Mechanisms for Generating Power-Law Size Distributions, Part 1

Principles of Complex Systems | @pocsvox
CSYS/MATH 300, Fall, 2015 | #FallPoCS2015

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont

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PoCS
Principles of Complex Systems
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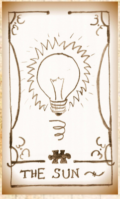
What's the Story?

The UNIVERSITY of VERMONT

Outline

The deal:



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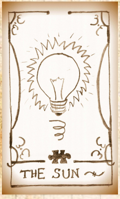

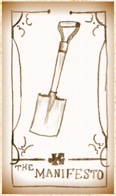

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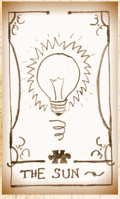
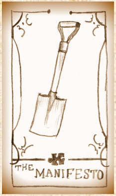
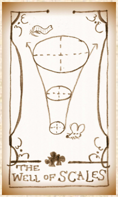
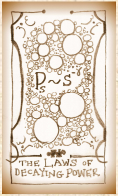

THE SUN


THE MANIFESTO


THE WELL OF SCALES


P_s ~ s^-γ
THE LAWS OF DECAYING POWER

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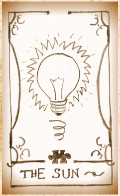
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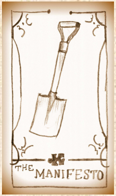
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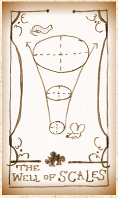
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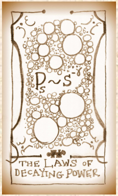
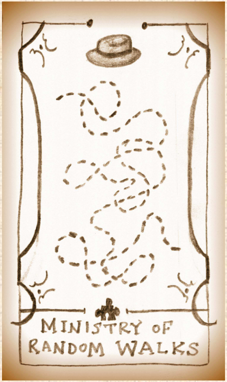
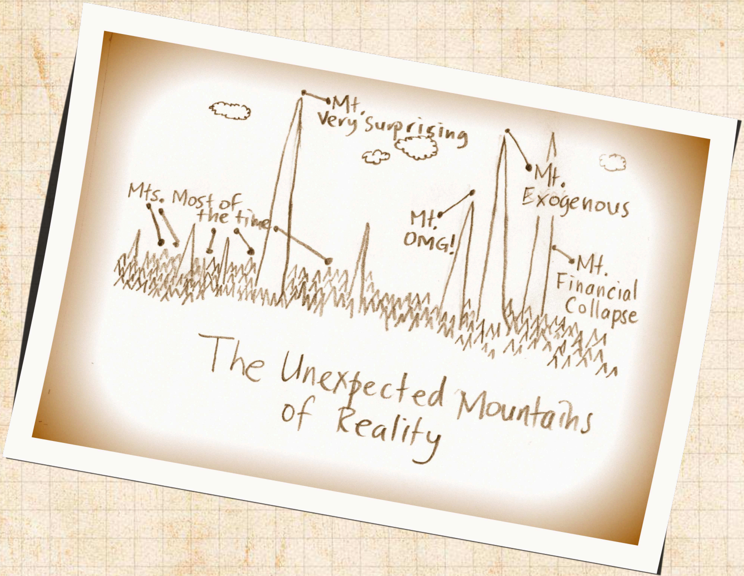

$P_s \sim s^{-\gamma}$
THE LAWS OF DECAYING POWER

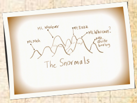

MINISTRY OF RANDOM WALKS

Great moments in Televised Random Walks:

Plinko! from the Price is Right.


Mt. Whatever
Mt. Zzzz
Mt. Who cares?
Mt. Meh
Mt. Quite boring
The Snormals


Mt. Very Surprising
Mts. Most of the time
Mt. OMG!
Mt. Exogenous
Mt. Financial Collapse
The Unexpected Mountains of Reality


The Snormals

Mechanisms:

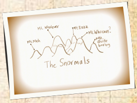
A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks.

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$



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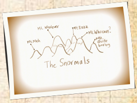

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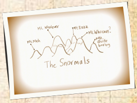

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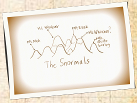

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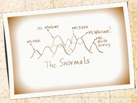

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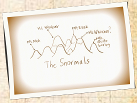

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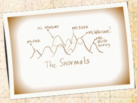
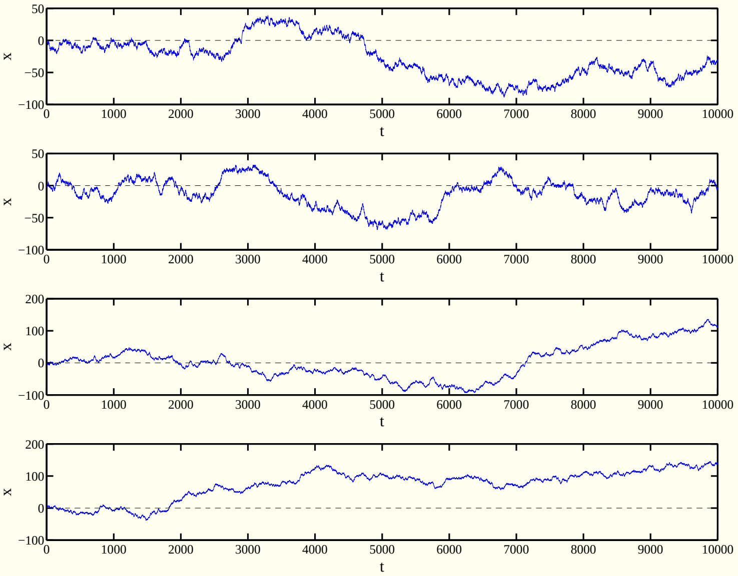
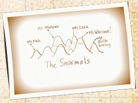


A few random random walks:





Random walks:

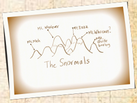
Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

- At any time step, we 'expect' our drunkard to be back at the pub.
- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?



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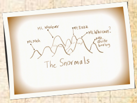

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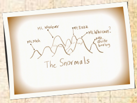

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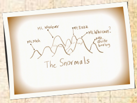

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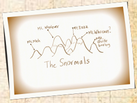

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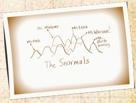

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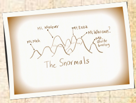


Variances sum:*

$$\mathrm{Var}(x_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

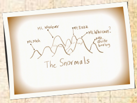
$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

So typical displacement from the origin scales as:

$$\sigma = t^{1/2}$$

- A non-trivial scaling law arises out of additive aggregation or accumulation.



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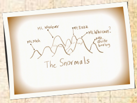

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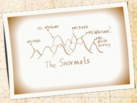

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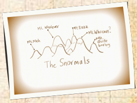

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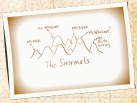

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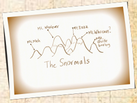
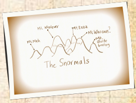


Stock Market randomness:

Also known as the bean machine, the quincunx (simulation), and the Galton box.

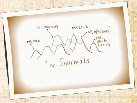


Random walk basics:

Counting random walks:

- Each specific random walk of length t appears with a chance $1/2^t$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as # distinct walks that start at $x = i$ and end at $x = j$ after t time steps.
- Random walk must displace by $+(j - i)$ after t steps.
- Insert question from assignment 3

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$



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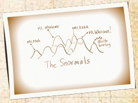

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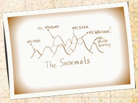

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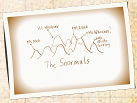

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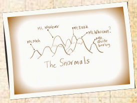

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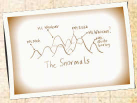

How does $P(x_t)$ behave for large t?

- Take time $t = 2n$ to help ourselves.
- $x_{2n} \in \{0, \pm 2, \pm 4, \dots, \pm 2n\}$
- x_{2n} is even so set $x_{2n} = 2k$.
- Using our expression $N(i, j, t)$ with $i = 0$, $j = 2k$, and $t = 2n$, we have

$$\mathbf{Pr}(x_{2n} \equiv 2k) \propto \binom{2n}{n+k}$$

- For large n, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$\mathbf{Pr}(x_t \equiv x) \simeq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

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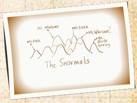
- The whole is different from the parts. #nutritious
- See also: Stable Distributions



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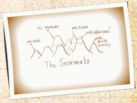

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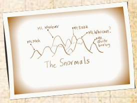

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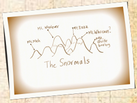

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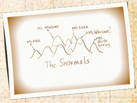

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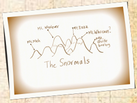

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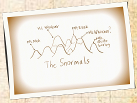

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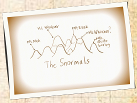
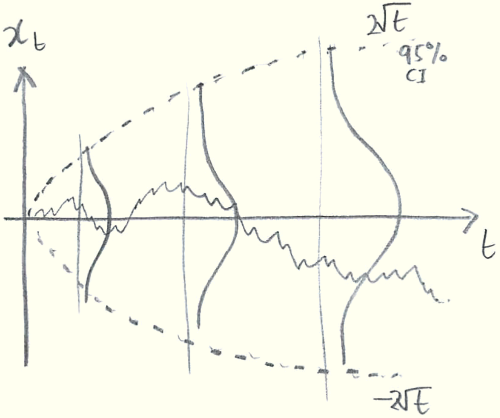
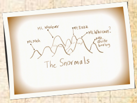


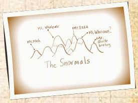
Universality is also not left-handed:



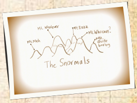
- This is Diffusion: the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.




Mt. Whatever
Mt. ZZZZ
Mt. Who cares?
Mt. Meh
Mt. Quite boring
The Snormals




t^1/2
t
The Snormals
The Random Road
through the Forests of Forgettable Events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Random walks are even weirder than you might think...

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
- In fact: $\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$
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See Feller, Intro to Probability Theory, Volume I [3]

Outline

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

Reasons for caring:

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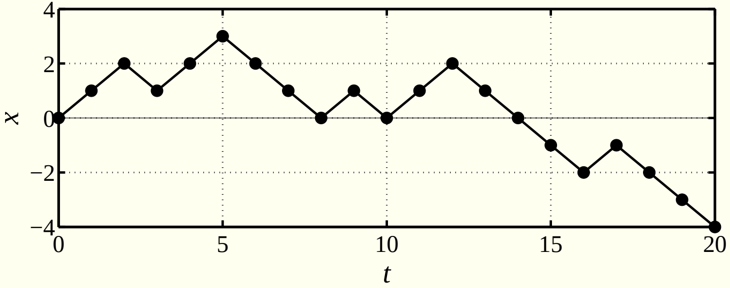

- A return to origin can only happen when $t = 2n$.
- In example above, returns occur at $t = 8$, 10, and 14.
- Call $P_{\mathrm{fr}(2n)}$ the probability of first return at $t = 2n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

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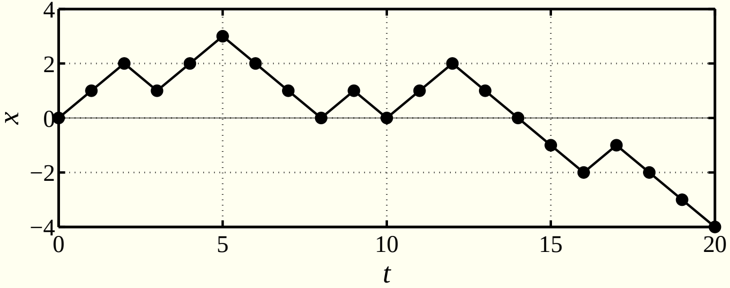

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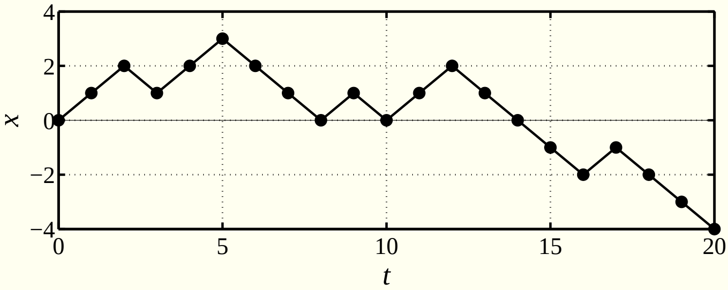

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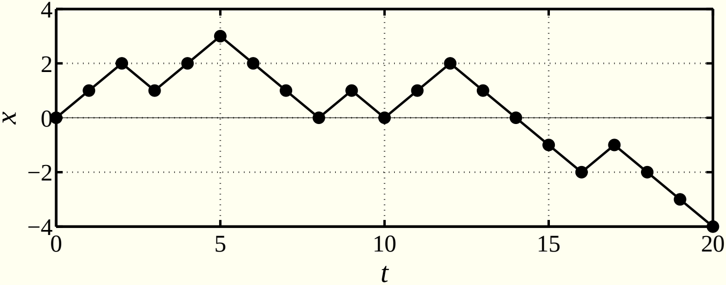

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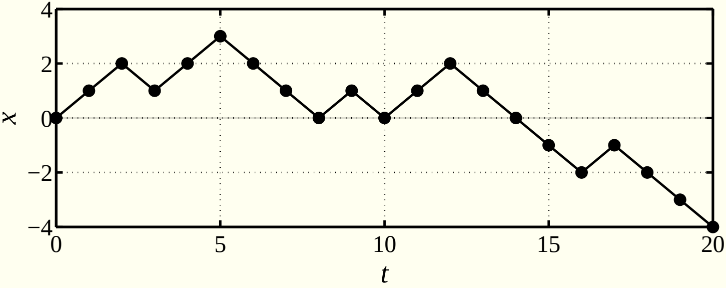

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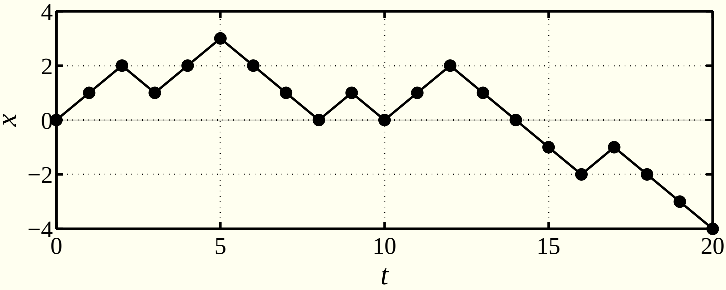

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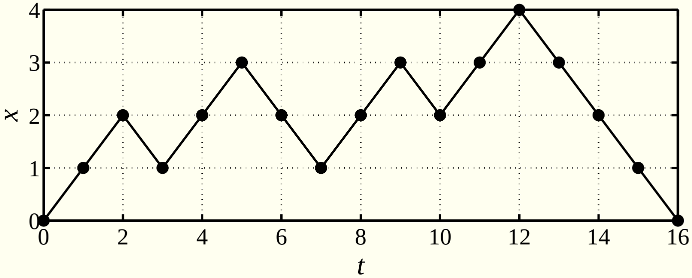

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- Observe walk first returning at $t = 16$ stays at or above $x = 1$ for $1 \le t \le 15$ (dashed red line).
- Now want walks that can return many times to $x = 1$.
- $P_{\text{fr}}(2n) =$
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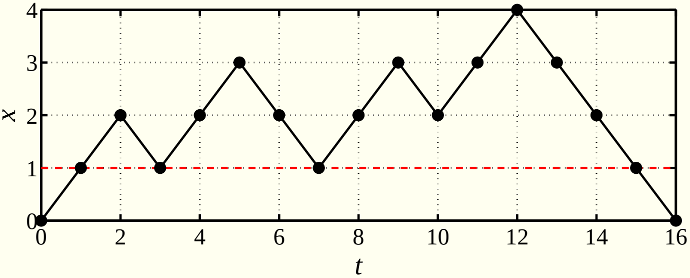
- The 2 accounts for drunkards that first lurch to $x = -1$.



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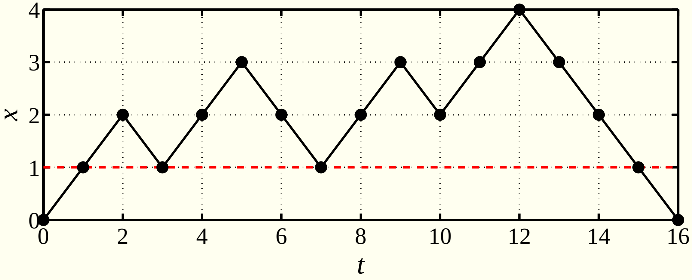

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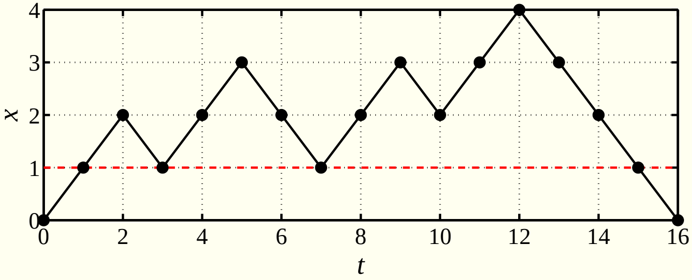
- The $\frac{1}{2}$ accounts for $x_{2n} = 2$ instead of 0.
- The 2 accounts for drunkards that first lurch to $x = -1$.



- Can assume drunkard first lurches to $x = 1$.
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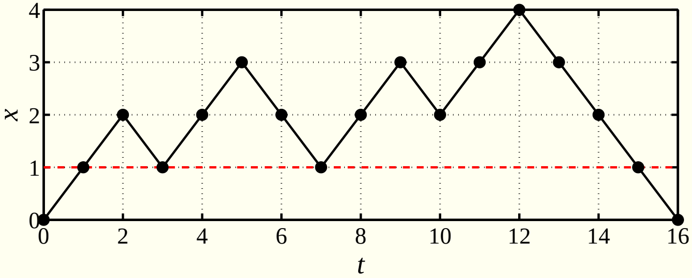

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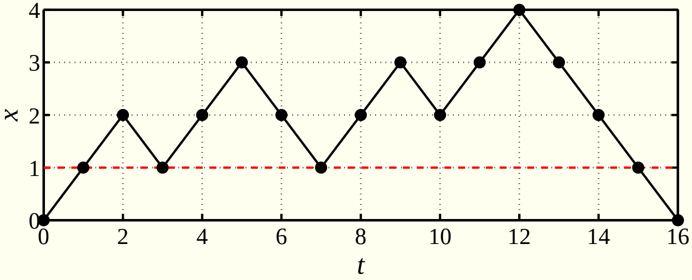

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Counting first returns:

Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Idea: If we can compute the number of walks that hit $x = 0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
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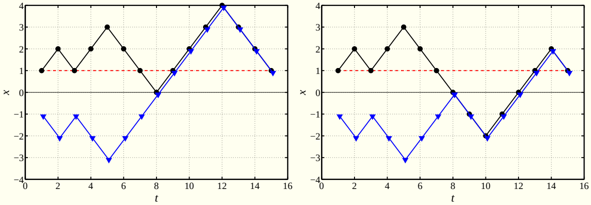

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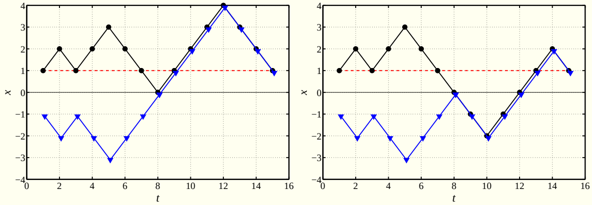

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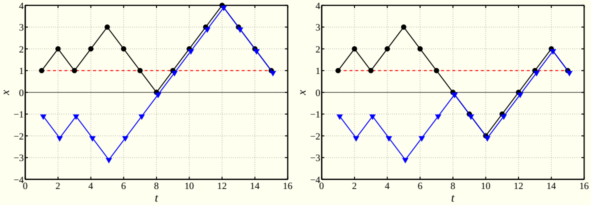

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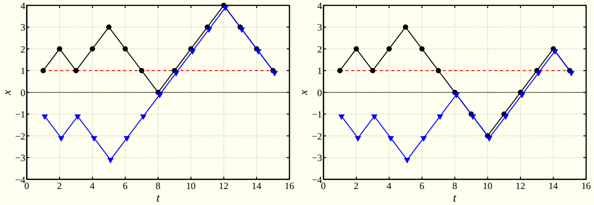

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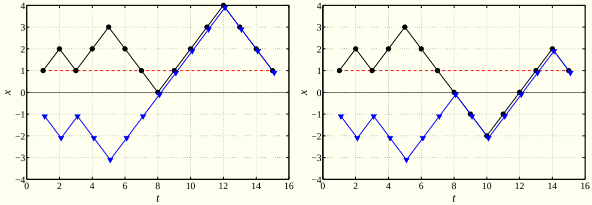

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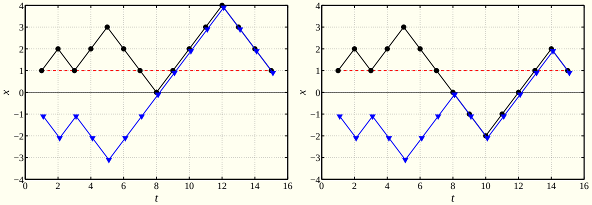

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On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system
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On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree #groovy
- Equal probability still present: walkers traverse edges with equal frequency. #totallygroovy

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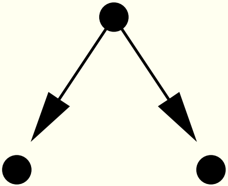
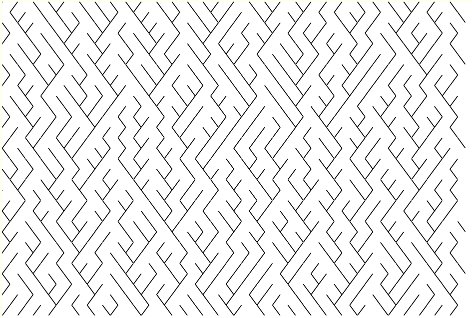
Scheidegger Networks [8, 2]





- Random directed network on triangular lattice.
- Toy model of real networks.
- ‘Flow’ is southeast or southwest with equal probability.

Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:

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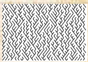

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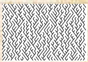

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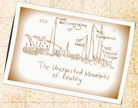
- Dispersion of suspended sediments in streams.
- Long times for clearing.



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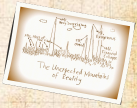

More than randomness

- Can generalize to Fractional Random Walks [6, 7, 5]
- Levy flights, Fractional Brownian Motion
- See Montroll and Shlesinger for example: [5]
  "On $1/f$ noise and other distributions with long tails."
  Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation σ scales as

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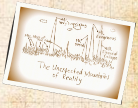
- Extensive memory of path now matters...



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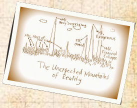

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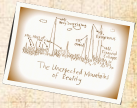

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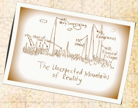

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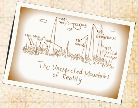
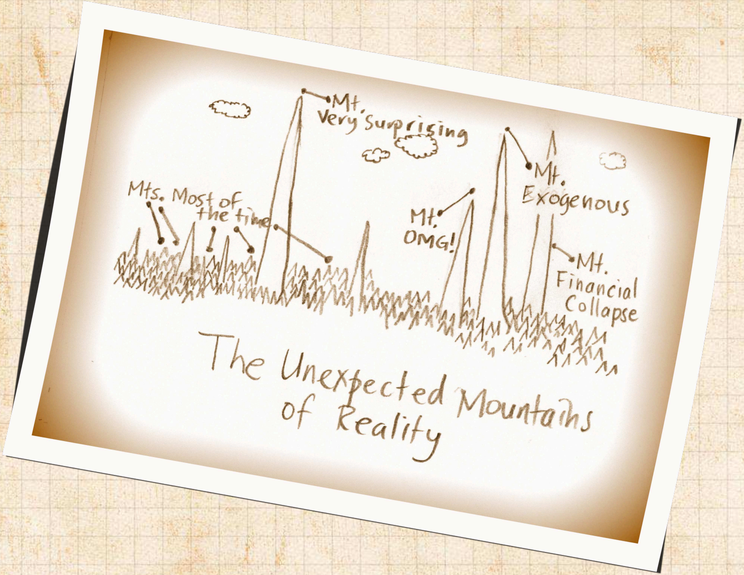
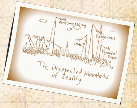
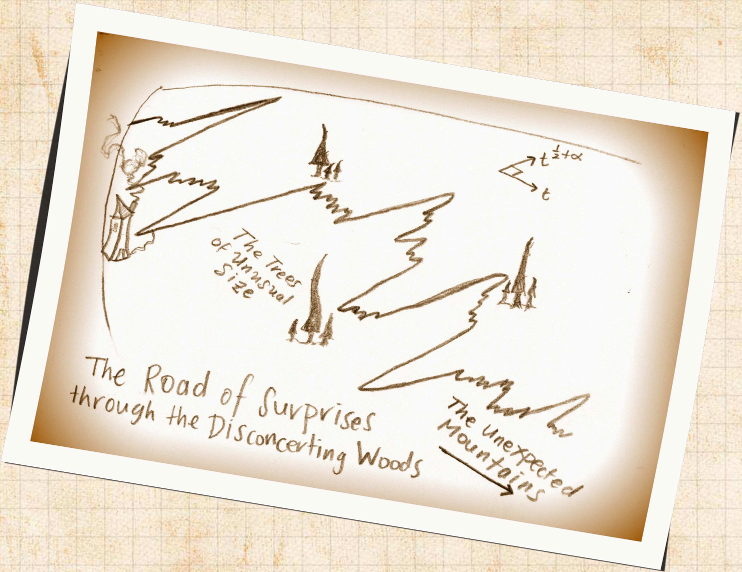
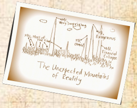



Mt. Very Surprising
Mts. Most of the time
Mt. OMG!
Mt. Exogenous
Mt. Financial Collapse
The Unexpected Mountains of Reality

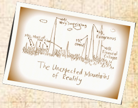



$t^{\frac{1}{2}+\alpha}$
t
The Trees of Unusual Size
The Road of Surprises through the Disconcerting Woods
The Unexpected Mountains


The Unexpected Mountains of Reality

Neural reboot (NR):

Desert rain frog/Squeaky toy:



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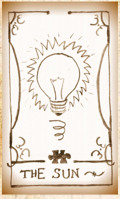

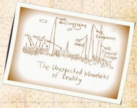

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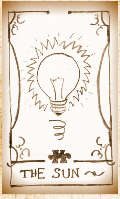

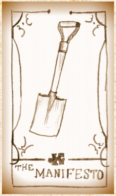

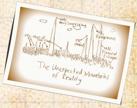

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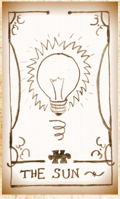

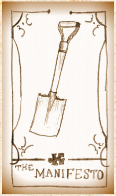

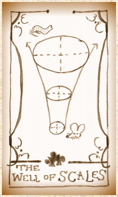

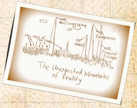

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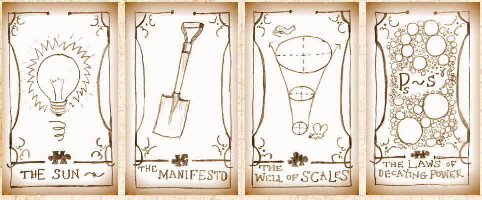
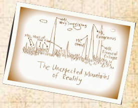

THE SUN
THE MANIFESTO
THE WELL OF SCALES
$P_s \sim s^{-\gamma}$
THE LAWS OF DECAYING POWER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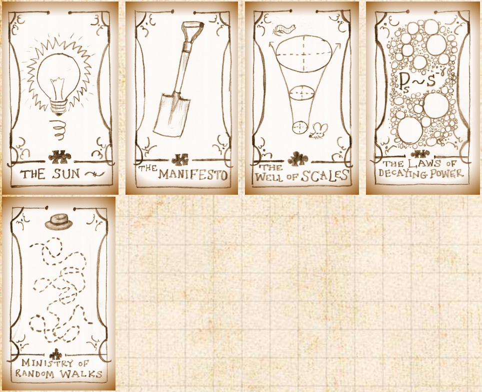
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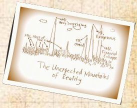
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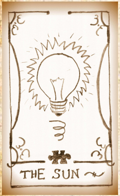
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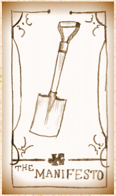
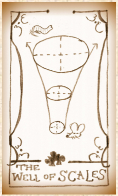

THE MANIFESTO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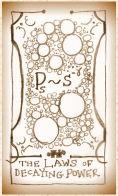
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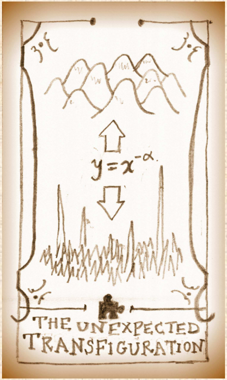
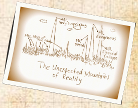
MINISTRY OF RANDOM WALKS


$y = x^{-\alpha}$
THE UNEXPECTED TRANSFIGURATION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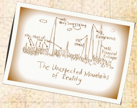
Outline



Variable Transformation

Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

- Random variable X with known distribution P_x
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- $P_Y(y)\mathrm{d}y = \sum_{x|f(x)=y} P_X(x)\mathrm{d}x = \sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
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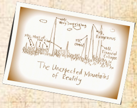

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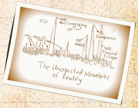

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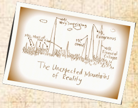

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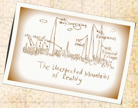

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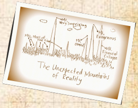

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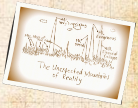

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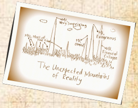

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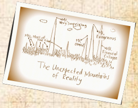

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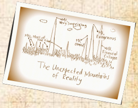

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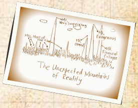

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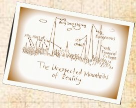

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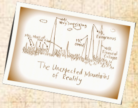

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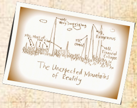

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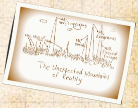

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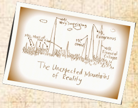

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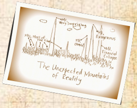

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$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_x(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- If $P_x(x) \rightarrow x^{\beta}$ as $x \rightarrow 0$ then

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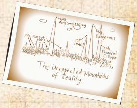

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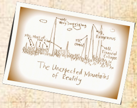

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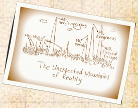

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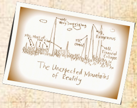


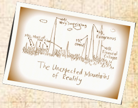
Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness (easy)...
- More later when we cover robustness.



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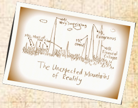

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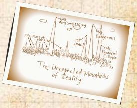

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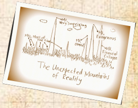

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- Select a random point in the universe $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
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Matter is concentrated in stars: [9]

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

- Also invert:
$\mathrm{d}F \propto \mathrm{d}(r^{-2}) \propto r^{-3}\mathrm{d}r \rightarrow \mathrm{d}r \propto r^3\mathrm{d}F \propto F^{-3/2}\mathrm{d}F$.

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Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$, and $\boxed{P_r(r) \propto r^2}$

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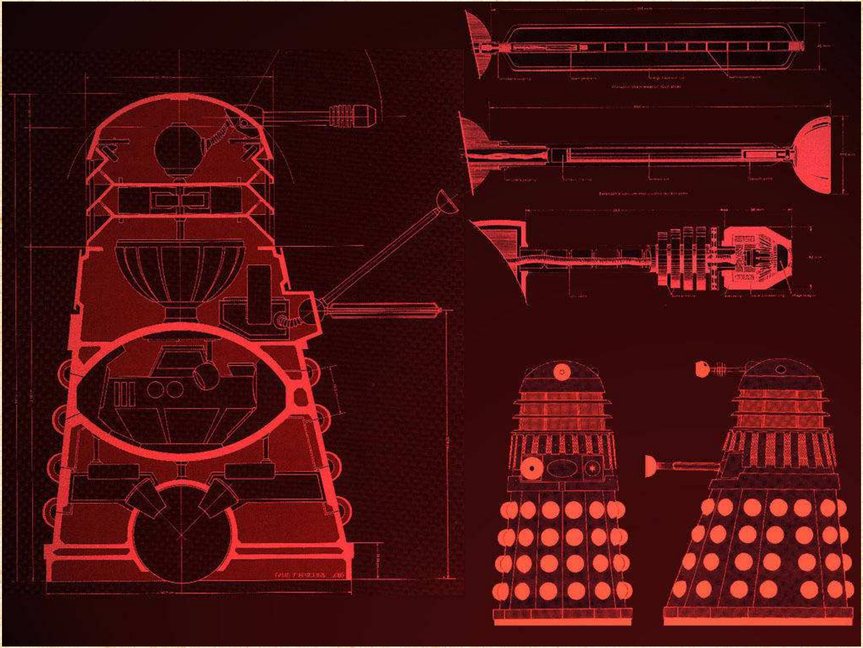
Doctorin’ the Tardis

☐ Todo: Build Dalek army.



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Extreme Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
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Neural reboot (NR):

Zoomage in slow motion

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