▶ Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

► Craziness...

# PoCS | @pocsvox

### Scale-free networks

# Scale-free networks

Main story

Model details

Analysis

more plan

mechanism

Robustness

Krapivisky & Redner's model

eneralized model

Analysis

# Universality?

kernels Superlinear attachme

ernels lutshell





# Outline

# Scale-free networks

Main story

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's mode

Generalized mode

Analysis

Universality

# Sublinear attachment kernels

Superlinear attachment kernels Nutshell

References

## PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

more plausible

nechanism

Robustness Krapivisky & Redner's

model

Analysis

Universality?

Sublinear attachment kerne

#### Superlinear attachment

kernels

ivucsilei







► Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

General finding by Krapivsky and Redner

 $n_k \sim k^{u} e^{-c_1 k^{1u} + {
m correction terms}}$ 

- Stretched exponentials (truncated power laws
- aka Welbull distributions.
- Universality: now details of kernel do not matter.
- ightharpoonup Distribution of degree is universal providing u < 1.

PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

echanism

Robustness

Krapivisky & Redner's model

Generalized model

Analysis

Universality?

### Sublinear attachment kerne

kernels

Nutshell





► Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

▶ General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + {\rm correction\ terms}}.$$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Sublinear attachment kerne







► Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

▶ General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + {\rm correction\ terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing  $\nu < 1$ .

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details

Analysis

echanism

Robustness

Krapivisky & Redner's model

Generalized model

Universality?

Sublinear attachment kern

Superlinear attachment

kernels Nutshell

Nutshell







► Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + {\rm correction\ terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- Distribution of degree is universal providing  $\nu < 1$

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Model details Analysis

A more plausib

echanism

Robustness

Krapivisky & Redner's model

Generalized model

Universality?

Sublinear attachment kern

Superlinear attachment

Nutshell







► Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + {\rm correction\ terms}}$$
 .

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- ▶ Universality: now details of kernel do not matter.
- Distribution of degree is universal providing  $\nu$ .

PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story

Analysis

A more plausible

echanism

Robustness Krapivisky & Redner's

nodel

Analysis

Universality?

Sublinear attachment kern

kernels

ivacsileir





Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

▶ General finding by Krapivsky and Redner: [4]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + {\rm correction \; terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing  $\nu$  < 1.

Scale-free

networks

Scale-free networks

Main story

Model details Analysis

Krapivisky & Redner's

Sublinear attachment kerr





# Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1+\mu} + \frac{\mu^2}{2} \frac{k^{1-2}}{1+2i}}$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Sublinear attachment kerne







# Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

▶ For  $1/3 < \nu < 1/2$ :

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Sublinear attachment kerne









# Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

▶ For  $1/3 < \nu < 1/2$ :

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

▶ And for  $1/(r+1) < \nu < 1/r$ , we have r pieces in exponential.

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

# Sublinear attachment kerne







# Outline

# Scale-free networks

Superlinear attachment kernels

### PoCS | @pocsvox

### Scale-free networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Superlinear attachment ker







▶ Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

### PoCS | @pocsvox

Scale-free networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Superlinear attachment ker







▶ Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

- Now a winner-take-all mechanism.

### PoCS | @pocsvox

Scale-free networks

#### Scale-free networks

Main story

Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Superlinear attachment ker









► Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- For  $\nu > 2$ , all but a finite # of nodes connect to one node.

### PoCS | @pocsvox

Scale-free networks

# Scale-free networks

Main story

Model details

Analysis

more plausible

Robustness

Krapivisky & Redner's

illouei

eneralized model

Analysis

Sublinear attachme

kernels

Superlinear attachment ker Nutshell





► Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- For  $\nu > 2$ , all but a finite # of nodes connect to one node.

### PoCS | @pocsvox

Scale-free networks

# Scale-free networks

Main story

Model details

Analysis

more plausible

Robustness

Krapivisky & Redner's

moder

delieralized model

Universality?

Sublinear attachmen

kernels Superlinear attachment ker

Nutshell





# Outline

# Scale-free networks

Main story

Model details

Analysis

A more plausible mechanism

Robustness

Krapivisky & Redner's model

Generalized mode

Analysis

Universality?

Sublinear attachment kernels

Nutshell

References

# PoCS | @pocsvox

### Scale-free networks

Scale-free networks

Main story

Model details Analysis

more plausible

echanism

Robustness

Krapivisky & Redner's

model

Analysis

Universality?

kernels

ernels

Nutshell

References







# Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.

PoCS | @pocsvox Scale-free

networks

Scale-free networks

Main story Model details

Analysis

Robustness

Krapivisky & Redner's

Universality?

Nutshell







# Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.

### PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

Krapivisky & Redner's

Universality?

kernels

Nutshell









# Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  - 1. Description: Characterizing very large networks
  - 2. Explanation: Micro story ⇒ Macro features

Scale-free networks

Scale-free networks

> Main story Model details

Analysis

Krapivisky & Redner's

Nutshell







# Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- ▶ But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- ▶ Two main areas of focus:
  - 1. Description: Characterizing very large networks
  - 2. Explanation: Micro story ⇒ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics...! Hexcrement

PoCS | @pocsvox Scale-free networks

Scale-free networks

Main story Model details

Analysis

more plansi

nechanism

Robustness

Krapivisky & Redner's

iodel

Seneralized mode

Universality?

sublinear attachm

uperlinear attachmer ernels

Nutshell





# Scale-free networks

# Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- ▶ But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- ► Two main areas of focus:
  - 1. Description: Characterizing very large networks
  - 2. Explanation: Micro story ⇒ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- ➤ Still much work to be done, especially with respect to dynamics...

### Scale-free networks

Main story Model details

Analysis

A more plausi

Poblistness

Krapivisky & Redner's

Generalized mode

nalysis

Sublinear attachn

Superlinear attachme kernels

Nutshell







# Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  - 1. Description: Characterizing very large networks
  - 2. Explanation: Micro story ⇒ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics... #excitement

PoCS | @pocsvox
Scale-free
networks

Scale-free networks

Main story Model details

Analysis

A more plaus

mechanism

Krapivisky & Redner's

Generalized model

nalysis

Universality?

kernels Superlinear attachme

Nutshell







# Neural reboot (NR):

Turning the corner:

### PoCS | @pocsvox

#### Scale-free networks

#### Scale-free networks

Main story

Model details Analysis

A more plausible

Robustness

Krapivisky & Redner's

Analysis

Universality? Sublinear attachment

kernels Superlinear attachment

kernels

### Nutshell











# References I

- [1] R. Albert, H. Jeong, and A.-L. Barabási.

  Error and attack tolerance of complex networks.

  Nature, 406:378–382, 2000. pdf
- [2] A.-L. Barabási and R. Albert.

  Emergence of scaling in random networks.

  Science, 286:509–511, 1999. pdf
- [3] J. Doyle, D. Alderson, L. Li, S. Low, M. Roughan, S. S., R. Tanaka, and W. Willinger.
  The "Robust yet Fragile" nature of the Internet.
  Proc. Natl. Acad. Sci., 2005:14497–14502, 2005.
  pdf
- [4] P. L. Krapivsky and S. Redner.
  Organization of growing random networks.
  Phys. Rev. E, 63:066123, 2001. pdf

PoCS | @pocsvox
Scale-free
networks

Scale-free
networks
Main story
Model details
Analysis
A more plausible
mechanism
Robustness
Krapivisty & Redners
mödel
Generalized model
Analysis
Universality?
Sublinear attachment
kernels
Superinear attachment
kernels



