Lognormals and friends Principles of Complex Systems | @pocsvox CSYS/MATH 300, Fall, 2015 | #FallPoCS2015

Prof. Peter Dodds | @peterdodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center Vermont Advanced Computing Core | University of Vermont











Principles of Complex Systems @pocsvox What's the Story?







Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan

These slides are brought to you by:

Sealie & Lambie Productions

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





20f24

Outline

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan

References

Lognormals Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





200 3 of 24

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





200 4 of 24



PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Outline

Lognormals Empirical Confusability





990 4 of 24

Alternative distributions

There are other 'heavy-tailed' distributions: 1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Alternative distributions

There are other 'heavy-tailed' distributions: 1. The Log-normal distribution ☑

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions

$$P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}} dx$$

 $CCDF = stretched exponential \square$.





PoCS | @pocsvox Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

うへで 5 of 24

UNIVERSITY

Alternative distributions

There are other 'heavy-tailed' distributions: 1. The Log-normal distribution ☑

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions

$$P(x) dx = rac{k}{\lambda} \left(rac{x}{\lambda}
ight)^{\mu-1} e^{-(x/\lambda)^{\mu}} dx$$

CCDF = stretched exponential C.3. Gamma distributions C, and more.



PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References



UNIVERSITY

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- In x is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.





PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan



Standard form reveals the mean μ and variance σ² of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

PoCS | @pocsvox Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

For lognormals

All moments of lognormals are





200 7 of 24

 Standard form reveals the mean μ and variance σ² of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

 $\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \quad \text{median}_{\text{lognormal}} = e^{\mu},$

$$\sigma_{
m lognormal} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$$

$$\mathsf{mode}_{\mathsf{lognormal}} = e^{\mu - \sigma^2}$$

PoCS



PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

 Standard form reveals the mean μ and variance σ² of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

 $\mu_{\rm lognormal} = e^{\mu + \frac{1}{2}\sigma^2}, \qquad {\rm median}_{\rm lognormal} = e^{\mu},$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}$$

All moments of lognormals are finite.

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Take *Y* as distributed normally:

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





200 8 of 24

Take *Y* as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





290 8 of 24

Take *Y* as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

PoCS | @pocsvox

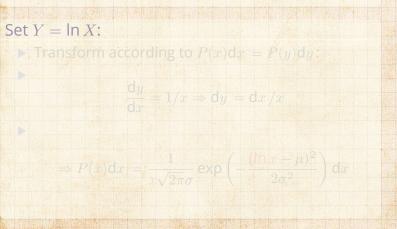
Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





29 C 8 of 24

Take *Y* as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Set $Y = \ln X$:

▶ Transform according to P(x)dx = P(y)dy:





Take *Y* as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Set $Y = \ln X$:

▶ Transform according to P(x)dx = P(y)dy:

 $\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$





20 8 of 24

Take *Y* as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Set $Y = \ln X$:

▶ Transform according to P(x)dx = P(y)dy:

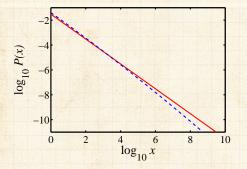
 $\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$

$$\Rightarrow P(x) dx = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$





Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan

References

For lognormal (blue), $\mu = 0$ and $\sigma = 10$.

For power law (red), $\gamma = 1$ and c = 0.03.





20 Pof 24

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

If the first term is relatively small,







$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

If the first term is relatively small,

$$\ln P(x) \sim -\left(1 - \frac{\mu}{\sigma^2}\right) \ln x + \text{const.}$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$=-\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\ln P(x) \sim -\left(1 + \frac{\mu}{\sigma^2}\right) \ln x + \text{const}$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$=-\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,



Random Multiplicative Growth Model

PoCS | @pocsvox Lognormals and friends

Random Growth with Variable Lifespan

References





$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$=-\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + {\rm const.}$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi}\sigma - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$=-\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\ln P(x) \sim -\left(1 - \frac{\mu}{\sigma^2}\right) \ln x + \text{const.} \Rightarrow \gamma = 1 - \frac{\mu}{\sigma^2}$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

 $-rac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(rac{\mu}{\sigma^2}-1
ight)\ln x$

lf $\mu < 0$, $\gamma > 1$ which is totally cool.

 $\Rightarrow \log_{10} x \lesssim 0.05 imes 2(\sigma^2 - \mu) \log_{10}$

If you find a -1 exponent, you may have a lognormal distribution...





nac 11 of 24

If μ < 0, γ > 1 which is totally cool.
If μ > 0, γ < 1, not so much.

Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$;

 $-rac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(rac{\mu}{\sigma^2} + 1
ight)\ln x$

 $\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10}$

⇒ If you find a -1 exponent, you may have a lognormal distribution.

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





• If $\mu < 0$, $\gamma > 1$ which is totally cool.

- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Expect -1 scaling to hold until $(\ln x)^2$ term become significant compared to $(\ln x)$: $-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} + 1\right) \ln x$ $\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$ $\Rightarrow \text{ If you find a -1 exponent,}$ you may have a lognormal distribution...

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





nac 11 of 24

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x):

If you find a -1 exponent, you may have a lognormal distribution..

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





200 11 of 24

• If $\mu < 0$, $\gamma > 1$ which is totally cool.

- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x):

$$-rac{1}{2\sigma^2}(\ln x)^2\simeq 0.05\left(rac{\mu}{\sigma^2}-1
ight)\ln x$$

If you find a -1 exponent, you may have a lognormal distribution..

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





• If $\mu < 0$, $\gamma > 1$ which is totally cool.

- lf $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x):
 - $-rac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(rac{\mu}{\sigma^2} 1
 ight) \ln x$
 - $\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 \mu) \log_{10} e \simeq 0.05 (\sigma^2 \mu)$

If you find a -1 exponent, you may have a lognormal distribution.

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





• If $\mu < 0$, $\gamma > 1$ which is totally cool.

- lf $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x):
 - $-rac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(rac{\mu}{\sigma^2} 1
 ight) \ln x$
 - $\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 \mu) \log_{10} e \simeq 0.05 (\sigma^2 \mu)$

If you find a -1 exponent, you may have a lognormal distribution

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x):
 - $-rac{1}{2\sigma^2}(\ln x)^2\simeq 0.05\left(rac{\mu}{\sigma^2}-1
 ight)\ln x$
 - $\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 \mu) \log_{10} e \simeq 0.05(\sigma^2 \mu)$
- ► ⇒ If you find a -1 exponent, you may have a lognormal distribution...

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Outline

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals

Random Multiplicative Growth Model





200 12 of 24

Generating lognormals:

Random multiplicative growth:

 $x_{n+1} = rx_n$ where r > 0 is a random growth variable (Shrinkage is allowed) In log space, growth is by addition: $\ln x_{n+1} = \ln r + \ln x_n$ $\Rightarrow \ln x_n$ is normally distributed $\Rightarrow x_n$ is lognormally distributed

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Variable Lifespan

References





na 13 of 24

Generating lognormals:

Random multiplicative growth:

 $x_{n+1} = rx_n$

where r > 0 is a random growth variable (Shrinkage is allowed)

 x_n is lognormally distributed

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Variable Lifespan





Generating lognormals:

Random multiplicative growth:

 $x_{n+1} = rx_n$

where r > 0 is a random growth variable
(Shrinkage is allowed)
In log space, growth is by addition:

 $\ln x_{n+1} = \ln r + \ln x_n$

 $\Rightarrow \ln x_n$ is normally distributed $\Rightarrow x$ is lognormally distributed



Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan





Generating lognormals:

Random multiplicative growth:

 $x_{n+1} = rx_n$

where r > 0 is a random growth variable

(Shrinkage is allowed)

In log space, growth is by addition:

 $\ln x_{n+1} = \ln r + \ln x_n$

 $\blacktriangleright \Rightarrow \ln x_n$ is normally distributed

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





nac 13 of 24

Generating lognormals:

Random multiplicative growth:

 $x_{n+1} = rx_n$

where r > 0 is a random growth variable

(Shrinkage is allowed)

In log space, growth is by addition:

 $\ln x_{n+1} = \ln r + \ln x_n$

⇒ ln x_n is normally distributed
 ⇒ x_n is lognormally distributed

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan





Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).

But Robert Axtell (2001) shows a power law fits th data very well with $\gamma = 2$, not $\gamma = 1$ (!) Problem of data censusing (missing small firms).

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusabilit

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References



VERMONT

20 14 of 24

One piece in Gibrat's model seems okay empirica Growth rate *r* appears to be independent of firm size.

- Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell ^[1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusabilit

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





na 14 of 24

One piece in Gibrat's model seems okay empirical Growth rate *r* appears to be independent of firm size.

- Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell ^[1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusabilit

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References

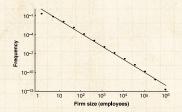




200 14 of 24

One piece in Gibrat's model seems okay empirical Growth rate *r* appears to be independent of firm size.

- Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell ^[1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).



Freq \propto (size)^{- γ} $\gamma \simeq 2$

One piece in Gibrat's model seems okay empirica Growth rate *r* appears to be independent of firm size.

PoCS | @pocsvox

Lognormals and friends

Lognormals

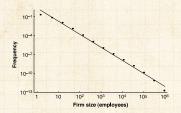
Empirical Confusabilit

Random Multiplicative Growth Model Random Growth with Variable Lifespan





- Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell ^[1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).



Freq \propto (size)^{- γ} $\gamma \simeq 2$

One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size.^[1].

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusabilit

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





20 14 of 24

Axtel cites Malcai et al.'s (1999) argument^[5] for why power laws appear with exponent $\gamma \simeq 2$

 $i(t+1) = \max(rx_i(t), c\langle x_i \rangle)$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan





Axtel cites Malcai et al.'s (1999) argument^[5] for why power laws appear with exponent $\gamma \simeq 2$ • The set up: N entities with size $x_i(t)$

 $r_i(t+1) = \max(rx_i(t), c\langle x_i \rangle)$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





200 15 of 24

- Axtel cites Malcai et al.'s (1999) argument^[5] for why power laws appear with exponent γ ≃ 2
- The set up: N entities with size $x_i(t)$
- Generally:

 $x_i(t+1) = rx_i(t)$

where r is drawn from some happy distribution Same as for loghormal but one extra piece Each x_r cannot drop too low with respect to the other sizes:

 $(t+1) = \max(rx_i(t), c\langle x_i \rangle)$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan





- Axtel cites Malcai et al.'s (1999) argument ^[5] for why power laws appear with exponent γ ≃ 2
- The set up: N entities with size $x_i(t)$

Generally:

 $x_i(t+1) = rx_i(t)$

where r is drawn from some happy distribution
Same as for lognormal but one extra piece.

Each x_i cannot drop too low with respect to the other sizes:

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan





- Axtel cites Malcai et al.'s (1999) argument^[5] for why power laws appear with exponent γ ≃ 2
- The set up: N entities with size $x_i(t)$

Generally:

 $x_i(t+1) = rx_i(t)$

where r is drawn from some happy distribution

- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

 $x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan





Some math later... Insert question from assignment 7 🗗

where γ is implicitly given by

N = total number of firms.

Now, if $c/N \ll 1$ and $\gamma > 2$ $N = \frac{(\gamma - 2)}{(\gamma - 1)}$

Vhich gives

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Variable Lifespan

References



VERMONT

Insert question from assignment 7 🖸

Find $P(x) \sim x^{-\gamma}$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Variable Lifespan

References



VIVERSITY S

Insert question from assignment 7 🖸

Find $P(x) \sim x^{-\gamma}$

• where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N =total number of firms.

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





Insert question from assignment 7 🖸

Find $P(x) \sim x^{-\gamma}$

• where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N =total number of firms.

Now, if $c/N \ll 1$ and $\gamma > 2$ $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$





PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan

Insert question from assignment 7 🖸

Find $P(x) \sim x^{-\gamma}$

• where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N =total number of firms.

Now, if
$$c/N \ll 1$$
 and $\gamma > 2$ $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$

Which gives
$$\gamma \sim 1 + \frac{1}{1-c}$$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





Insert question from assignment 7 🖸

Find $P(x) \sim x^{-\gamma}$

• where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N =total number of firms.

Now, if
$$c/N \ll 1$$
 and $\gamma > 2$ $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left\lfloor \frac{-1}{-(c/N)} \right\rfloor$

Which gives $\gamma \sim 1 + \frac{1}{1-c}$

• Groovy... $c \text{ small} \Rightarrow \gamma \simeq 2$

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References





Outline

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals

Random Growth with Variable Lifespan





200 17 of 24

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i vary
- Example: $P(t)dt = ae^{-at}dt$ where t = ageBack to no bottom limit: each x_i follows a lognormal

(Assume for this example that $\sigma \sim t$ and $\mu = \ln n$ Now averaging different lognormal distributions

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





20 18 of 24

Ages of firms/people/... may not be the same

- > Allow the number of updates for each size x_i to vary

PoCS | @pocsvox

Lognormals and friends

Lognormals

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





29 CP 18 of 24

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- **•** Example: $P(t)dt = ae^{-at}dt$ where t = age.

Back to no bottom limit: each x_i follows a lognormal Sizes are distributed as

(Assume for this example that $\sigma \sim t$ and $\mu = \ln p$ Now averaging different lognormal distributions

PoCS | @pocsvox

Lognormals and friends

Lognormals

mpirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





na 18 of 24

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- **•** Example: $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal

(Assume for this example that $\sigma \sim t$ and $\mu = \ln n$ Now averaging different lognormal distributions

PoCS | @pocsvox

Lognormals and friends

Lognormals

mpirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





na 18 of 24

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- **•** Example: $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal

Sizes are distributed as ^[6]

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- **•** Example: $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal

Sizes are distributed as ^[6]

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$) Now averaging different lognormal distributions.

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Averaging lognormals

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

 $P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) dt$

Insert fabulous calculation (team is spared) Some enjoyable suffering leads to:





20 19 of 24

Averaging lognormals

PoCS | @pocsvox

Lognormals and friends

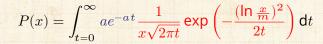
Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References



Insert fabulous calculation (team is spared).





20 P 19 of 24

Averaging lognormals

PoCS | @pocsvox

Lognormals and friends

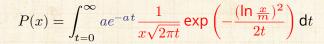
Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References



Insert fabulous calculation (team is spared).
Some enjoyable suffering leads to:

 $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}}$





200 19 of 24

 $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x)^2}}$

Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$

Break in scaling (not uncommon) Double- Preto discourse of the second se

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

• Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

Break in scaling (not uncommon) Double-Parend Scholarion Control First noticed by Montroll and Shlesinger Later: Huberman and Adamic : Number pages per website

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

• Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1\\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$

Break in scaling (not uncommon) Double-First noticed by Montroll and Shlesinger Later: Huberman and Adamic : Number pages per website

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

• Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1\\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$

'Break' in scaling (not uncommon)

First noticed by Montroll and Shlesinger Later: Huberman and Adamic : Number of pages per website

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

• Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1\\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$

'Break' in scaling (not uncommon)
 Double-Pareto distribution C
 First noticed by Montroll and Shlesing
 Later: Huberman and Adamic

pages per website

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References





$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

• Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1\\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$

- 'Break' in scaling (not uncommon)
- Double-Pareto distribution I and the second seco
- First noticed by Montroll and Shlesinger^[7, 8]

Later: Huberman and Adamic Number of pages per website

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda (\ln \frac{x}{m})^2}}$$

• Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.

$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1\\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$

- 'Break' in scaling (not uncommon)
- ▶ Double-Pareto distribution C
- First noticed by Montroll and Shlesinger^[7, 8]
- Later: Huberman and Adamic^[3, 4]: Number of pages per website

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative

Growth Model

Random Growth with Variable Lifespan

References

Lognormals and power laws can be awfully similar
 Random Multiplicative Growth leads to lognormal distributions
 Enforcing a minimum size leads to a power law tail
 With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
 Take-home message: Be careful out there...





うへで 21 of 24

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative

Growth Model

Random Growth with Variable Lifespan

References

- Lognormals and power laws can be awfully similar
 Random Multiplicative Growth leads to lognormal distributions
 - Enforcing a minimum size leads to a power law tail With no minimum size but a distribution of lifetimes, the double Pareto distribution appears Take-home message: Be careful out there...





na @ 21 of 24

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative

Growth Model

Random Growth with Variable Lifespan

References

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
 - With no minimum size but a distribution of lifetimes, the double Pareto distribution appears Take-home message: Be careful out there...





29 c 21 of 24

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative

Growth Model

Random Growth with Variable Lifespan

References

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears



VERMONT

20 0 21 of 24

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative

Growth Model

Random Growth with Variable Lifespan

References

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...





29 c 21 of 24

References I

[1] R. Axtell. Zipf distribution of U.S. firm sizes. Science, 293(5536):1818–1820, 2001. pdf 2

[2] R. Gibrat. Les inégalités économiques. Librairie du Recueil Sirey, Paris, France, 1931.

[3] B. A. Huberman and L. A. Adamic. Evolutionary dynamics of the World Wide Web. Technical report, Xerox Palo Alto Research Center, 1999.

[4] B. A. Huberman and L. A. Adamic. The nature of markets in the World Wide Web. <u>Quarterly Journal of Economic Commerce</u>, 1:5–12, 2000.

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan





References II

[5] O. Malcai, O. Biham, and S. Solomon. Power-law distributions and lévy-stable intermittent fluctuations in stochastic systems of many autocatalytic elements. Phys. Rev. E, 60(2):1299-1303, 1999. pdf [6] M. Mitzenmacher. A brief history of generative models for power law and lognormal distributions. Internet Mathematics, 1:226–251, 2003. pdf [7] E. W. Montroll and M. W. Shlesinger. On 1/f noise and other distributions with long tails. Proc. Natl. Acad. Sci., 79:3380-3383, 1982. pdf

PoCS | @pocsvox

Lognormals and friends

Lognormals Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Ideenan

References





References III

PoCS | @pocsvox

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan



 [8] E. W. Montroll and M. W. Shlesinger. Maximum entropy formalism, fractals, scaling phenomena, and 1/f noise: a tale of tails.
 J. Stat. Phys., 32:209–230, 1983.



