# Mechanisms for Generating Power-Law Size Distributions, Part 2

## Principles of Complex Systems | @pocsvox
## CSYS/MATH 300, Fall, 2015 | #FallPoCS2015

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
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What's the Story?

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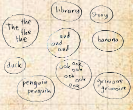
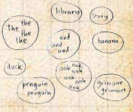


# Outline



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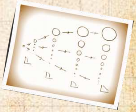

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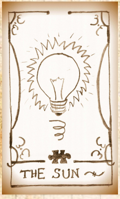






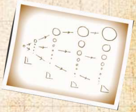

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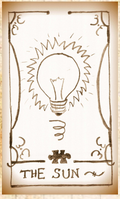


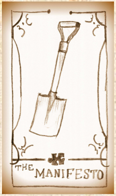






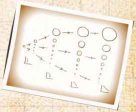

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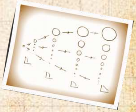

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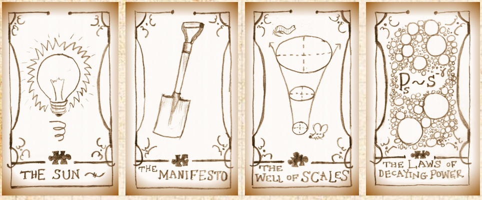
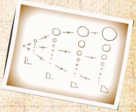


THE SUN
THE MANIFESTO
THE WELL OF SCALES
Ps ~ s-γ
THE LAWS OF DECAYING POWER



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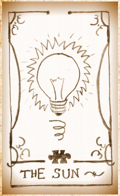
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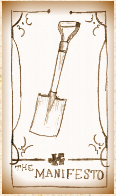
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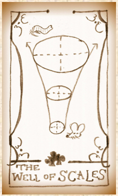
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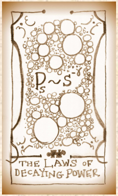
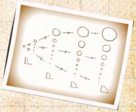

THE LAWS OF DECAYING POWER


MINISTRY OF RANDOM WALKS





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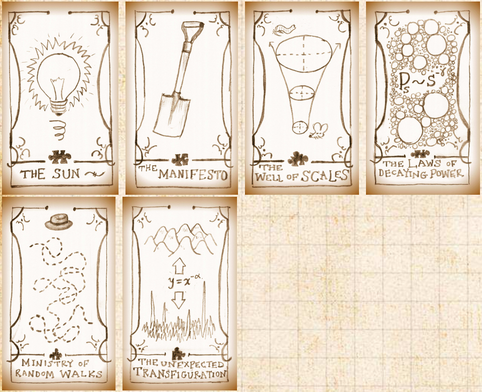
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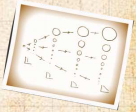
y=x^-α
THE UNEXPECTED TRANSFIGURATION







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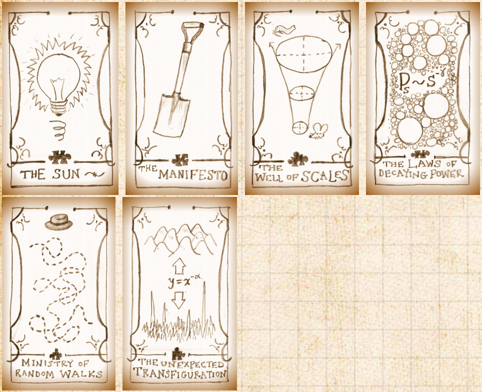
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$P_s \sim s^{-\gamma}$
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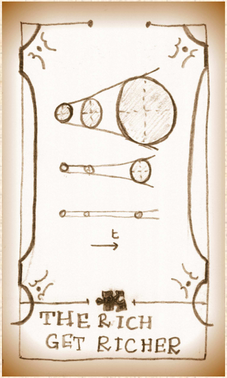
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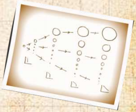
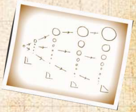


# Aggregation:

- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
- Competing mechanisms (trickiness)





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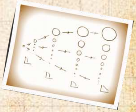

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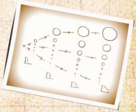

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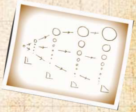

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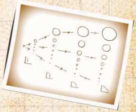

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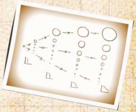
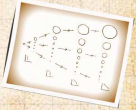


# Pre-Zipf's law observations of Zipf's law

- 1910s: Word frequency examined re Stenography (or shorthand or brachygraphy or tachygraphy), Jean-Baptiste Estoup [10].
- 1910s: Felix Auerbach pointed out the Zipfitude of city sizes in "Das Gesetz der Bevölkerungskonzentration" ("The Law of Population Concentration") [1].
- 1924: G. Udny Yule [27]:
  # Species per Genus
- 1926: Lotka [14]:
  # Scientific papers per author (Lotka's law)







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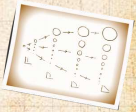

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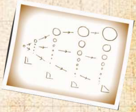

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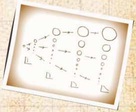

# Theoretical Work of Yore:

- 1953: Mandelbrot [16]:
  Optimality argument for Zipf's law; focus on language.
- 1955: Herbert Simon [23, 29]:
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- 1965/1976: Derek de Solla Price [7, 8]:
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- 1999: Barabasi and Albert [2]:
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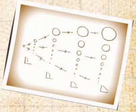

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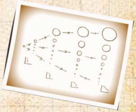

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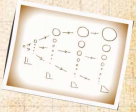

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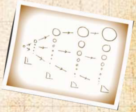

## Herbert Simon (1916–2001):

- Political scientist (and much more)
- Involved in Cognitive Psychology, Computer Science, Public Administration, Economics, Management, Sociology
- Coined ‘bounded rationality’ and ‘satisficing’
- Nearly 1000 publications (see Google Scholar)
- An early leader in Artificial Intelligence, Information Processing, Decision-Making, Problem-Solving, Attention Economics, Organization Theory, Complex Systems, And Computer Simulation Of Scientific Discovery.
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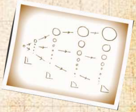

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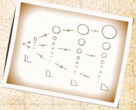

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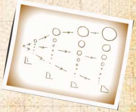

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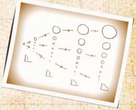

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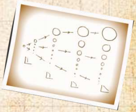

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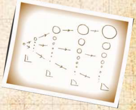


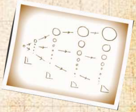
# Essential Extract of a Growth Model:



## Random Competitive Replication (RCR):

1. Start with 1 elephant (or element) of a particular flavor at $t = 1$
2. At time $t = 2, 3, 4, \ldots$, add a new elephant in one of two ways:
   - With probability $\rho$, create a new elephant with a new flavor
   - With probability $1 - \rho$, randomly choose from all existing elephants, and make a copy.
   - Elephants of the same flavor form a group





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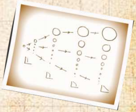

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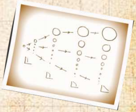

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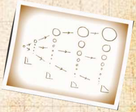

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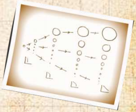

## Random Competitive Replication (RCR):

1. Start with 1 elephant (or element) of a particular flavor at $t = 1$
2. At time $t = 2, 3, 4, \ldots$, add a new elephant in one of two ways:
   - With probability $\rho$, create a new elephant with a new flavor
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   - Elephants of the same flavor form a group

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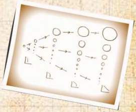

# Random Competitive Replication:



## Example: Words appearing in a language

- Consider words as they appear sequentially.
- With probability $\rho$, the next word has not previously appeared
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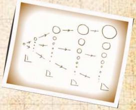
Note: This is a terrible way to write a novel.





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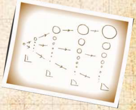

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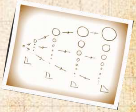

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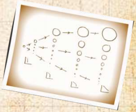

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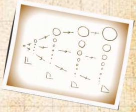

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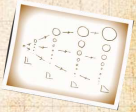

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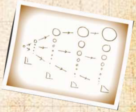

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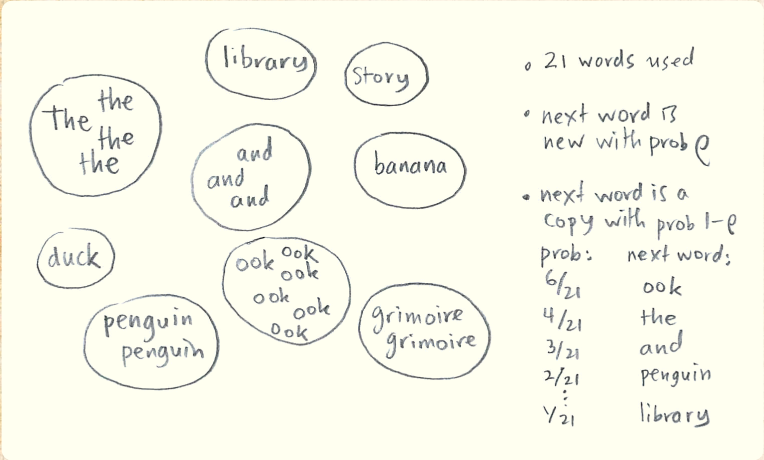


the the the the
library
story
and and and
banana
duck
ook ook ook ook ook ook
penguin penguin
grimoire grimoire
• 21 words used
• next word is new with prob ρ
• next word is a copy with prob 1−ρ
prob: next word:
6/21 ook
4/21 the
3/21 and
2/21 penguin
⋮
1/21 library

## Some observations:

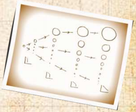
- Fundamental Rich-get-Richer story;
- Competition for replication between individual elephants is random;
- Competition for growth between groups of matching elephants is not random;
- Selection on groups is biased by size;
- Random selection sounds easy;
- Possible that no great knowledge of system needed (but more later ...).

## Your free set of tofu knives:

- Related to Pólya's Urn Model, a special case of problems involving urns and colored balls.
- Sampling with super-duper replacement and sneaky sneaking in of new colors.







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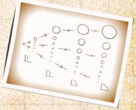
Rich-Get-Richer Mechanism
Simon's Model
Analysis
Words
Catchphrases
Optimization
Minimal Cost
Mandelbrot vs. Simon
Assumptions
Model
Analysis
And the winner is...?
Nutshell
Extra
References



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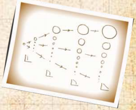

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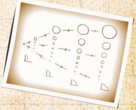

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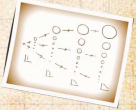

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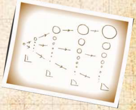

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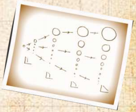

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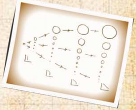

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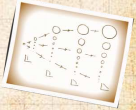

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## Some observations:

- Steady growth of system: +1 elephant per unit time.
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  2. Elephants moving between groups
  3. Variable innovation rate $\rho$
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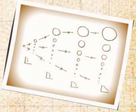

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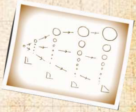

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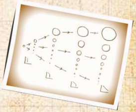

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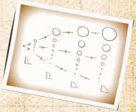

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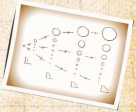

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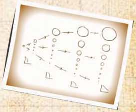

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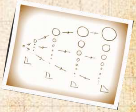

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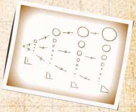

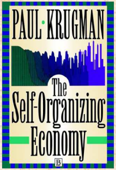


"The Self-Organizing Economy" by Paul Krugman (1996). [13]

### Ch. 3: An Urban Mystery, p. 46

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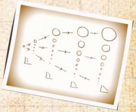

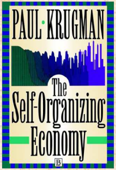


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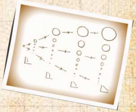

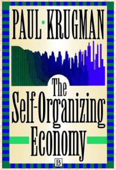


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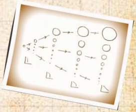

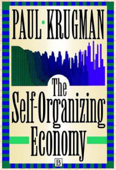


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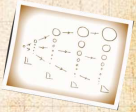
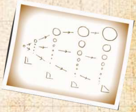


# Outline







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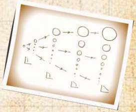
- $N_{k,t}$ = # groups containing $k$ elephants at time $t$.





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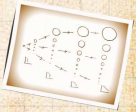

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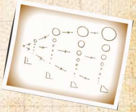

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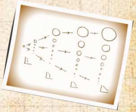

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First: $\sum_k kN_{k,t} = t =$ number of elephants at time $t$

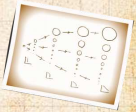
# Random Competitive Replication:



$P_k(t)$ = Probability of choosing an elephant that belongs to a group of size $k$:

- $N_{k,t}$ size $k$ groups
- $\Rightarrow kN_{k,t}$ elephants in size $k$ groups
- $t$ elephants overall





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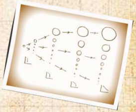

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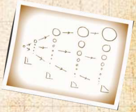

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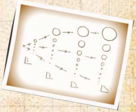

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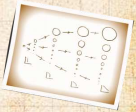
$$P_k(t) = \frac{kN_{k,t}}{t}.$$







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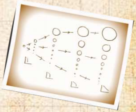

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1. An elephant belonging to a group with $k$ elephants is replicated:

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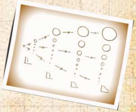

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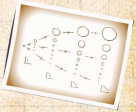

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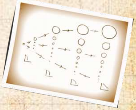

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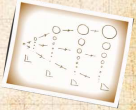

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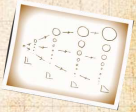

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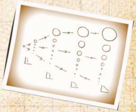

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1. The new elephant is a new flavor:

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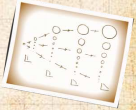

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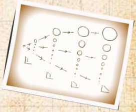

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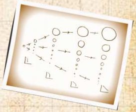

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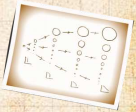

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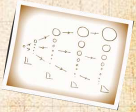

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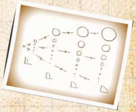

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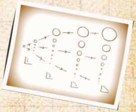


# Random Competitive Replication:

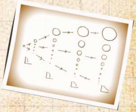
## Putting everything together:

For $k > 1$:

$$\left\langle N_{k,t+1} - N_{k,t} \right\rangle = (1-\rho)\left( (+1)(k-1)\frac{N_{k-1,t}}{t} + (-1)k\frac{N_{k,t}}{t} \right)$$







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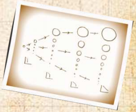

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Assume distribution stabilizes: $N_{k,t} = n_k t$

(Reasonable for $t$ large)

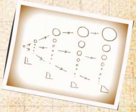
- Drop expectations
- Numbers of elephants now fractional
- Okay over large time scales

- For later: the fraction of groups that have size $k$ is $n_k/\rho$ since

$$\frac{N_{k,t}}{\rho t} = \frac{n_k \cancel{t}}{\rho \cancel{t}} = \frac{n_k}{\rho}.$$







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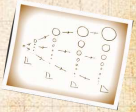

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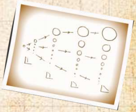

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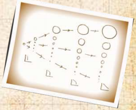

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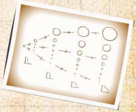

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## Stochastic difference equation:

$$\left\langle N_{k,t+1} - N_{k,t} \right\rangle = (1-\rho)\left( (k-1)\frac{N_{k-1,t}}{t} - k\frac{N_{k,t}}{t} \right)$$

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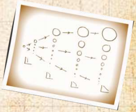

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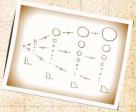

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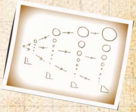

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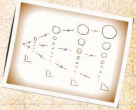
$$\Rightarrow n_k\left(1 + (1-\rho)k\right) = (1-\rho)(k-1)n_{k-1}$$







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  Insert question from assignment 4

- For just the tail: Expand as a series of powers of $1/k$

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  We (okay, you) find

$$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

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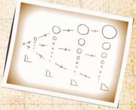

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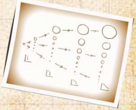

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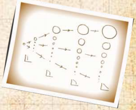

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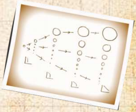

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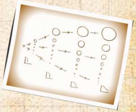

# Random Competitive Replication:

## We have a simple recursion:

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- Interested in $k$ large (the tail of the distribution)
- Can be solved exactly.

  Insert question from assignment 4

- For just the tail: Expand as a series of powers of $1/k$

  Insert question from assignment 4

  We (okay, you) find

$$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

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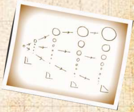

- Micro-to-Macro story with $\rho$ and $\gamma$ measurable.

$$\boxed{\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}}$$

- Observe $2 < \gamma < \infty$ for $0 < \rho < 1$.
- For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- 'Wild' power-law size distribution of group sizes, bordering on 'infinite' mean.
- For $\rho \simeq 1$ (high innovation rate):

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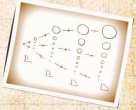
- All elephants have different flavors.
- Upshot: Tunable mechanism producing a family of universality classes.







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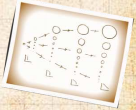

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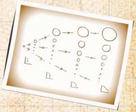

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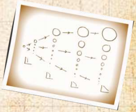

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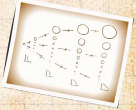

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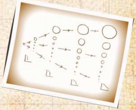

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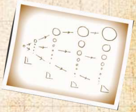

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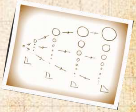

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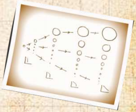

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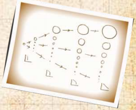

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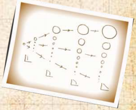

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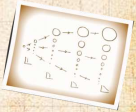

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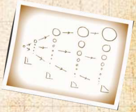

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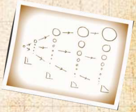

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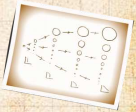

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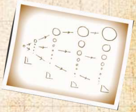
## We had one other equation:

- $$\langle N_{1,t+1} - N_{1,t} \rangle = \rho - (1-\rho) 1 \cdot \frac{N_{1,t}}{t}$$
- As before, set $N_{1,t} = n_1 t$ and drop expectations
- $$n_1(t+1) - n_1 t = \rho - (1-\rho) 1 \cdot \frac{n_1 t}{t}$$
- $$n_1 = \rho - (1-\rho) n_1$$
- Rearrange:
$$n_1 + (1-\rho) n_1 = \rho$$
- $$\boxed{n_1 = \frac{\rho}{2-\rho}}$$







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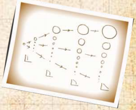

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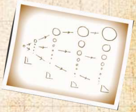

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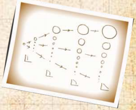

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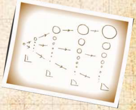

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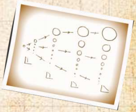
$$\frac{1}{\rho t} N_{1,t} = \frac{1}{\rho t} \frac{\rho t}{2-\rho} = \frac{1}{2-\rho}$$

(also = fraction of groups of size 1)

- For $\rho$ small, fraction of unique elephants $\sim 1/2$
- Roughly observed for real distributions
- $\rho$ increases, fraction increases
- Can show fraction of groups with two elephants $\sim 1/6$
- Model works well for large and small $k$ #awesome





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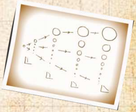

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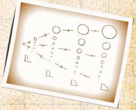

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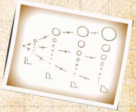

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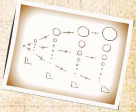

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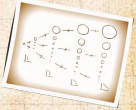

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- Recall number of distinct elephants = $\rho t$.
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- For $\rho$ small, fraction of unique elephants $\sim 1/2$
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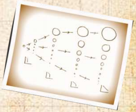

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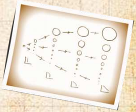

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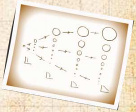
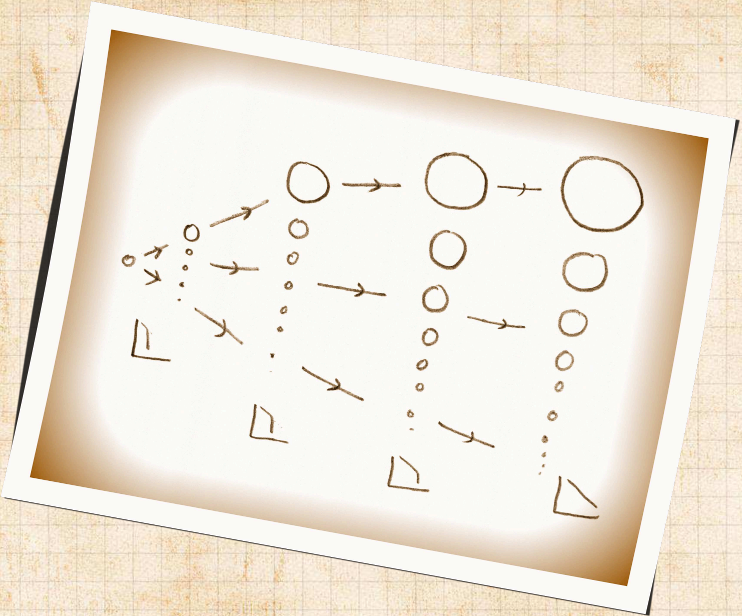
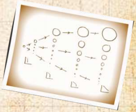






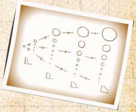




# Outline







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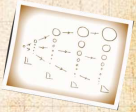

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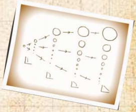
| $N_1$ (real) | $N_1$ (est) | $N_2$ (real) | $N_2$ (est) |
|---|---|---|---|
| 16,432 | 15,850 | 4,776 | 4,870 |







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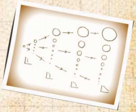
- Rich-Get-Richer Mechanism
  - Simon's Model
  - Analysis
  - Words
  - Catchphrases
- Optimization
  - Minimal Cost
  - Mandelbrot vs. Simon
  - Assumptions
  - Model
  - Analysis
  - And the winner is...?
- Nutshell
- Extra
- References





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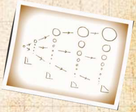

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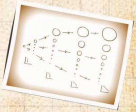

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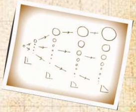

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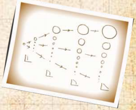

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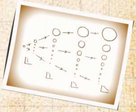

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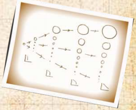

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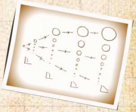

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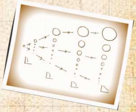

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- First to study network evolution with these kinds of models.
- Citation network of scientific papers
- Price's term: Cumulative Advantage
- Idea: papers receive new citations with probability proportional to their existing # of citations
- Directed network
- Two (surmountable) problems:
  1. New papers have no citations
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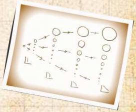

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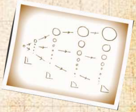

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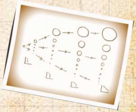

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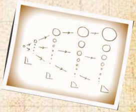

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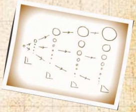

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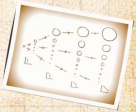

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- Studied careers of scientists and found credit flowed disproportionately to the already famous

- (Hath = suggested unit of purchasing power.)
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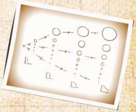

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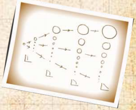

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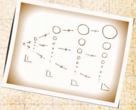

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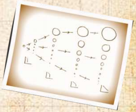

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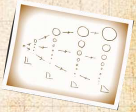

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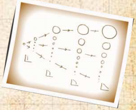

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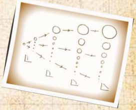

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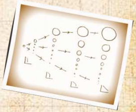

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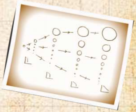

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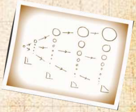

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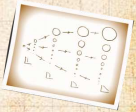

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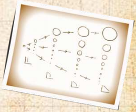

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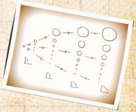

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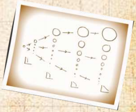

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Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.

# Evolution of catch phrases:



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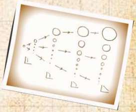

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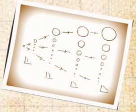

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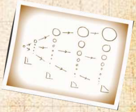

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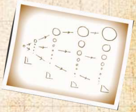

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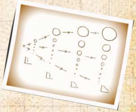

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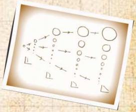

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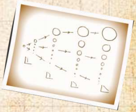

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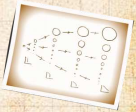

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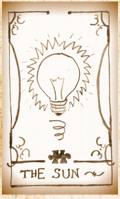






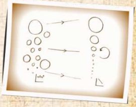

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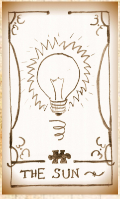


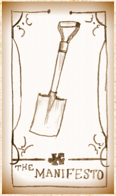




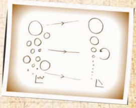

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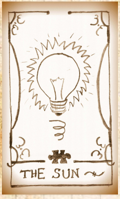


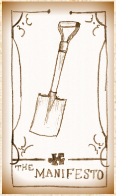


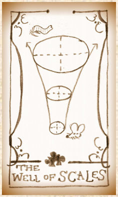






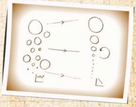

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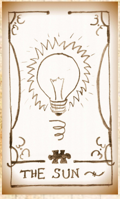
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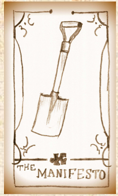
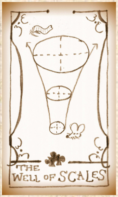

THE MANIFESTO

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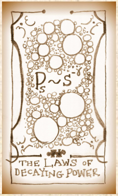
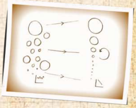

THE LAWS OF DECAYING POWER







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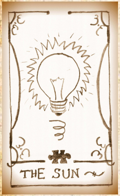
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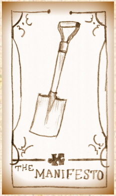
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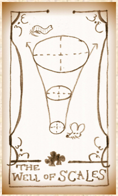
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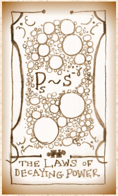
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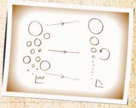
MINISTRY OF RANDOM WALKS







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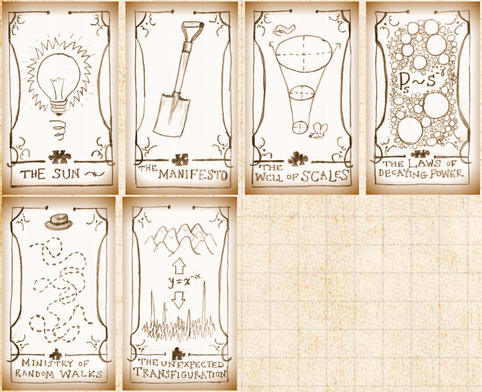
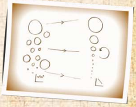

THE SUN
THE MANIFESTO
THE WELL OF SCALES
$P_s \sim s^{-\gamma}$
THE LAWS OF DECAYING POWER
MINISTRY OF RANDOM WALKS
$y = x^{-\alpha}$
THE UNEXPECTED TRANSFIGURATION





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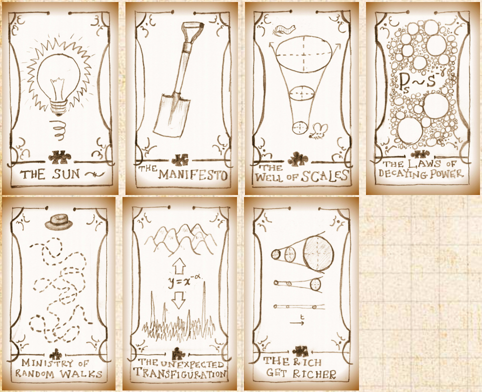
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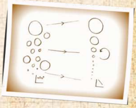
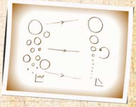
THE RICH GET RICHER









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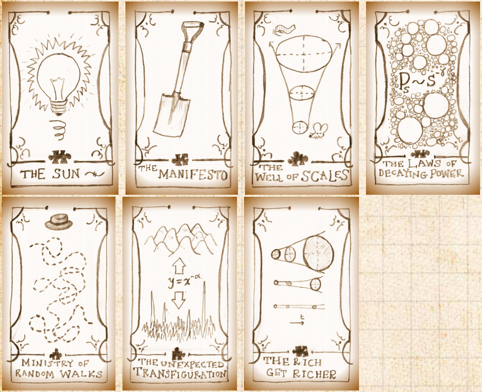
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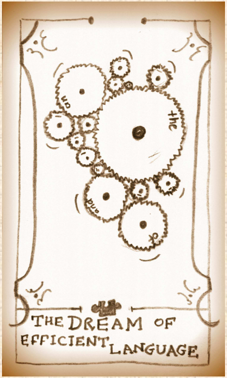
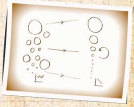

THE DREAM OF EFFICIENT LANGUAGE

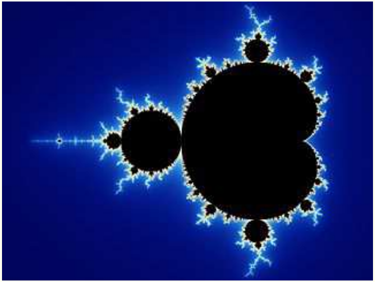
# Outline







# Benoît Mandelbrot



- Mandelbrot = father of fractals
- Mandelbrot = almond bread
- Bonus Mandelbrot set action: here.

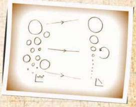
# Another approach:

## Benoît Mandelbrot

- Derived Zipf's law through optimization [16]
- Idea: Language is efficient
- Communicate as much information as possible for as little cost
- Need measures of information ($H$) and average cost ($C$)...
- Language evolves to maximize $H/C$, the amount of information per average cost.
- Equivalently: minimize $C/H$.
- Recurring theme: what role does optimization play in complex systems?







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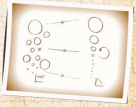

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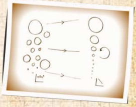

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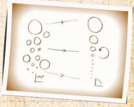

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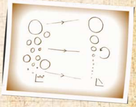

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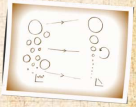

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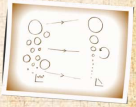

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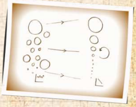

# The Quickening—Mandelbrot v. Simon:

## There Can Be Only One:


- Things there should be only one of: Theory, Highlander Films.
- Feel free to play Queen's It's a Kind of Magic in your head (funding remains tight).

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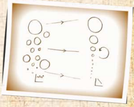

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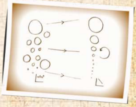

# Now let us enjoy the Trailer for Highlander:

*We were born to be Princes of the Universe*

## Mandelbrot vs. Simon:

- Mandelbrot (1953): "An Informational Theory of the Statistical Structure of Languages" [16]
- Simon (1955): "On a class of skew distribution functions" [23]
- Mandelbrot (1959): "A note on a class of skew distribution functions: analysis and critique of a paper by H.A. Simon" [17]
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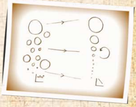

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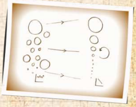

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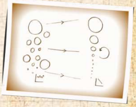

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*We were born to be Princes of the Universe*

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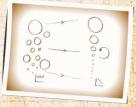

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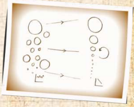

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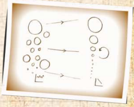

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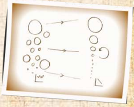

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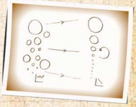

*I am immortal, I have inside me blood of kings*

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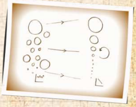

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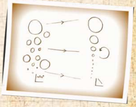

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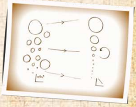

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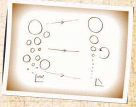

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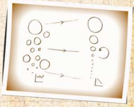
"You can't do this to me, I WENT TO COLLEGE!"





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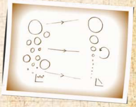

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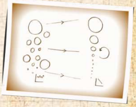
“You can’t do this to me, I WENT TO COLLEGE!” “You weak minded fool!” “You just lost your brain privileges,” etc.



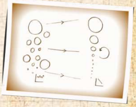


# Tho theories enter, one theory leaves

# Outline







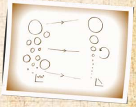
# Zipfarama via Optimization:

## Mandelbrot's Assumptions:

- Language contains $n$ words: $w_1, w_2, \ldots, w_n$.
- $i$th word appears with probability $p_i$
- Words appear randomly according to this distribution (obviously not true...)
- Words = composition of letters is important
- Alphabet contains $m$ letters
- Words are ordered by length (shortest first)







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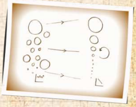

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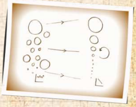

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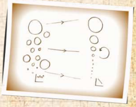

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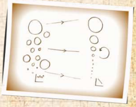

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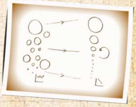

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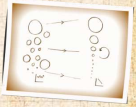

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## Word Cost

- Length of word (plus a space)
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## Objection

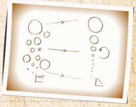
- Real words don't use all letter sequences

## Objections to Objection

- Maybe real words roughly follow this pattern (?)
- Words can be encoded this way
- Na na na-na naaaaa...







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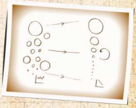

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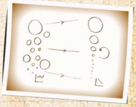

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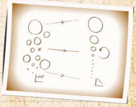

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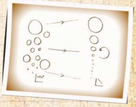

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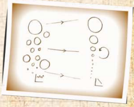

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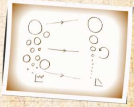

# Zipfarama via Optimization:



## Binary alphabet plus a space symbol

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| word | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 |
| length | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| $1 + \log_2 i$ | 1 | 2 | 2.58 | 3 | 3.32 | 3.58 | 3.81 | 4 |

- Word length of $2^k$th word: $= k + 1$
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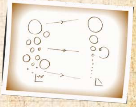

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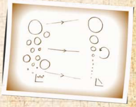

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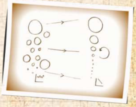

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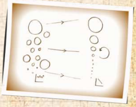

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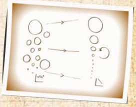
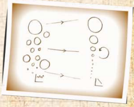


# Outline







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## Total Cost $C$

- Cost of the $i$th word: $C_i \simeq 1 + \log_m i$
- Cost of the $i$th word plus space: $C_i \simeq 1 + \log_m (i+1)$
- Subtract fixed cost: $C'_i = C_i - 1 \simeq \log_m (i+1)$
- Simplify base of logarithm:

$$C'_i \simeq \log_m (i+1) = \frac{\log_e (i+1)}{\log_e m}$$

- Total Cost:

$$C \sim \sum_{i=1}^{n} p_i C'_i \propto \sum_{i=1}^{n} p_i \ln(i+1)$$

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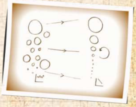
$$C \sim \sum_{i=1}^{n} p_i C'_i \propto \sum_{i=1}^{n} p_i \ln(i+1)$$







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## Total Cost $C$

- Cost of the $i$th word: $C_i \simeq 1 + \log_m i$
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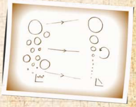

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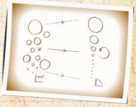

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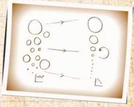

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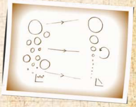

# Zipfarama via Optimization:

## Information Measure

- Use Shannon's Entropy (or Uncertainty):

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i$$

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- Proportional to average number of bits needed to encode each 'word' based on frequency of occurrence
- $-\log_2 p_i = \log_2 1/p_i$ = minimum number of bits needed to distinguish event $i$ from all others
- If $p_i = 1/2$, need only 1 bit ($\log_2 1/p_i = 1$)
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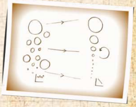

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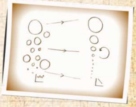

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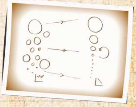

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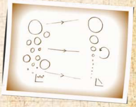

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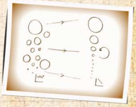

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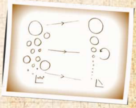

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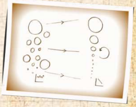

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- Use a slightly simpler form:

$$H = -\sum_{i=1}^{n} p_i \log_e p_i / \log_e 2 = -g \sum_{i=1}^{n} p_i \ln p_i$$

where $g = 1/\ln 2$

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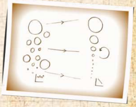

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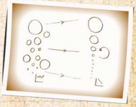

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$$F(p_1, p_2, \ldots, p_n) = C/H$$

subject to constraint

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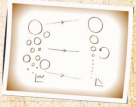

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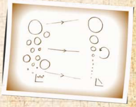

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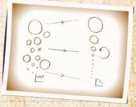
(2) Longer words are more informative (rarer)

# Outline









# Zipfarama via Optimization:

## Time for Lagrange Multipliers:

- Minimize

$$\Psi(p_1, p_2, \dots, p_n) =$$

$$F(p_1, p_2, \dots, p_n) + \lambda G(p_1, p_2, \dots, p_n)$$

where

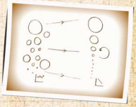
$$F(p_1, p_2, \dots, p_n) = \frac{C}{H} = \frac{\sum_{i=1}^{n} p_i \ln(i+1)}{-g \sum_{i=1}^{n} p_i \ln p_i}$$

and the constraint function is

$$G(p_1, p_2, \dots, p_n) = \sum_{i=1}^{n} p_i - 1 (= 0)$$







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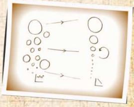

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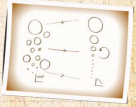
Insert question from assignment 3







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## Some mild suffering leads to:

- $$p_j = e^{-1-\lambda H^2/gC}(j+1)^{-H/gC} \propto (j+1)^{-H/gC}$$
- A power law appears [applause]: $\alpha = H/gC$
- Next: sneakily deduce $\lambda$ in terms of $g$, $C$, and $H$.
- Find

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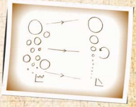

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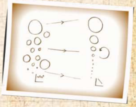

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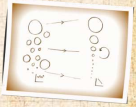

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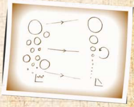

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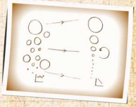

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## Finding the exponent

- Now use the normalization constraint:

$$1 = \sum_{j=1}^{n} p_j = \sum_{j=1}^{n} (j+1)^{-H/gC} = \sum_{j=1}^{n} (j+1)^{-\alpha}$$

- As $n \to \infty$, we end up with $\zeta(H/gC) = 2$ where $\zeta$ is the Riemann Zeta Function
- Gives $\alpha \simeq 1.73$ ($> 1$, too high) or $\gamma = 1 + \frac{1}{\alpha} \simeq 1.58$ (very wild)
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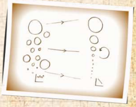

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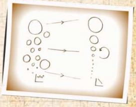

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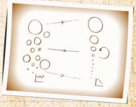

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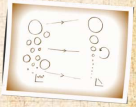

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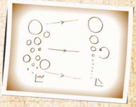

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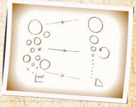


# Zipfarama via Optimization:



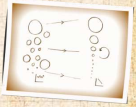
## Finding the exponent

- Now use the normalization constraint:

$$1 = \sum_{j=1}^{n} p_j = \sum_{j=1}^{n} (j+1)^{-H/gC} = \sum_{j=1}^{n} (j+1)^{-\alpha}$$

- As $n \to \infty$, we end up with $\zeta(H/gC) = 2$ where $\zeta$ is the Riemann Zeta Function
- Gives $\alpha \simeq 1.73$ ($> 1$, too high) or $\gamma = 1 + \frac{1}{\alpha} \simeq 1.58$ (very wild)
- If cost function changes ($j + 1 \to j + a$) then exponent is tunable
- Increase $a$, decrease $\alpha$



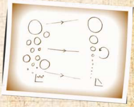


# Zipfarama via Optimization:

## All told:

- Reasonable approach: Optimization is at work in evolutionary processes
- But optimization can involve many incommensurate elephants: monetary cost, robustness, happiness,...
- Mandelbrot's argument is not super convincing
- Exponent depends too much on a loose definition of cost





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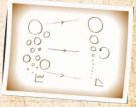

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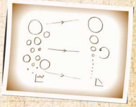

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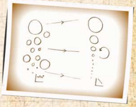
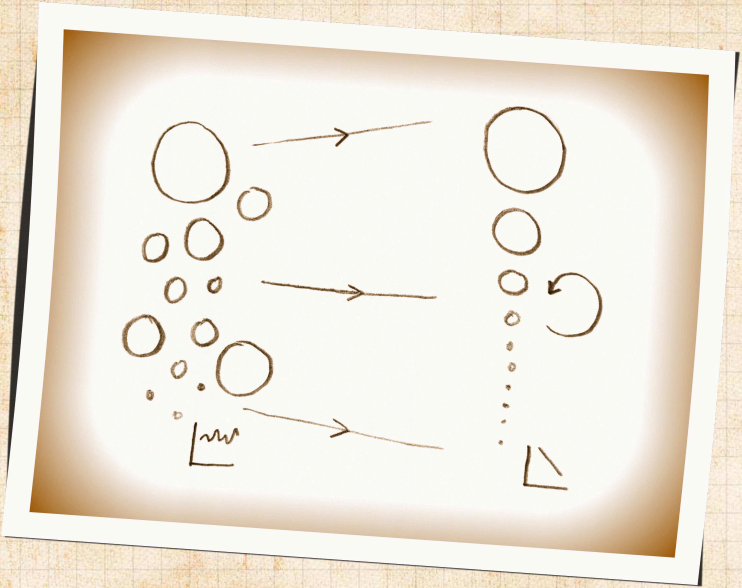
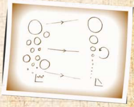




## From the discussion at the end of Mandelbrot's paper:

- A. S. C. Ross: "M. Mandelbrot states that 'the actual direction of evolution (sc. of language) is, in fact, towards fuller and fuller utilization of places'. We are, in fact, completely without evidence as to the existence of any 'direction of evolution' in language, and it is axiomatic that we shall remain so. Many philologists would deny that a 'direction of evolution' could be theoretically possible; thus I myself take the view that a language develops in what is essentially a purely random manner."
- Mandelbrot: "As to the 'fundamental linguistic units being the least possible differences between pairs of utterances' this is a logical consequence of the fact that two is the least integer greater than one."





# More:

## Reconciling Mandelbrot and Simon

- Mixture of local optimization and randomness
- Numerous efforts...

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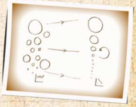

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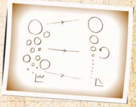

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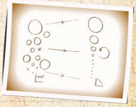

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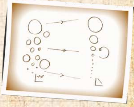

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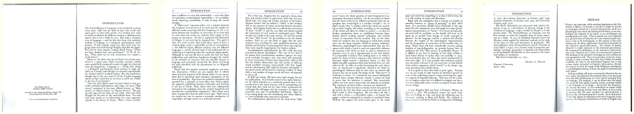

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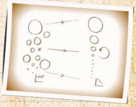

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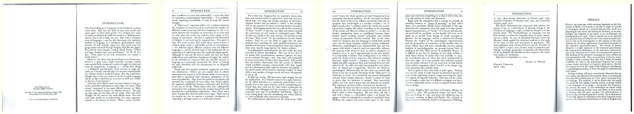

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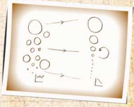

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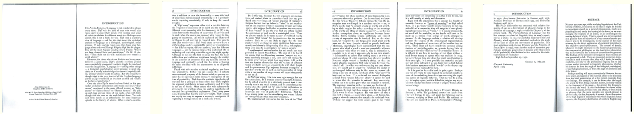

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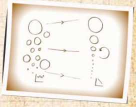

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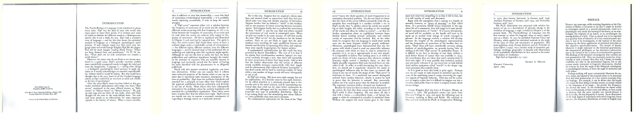

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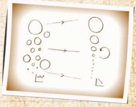

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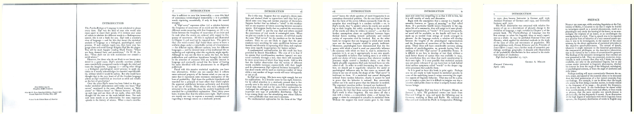

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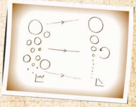
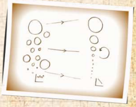


# What Shannon said about meaning in his 1948 paper "A mathematical theory of communication": [22]

> The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one *selected from a set* of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.



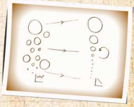


# INTRODUCTION

*The Psycho-Biology of Language* is not calculated to please every taste. Zipf was the kind of man who would take roses apart to count their petals; if it violates your sense of values to tabulate the different words in a Shakespearean sonnet, this is not a book for you. Zipf took a scientist's view of language — and for him that meant the statistical analysis of language as a biological, psychological, social process. If such analysis repels you, then leave your language alone and avoid George Kingsley Zipf like the plague. You will be much happier reading Mark Twain: "There are liars, damned liars, and statisticians." Or W. H. Auden: "Thou shalt not sit with statisticians nor commit a social science."

However, for those who do not flinch to see beauty murdered in a good cause, Zipf's scientific exertions yielded some wonderfully unexpected results to boggle the mind and tease the imagination. Language *is* — among other things — a biological, psychological, social process; to apply statistics to it merely acknowledges its essential unpredictability, without which it would be useless. But who would have thought that in the very heart of all the freedom language allows us Zipf would find an invariant as solid and reliable as the law of gravitation?



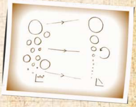


Put it this way. Suppose that we acquired a dozen monkeys and chained them to typewriters until they had produced some very long and random sequence of characters. Suppose further that we defined a "word" in this monkey-text as any sequence of letters occurring between successive spaces. And suppose finally that we counted the occurrences of these "words" in just the way Zipf and others counted the occurrences of real words in meaningful texts. When we plot our results in the same manner, we will find exactly the same "Zipf curves" for the monkeys as for the human authors. Since we are not likely to argue that the poor monkeys were searching for some equilibrium between uniformity and diversity in expressing their ideas, such explanations seem equally inappropriate for human authors.

A mathematical rationalization for this result has been provided by Benoit Mandelbrot. The crux of it is that if we assume that word-boundary markers (spaces) are scattered randomly through a text, then there will necessarily be more occurrences of short than long words. Add to this fact the further observation that the variety of different words available increases exponentially with their length and the phenomenon Zipf reported becomes inescapable: a few short words will be used an enormous number of times while a vast number of longer words will occur infrequently or not at all.

So Zipf was wrong. His facts were right enough, but not his explanations. In a broader sense he was right, however, for he called attention to a stochastic process that is frequently seen in the social sciences, and by accumulating statistical data that cried out for some better explanation he challenged his colleagues and his successors to explore an important new type of probability distribution. Zipf belongs among those rare but stimulating men whose failures are more profitable than most men's successes.





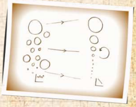
# So who's right?



Bornholdt and Ebel (PRE), 2001:
"World Wide Web scaling exponent from Simon's 1955 model" [4].

- Show Simon's model fares well.
- Recall $\rho$ = probability new flavor appears.
- Alta Vista crawls in approximately 6 month period in 1999 give $\rho \simeq 0.10$
- Leads to $\gamma = 1 + \frac{1}{1-\rho} \simeq 2.1$ for in-link distribution.
- Cite direct measurement of $\gamma$ at the time: $2.1 \pm 0.1$ and 2.09 in two studies.





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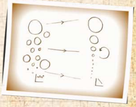

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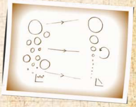

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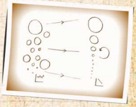

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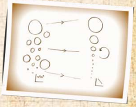

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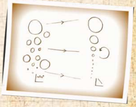
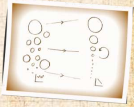


# Outline







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## Recent evidence for Zipf's law...

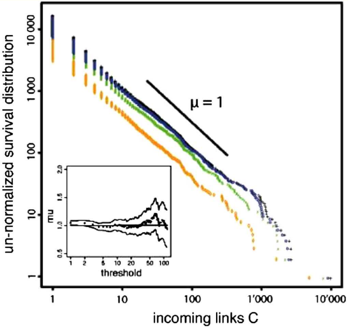


FIG. 1 (color online). (Color Online) Log-log plot of the number of packages in four Debian Linux Distributions with more than $C$ in-directed links. The four Debian Linux Distributions are Woody (19.07.2002) (orange diamonds), Sarge (06.06.2005) (green crosses), Etch (15.08.2007) (blue circles), Lenny (15.12.2007) (black+'s). The inset shows the maximum likelihood estimate (MLE) of the exponent $\mu$ together with two boundaries defining its 95% confidence interval (approximately given by $1 \pm 2/\sqrt{n}$, where $n$ is the number of data points using in the MLE), as a function of the lower threshold. The MLE has been modified from the standard Hill estimator to take into account the discreteness of $C$.

Maillart et al., PRL, 2008:
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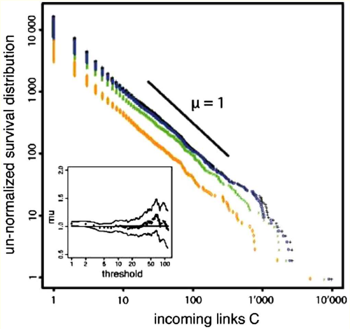


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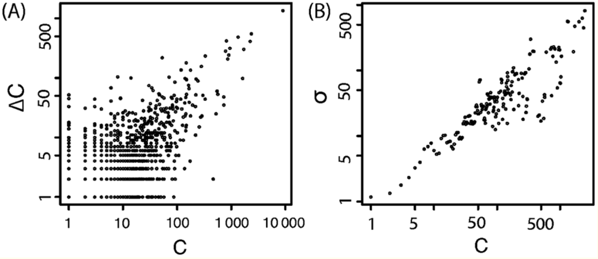
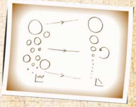



FIG. 2. Left panel: Plots of $\Delta C$ versus $C$ from the Etch release (15.08.2007) to the latest Lenny version (05.05.2008) in double logarithmic scale. Only positive values are displayed. The linear regression $\Delta C = R \times C + C_0$ is significant at the 95% confidence level, with a small value $C_0 = 0.3$ at the origin and $R = 0.09$. Right panel: same as left panel for the standard deviation of $\Delta C$.

- Rough, approximately linear relationship between $C$ number of in-links and $\Delta C$.







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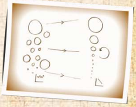
## Nutshell:

- Simonish random 'rich-get-richer' models agree in detail with empirical observations.
- Power-lawfulness: Mandelbrot's optimality is still apparent.
- Optimality arises for free in Random Competitive Replication models.







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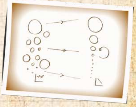
- Simonish random 'rich-get-richer' models agree in detail with empirical observations.
- Power-lawfulness: Mandelbrot's optimality is still apparent.
- Optimality arises for free in Random Competitive Replication models.







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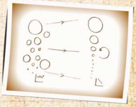

# Neural reboot (NR):

## Walking with a baby robin:

# Others are also not happy:

## Krugman and Simon

- "The Self-Organizing Economy" (Paul Krugman, 1996) [13]
- Krugman touts Zipf's law for cities, Simon's model
- "Déjà vu, Mr. Krugman" (Berry, 1999)
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# Who needs a hug?



## From Berry [3]

- Déjà vu, Mr. Krugman. Been there, done that. The Simon-Ijiri model was introduced to geographers in 1958 as an explanation of city size distributions, the first of many such contributions dealing with the steady states of random growth processes, ...
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