

Random Networks

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2015 | #FallPoCS2015

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Random
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Sealie & Lambie Productions



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Outline

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Some important models:

1. Generalized random networks;
2. Small-world networks;
3. Generalized affiliation networks;
4. Scale-free networks;
5. Statistical generative models (p^*).

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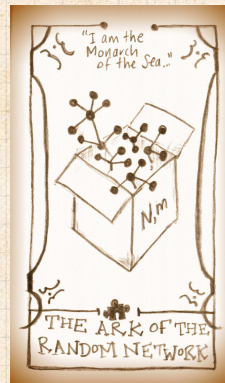
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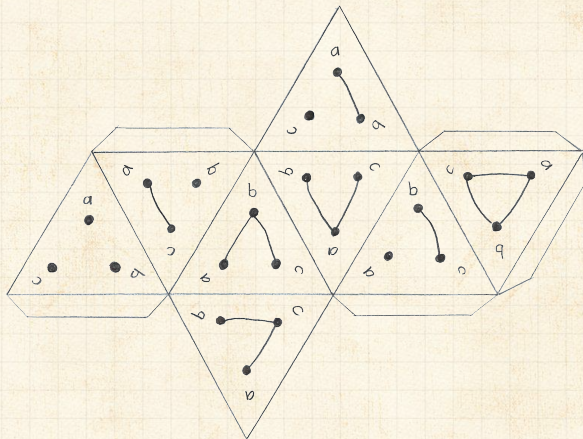
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Random network generator for $N = 3$:



- Get your own exciting generator [here](#)
- As $N \nearrow$, polyhedral die rapidly becomes a ball...

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Pure, abstract random networks:

- ▶ Consider set of all networks with N labelled nodes and m edges.
- ▶ Standard random network = one **randomly chosen** network from this set.
- ▶ To be clear: each network is **equally** probable.
- ▶ Sometimes equiprobability is a good assumption, but it is always an assumption.
- ▶ Known as Erdős-Rényi random networks or **ER graphs**.

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Random networks—basic features:

- ▶ Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

- ▶ Limit of $m = 0$: empty graph.
- ▶ Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- ▶ Number of possible networks with N labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N^2}.$$

- ▶ Given m edges, there are $\binom{\binom{N}{2}}{m}$ different possible networks.
- ▶ Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- ▶ **Real world**: links are usually costly so real networks are almost always **sparse**.



How to build standard random networks:

- ▶ Given N and m .
 - ▶ Two probabilistic methods (we'll see a third later on)
1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p .
 - ▶ **Useful for theoretical work.**
 2. Take N nodes and add exactly m links by selecting edges without replacement.
 - ▶ **Algorithm:** Randomly choose a pair of nodes i and j , $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
 - ▶ Best for adding relatively small numbers of links (most cases).
 - ▶ 1 and 2 are effectively equivalent for large N .

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A few more things:

- For method 1, # links is probabilistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

- So the expected or **average degree** is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{2}{N} p \frac{1}{2} \cancel{N} (N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \rightarrow 0$ as $N \rightarrow \infty$.



Random networks: examples

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Next slides:

Example realizations of random networks

- ▶ $N = 500$
- ▶ Vary m , the number of edges from 100 to 1000.
- ▶ Average degree $\langle k \rangle$ runs from 0.4 to 4.
- ▶ Look at full network plus the largest component.

Random networks: examples for $N=500$

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$m = 100$
 $\langle k \rangle = 0.4$



$m = 200$
 $\langle k \rangle = 0.8$



$m = 230$
 $\langle k \rangle = 0.92$



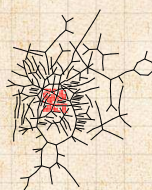
$m = 240$
 $\langle k \rangle = 0.96$



$m = 250$
 $\langle k \rangle = 1$



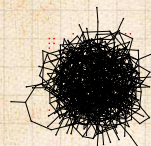
$m = 260$
 $\langle k \rangle = 1.04$



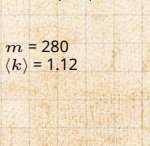
$m = 280$
 $\langle k \rangle = 1.12$



$m = 300$
 $\langle k \rangle = 1.2$



$m = 500$
 $\langle k \rangle = 2$



$m = 1000$
 $\langle k \rangle = 4$

Random networks: largest components

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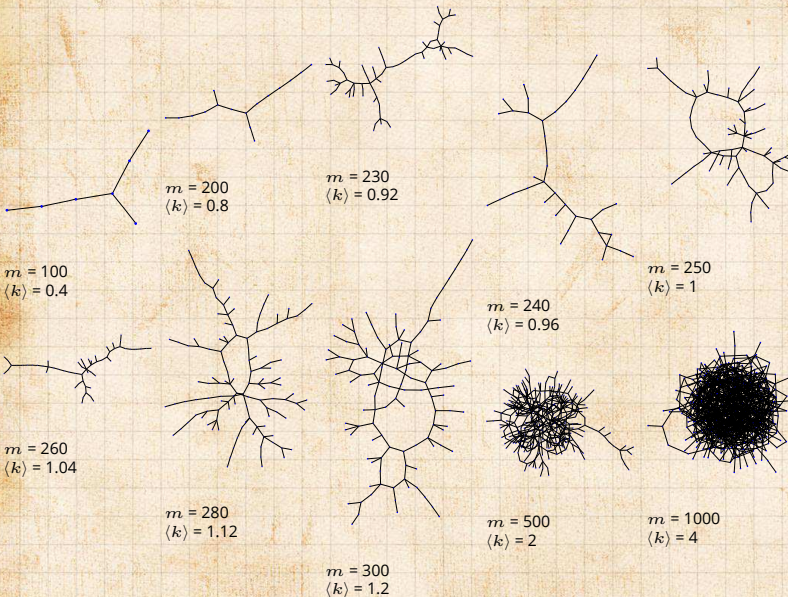
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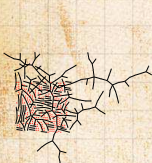
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$m = 250$
 $\langle k \rangle = 1$



$m = 250$
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$m = 250$
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$m = 250$
 $\langle k \rangle = 1$



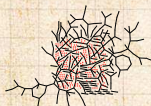
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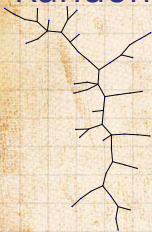
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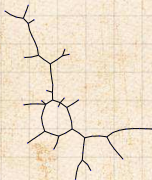
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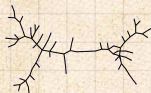
References



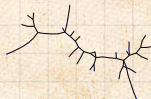
$m = 250$
 $\langle k \rangle = 1$



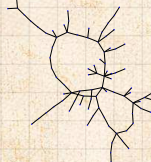
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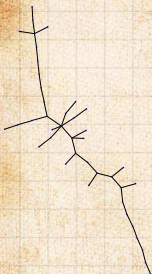
$m = 250$
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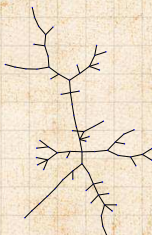
$m = 250$
 $\langle k \rangle = 1$



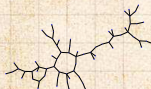
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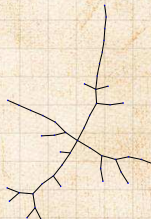
$m = 250$
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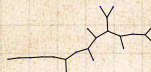
$m = 250$
 $\langle k \rangle = 1$



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 $\langle k \rangle = 1$

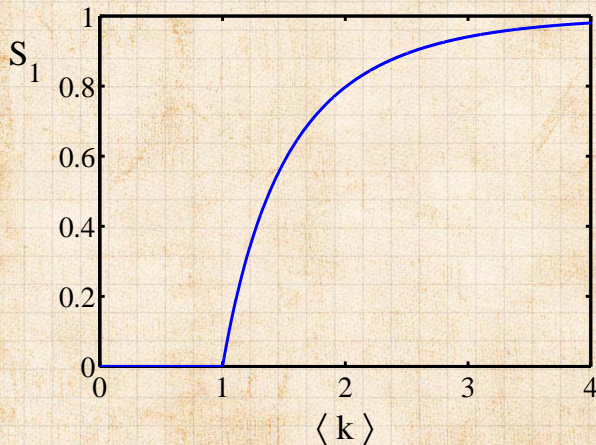
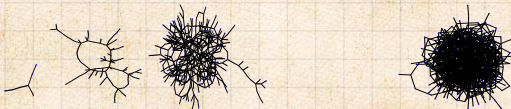


$m = 250$
 $\langle k \rangle = 1$

Giant component

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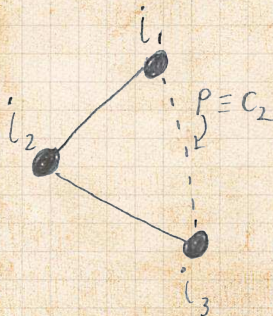
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Clustering in random networks:

- ▶ For construction method 1, what is the clustering coefficient for a finite network?
- ▶ Consider triangle/triple clustering coefficient: [5]

$$C_2 = \frac{3 \times \text{\#triangles}}{\text{\#triples}}$$



- ▶ Recall: C_2 = probability that two friends of a node are also friends.
- ▶ Or: C_2 = probability that a triple is part of a triangle.
- ▶ For standard random networks, we have simply that

$$C_2 = p.$$



Clustering in random networks:



- ▶ So for large random networks ($N \rightarrow \infty$), clustering drops to zero.
- ▶ Key structural feature of random networks is that they locally look like **pure branching networks**
- ▶ No small loops.

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Degree distribution:

- ▶ Recall P_k = probability that a randomly selected node has degree k .
- ▶ Consider method 1 for constructing random networks: each possible link is realized with probability p .
- ▶ Now consider one node: there are ' $N - 1$ choose k ' ways the node can be connected to k of the other $N - 1$ nodes.
- ▶ Each connection occurs with probability p , each non-connection with probability $(1 - p)$.
- ▶ Therefore have a binomial distribution :

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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Limiting form of $P(k; p, N)$:

- ▶ Our degree distribution:
 $P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
- ▶ What happens as $N \rightarrow \infty$?
- ▶ We must end up with the normal distribution right?
- ▶ If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \rightarrow \infty$.
- ▶ But we want to keep $\langle k \rangle$ fixed...
- ▶ So examine limit of $P(k; p, N)$ when $p \rightarrow 0$ and $N \rightarrow \infty$ with $\langle k \rangle = p(N-1) = \text{constant}$.

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k} \rightarrow \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

- ▶ This is a Poisson distribution  with mean $\langle k \rangle$.

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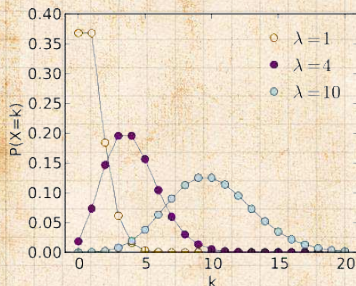
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Poisson basics:

$$P(k; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$



- ▶ $\lambda > 0$
- ▶ $k = 0, 1, 2, 3, \dots$
- ▶ Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- ▶ e.g.:
phone calls/minute,
horse-kick deaths.
- ▶ 'Law of small numbers'

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- Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$

- Checking:

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!}$$

$$= e^{-\langle k \rangle} e^{\langle k \rangle} = 1$$

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Poisson basics:

- Mean degree: we must have

$$\langle k \rangle = \sum_{k=0}^{\infty} k P(k; \langle k \rangle).$$

- Checking:

$$\begin{aligned} \sum_{k=0}^{\infty} k P(k; \langle k \rangle) &= \sum_{k=0}^{\infty} k \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \\ &= e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^k}{(\underline{k-1})!} \\ &= \underline{\langle k \rangle} e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^{k-1}}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{i=0}^{\infty} \frac{\langle k \rangle^i}{i!} = \langle k \rangle e^{-\langle k \rangle} e^{\langle k \rangle} = \langle k \rangle \end{aligned}$$

- In CoNKs, we get to a better and crazier way of doing this...

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Poisson basics:

- ▶ The **variance** of degree distributions for random networks turns out to be **very important**.
- ▶ Using calculation similar to one for finding $\langle k \rangle$ we find the **second moment** to be:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- ▶ Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

- ▶ So standard deviation σ is equal to $\sqrt{\langle k \rangle}$.
- ▶ Note: This is a special property of Poisson distribution and can trip us up...

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General random networks

- ▶ So... standard random networks have a Poisson degree distribution
- ▶ Generalize to arbitrary degree distribution P_k .
- ▶ Also known as the **configuration model**.^[5]
- ▶ Can generalize construction method from ER random networks.
- ▶ Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- ▶ But we'll be more interested in
 1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
 2. Examining mechanisms that lead to networks with certain degree distributions.

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Coming up:

Example realizations of random networks with power law degree distributions:

- ▶ $N = 1000$.
- ▶ $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- ▶ Set $P_0 = 0$ (no isolated nodes).
- ▶ Vary exponent γ between 2.10 and 2.91.
- ▶ Again, look at full network plus the largest component.
- ▶ Apart from degree distribution, wiring is random.

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Random networks: examples for $N=1000$

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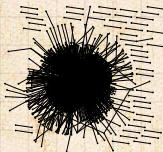
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$\gamma = 2.1$
 $\langle k \rangle = 3.448$



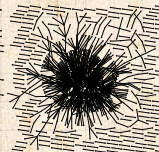
$\gamma = 2.19$
 $\langle k \rangle = 2.986$



$\gamma = 2.28$
 $\langle k \rangle = 2.306$



$\gamma = 2.37$
 $\langle k \rangle = 2.504$



$\gamma = 2.46$
 $\langle k \rangle = 1.856$



$\gamma = 2.55$
 $\langle k \rangle = 1.712$



$\gamma = 2.64$
 $\langle k \rangle = 1.6$



$\gamma = 2.73$
 $\langle k \rangle = 1.862$



$\gamma = 2.82$
 $\langle k \rangle = 1.386$



$\gamma = 2.91$
 $\langle k \rangle = 1.49$

Random networks: largest components

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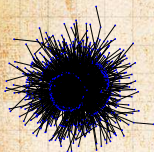
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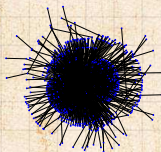
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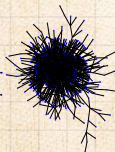
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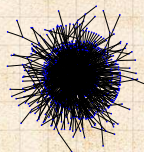
$\gamma = 2.1$
 $\langle k \rangle = 3.448$



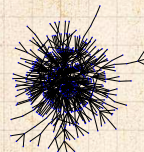
$\gamma = 2.19$
 $\langle k \rangle = 2.986$



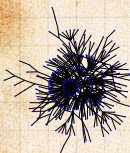
$\gamma = 2.28$
 $\langle k \rangle = 2.306$



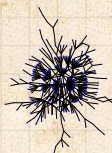
$\gamma = 2.37$
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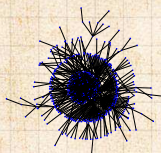
$\gamma = 2.46$
 $\langle k \rangle = 1.856$



$\gamma = 2.55$
 $\langle k \rangle = 1.712$



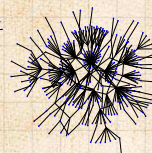
$\gamma = 2.64$
 $\langle k \rangle = 1.6$



$\gamma = 2.73$
 $\langle k \rangle = 1.862$



$\gamma = 2.82$
 $\langle k \rangle = 1.386$



$\gamma = 2.91$
 $\langle k \rangle = 1.49$

Generalized random networks:

- ▶ Arbitrary degree distribution P_k .
- ▶ Create (unconnected) nodes with degrees sampled from P_k .
- ▶ Wire nodes together randomly.
- ▶ Create ensemble to test deviations from randomness.

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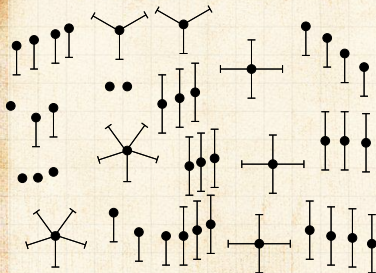
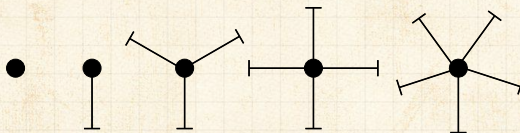
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Building random networks: Stubs

Phase 1:

- **Idea:** start with a soup of unconnected nodes with **stubs** (half-edges):



- Randomly select stubs (not nodes!) and connect them.
- Must have an even number of stubs.
- Initially allow **self-** and **repeat** connections.

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Building random networks: First rewiring

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Random Networks

Phase 2:

- ▶ Now find any (A) self-loops and (B) repeat edges and **randomly rewire** them.



- ▶ **Being careful:** we can't change the degree of any node, so we can't simply move links around.
- ▶ **Simplest solution:** randomly rewire **two edges** at a time.

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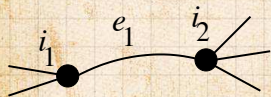
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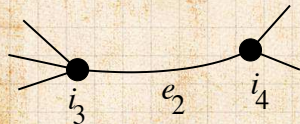
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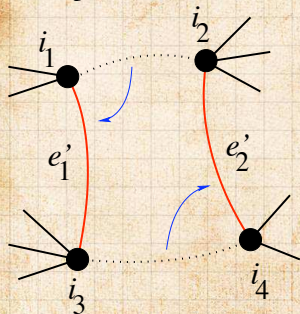
General random rewiring algorithm



- ▶ Randomly choose **two edges**.
(Or choose problem edge and a random edge)



- ▶ Check to make sure edges are **disjoint**.



- ▶ Rewire one end of each edge.
- ▶ Node degrees **do not change**.
- ▶ Works if e_1 is a self-loop or repeated edge.
- ▶ Same as finding on/off/on/off 4-cycles. and rotating them.

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Sampling random networks

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Random
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Phase 2:

- Use rewiring algorithm to remove all self and repeat loops.

Phase 3:

- **Randomize network** wiring by applying rewiring algorithm liberally.
- **Rule of thumb:** # Rewirings $\simeq 10 \times \# \text{ edges}$ [3].

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The edge-degree distribution:

- ▶ The degree distribution P_k is fundamental for our description of many complex networks
- ▶ Again: P_k is the degree of **randomly chosen node**.
- ▶ A second very important distribution arises from **choosing randomly on edges** rather than on nodes.
- ▶ Define Q_k to be the probability the node at a **random end** of a **randomly chosen edge** has degree k .
- ▶ Now choosing nodes based on their degree (i.e., size):

$$Q_k \propto kP_k$$

- ▶ Normalized form:

$$Q_k = \frac{kP_k}{\sum_{k'=0}^{\infty} k'P_{k'}} = \frac{kP_k}{\langle k \rangle}.$$



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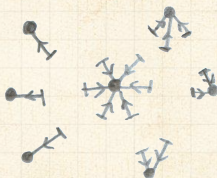
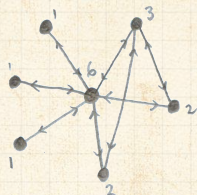
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- ▶ Probability of randomly selecting a node of degree k by choosing from nodes:
 $P_1 = 3/7$, $P_2 = 2/7$, $P_3 = 1/7$,
 $P_6 = 1/7$.

- ▶ Probability of landing on a node of degree k after randomly selecting an edge and then randomly choosing one direction to travel:
 $Q_1 = 3/16$, $Q_2 = 4/16$,
 $Q_3 = 3/16$, $Q_6 = 6/16$.

- ▶ Probability of finding # outgoing edges = k after randomly selecting an edge and then randomly choosing one direction to travel:
 $R_0 = 3/16$, $R_1 = 4/16$,
 $R_2 = 3/16$, $R_5 = 6/16$.



The edge-degree distribution:

- ▶ For random networks, Q_k is also the probability that a friend (neighbor) of a random node has k friends.
- ▶ Useful variant on Q_k :

R_k = probability that a friend of a random node has k other friends.

$$R_k = \frac{(k+1)P_{k+1}}{\sum_{k'=0} (k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

- ▶ Equivalent to friend having degree $k+1$.
- ▶ **Natural question:** what's the expected number of other friends that one friend has?

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The edge-degree distribution:

- Given R_k is the probability that a friend has k other friends, then the average number of **friends' other friends** is

$$\begin{aligned}\langle k \rangle_R &= \sum_{k=0}^{\infty} k R_k = \sum_{k=0}^{\infty} k \frac{(k+1) P_{k+1}}{\langle k \rangle} \\&= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k(k+1) P_{k+1} \\&= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} ((k+1)^2 - (k+1)) P_{k+1}\end{aligned}$$

(where we have sneakily matched up indices)

$$\begin{aligned}&= \frac{1}{\langle k \rangle} \sum_{j=0}^{\infty} (j^2 - j) P_j \quad (\text{using } j = k+1) \\&= \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle)\end{aligned}$$

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The edge-degree distribution:

- Note: our result, $\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle)$, is true for **all** random networks, **independent of degree distribution**.

- For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k \rangle^2 + \langle k \rangle - \langle k \rangle) = \langle k \rangle$$

- Again, neatness of results is a special property of the Poisson distribution.
- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...



The edge-degree distribution:

- ▶ In fact, R_k is rather special for pure random networks ...
- ▶ Substituting

$$P_k = \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

into

$$R_k = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

we have

$$\begin{aligned} R_k &= \frac{(k+1)}{\langle k \rangle} \frac{\langle k \rangle^{(k+1)}}{(k+1)!} e^{-\langle k \rangle} = \frac{\cancel{(k+1)}}{\langle k \rangle} \frac{\langle k \rangle^{(k+1)}}{\cancel{(k+1)}k!} e^{-\langle k \rangle} \\ &= \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \equiv P_k. \end{aligned}$$

- ▶ #samesies.



Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle) = \langle k^2 \rangle - \langle k \rangle.$$

- Key: Average depends on the **1st and 2nd moments** of P_k and **not just the 1st moment**.
- Three peculiarities:
 1. We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
 2. If P_k has a **large second moment**, then $\langle k_2 \rangle$ will be big.
(e.g., in the case of a power-law distribution)
 3. Your friends really are different from you... [2, 4]
 4. See also: class size paradoxes (nod to: Gelman)

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Two reasons why this matters

More on peculiarity #3:

- ▶ A node's average # of friends: $\langle k \rangle$
- ▶ Friend's average # of friends: $\frac{\langle k^2 \rangle}{\langle k \rangle}$
- ▶ Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2} \right) \geq \langle k \rangle$$

- ▶ So only if everyone has the same degree (variance = $\sigma^2 = 0$) can a node be the same as its friends.
- ▶ Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.

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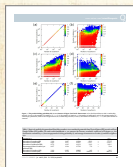
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"Generalized friendship paradox in complex networks: The case of scientific collaboration"

Eom and Jo,
Nature Scientific Reports, **4**, 4603, 2014. ^[1]

Your friends really are ~~monsters~~ #winners:¹

- ▶ **Go on, hurt me:** Friends have more coauthors, citations, and publications.
- ▶ **Other horrific studies:** your connections on Twitter have more followers than you, your sexual partners more partners than you, ...
- ▶ **The hope:** Maybe they have more enemies and diseases too.

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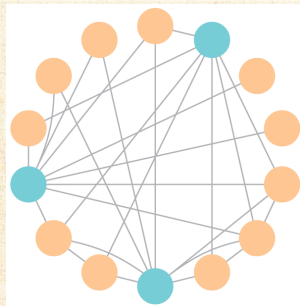
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¹Some press [here](#)  [MIT Tech Review].

Related disappointment:



- ▶ Nodes see their friends' color choices.
- ▶ Which color is more popular?¹
- ▶ Again: thinking in edge space changes everything.

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¹<https://www.washingtonpost.com/graphics/business/wonkblog/majority-illusion/>

Two reasons why this matters

(Big) Reason #2:

- ▶ $\langle k \rangle_R$ is **key** to understanding how well random networks are connected together.
- ▶ e.g., we'd like to know what's the size of the largest component within a network.
- ▶ As $N \rightarrow \infty$, does our network have a **giant component**?
- ▶ **Defn:** Component = connected subnetwork of nodes such that \exists path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
- ▶ **Defn:** Giant component = component that comprises a non-zero fraction of a network as $N \rightarrow \infty$.
- ▶ Note: Component = Cluster

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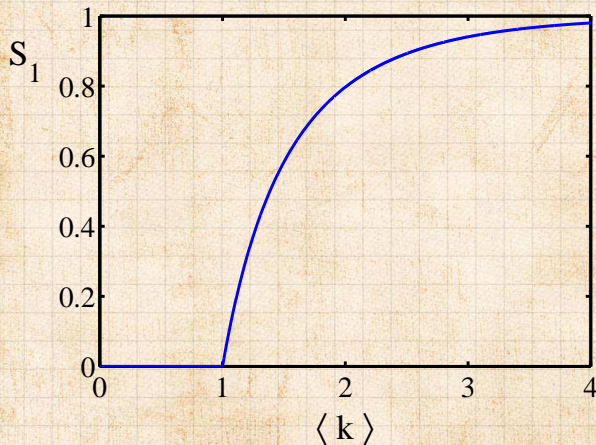
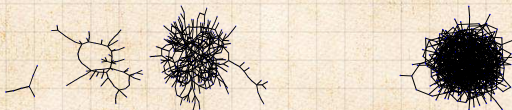
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Giant component

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Giant component for standard random networks:

- ▶ Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- ▶ Determine condition for giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle - \langle k \rangle}{\langle k \rangle} = \langle k \rangle$$

- ▶ Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
- ▶ When $\langle k \rangle < 1$, all components are finite.
- ▶ Fine example of a continuous phase transition .
- ▶ We say $\langle k \rangle = 1$ marks the critical point of the system.

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Random networks with skewed P_k :

- ▶ e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$, $k \geq 1$, then

$$\langle k^2 \rangle = c \sum_{k=1}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=1}^{\infty} x^{2-\gamma} dx$$

$$\propto x^{3-\gamma} \Big|_{x=1}^{\infty} = \infty \quad (\gg \langle k \rangle).$$

- ▶ So giant component **always exists** for these kinds of networks.
- ▶ Cutoff scaling is k^{-3} : if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.
- ▶ How about $P_k = \delta_{kk_0}$?
- ▶ CoNKs: We figure out the final size and complete dynamics.



Neural reboot (NR):

Falling maple leaf

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References I

- [1] Y.-H. Eom and H.-H. Jo.
Generalized friendship paradox in complex networks: The case of scientific collaboration.
[Nature Scientific Reports, 4:4603, 2014. pdf](#) 
- [2] S. L. Feld.
Why your friends have more friends than you do.
[Am. J. of Sociol., 96:1464–1477, 1991. pdf](#) 
- [3] R. Milo, N. Kashtan, S. Itzkovitz, M. E. J. Newman,
and U. Alon.
On the uniform generation of random graphs with
prescribed degree sequences, 2003. [pdf](#) 
- [4] M. E. J. Newman.
Ego-centered networks and the ripple effect.,
[Social Networks, 25:83–95, 2003. pdf](#) 

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[5] M. E. J. Newman.

The structure and function of complex networks.

SIAM Rev., 45(2):167–256, 2003. pdf 

References

