

# The Amusing Law of Benford

Principles of Complex Systems | @pocsvox  
 CSYS/MATH 300, Fall, 2015 | #FallPoCS2015

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## Benford's Law — The Law of First Digits

- ▶  $P(\text{first digit} = d) \propto \log_b \left(1 + \frac{1}{d}\right)$
- for certain sets of 'naturally' occurring numbers in base  $b$
- ▶ Around 30.1% of first digits are '1', compared to only 4.6% for '9'.
- ▶ First observed by [Simon Newcomb](#) [2] in 1881 "Note on the Frequency of Use of the Different Digits in Natural Numbers"
- ▶ Independently discovered in 1938 by [Frank Benford](#).
- ▶ Newcomb almost always noted but Benford gets the stamp, according to [Stigler's Law of Eponymy](#).

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## Benford's Law—The Law of First Digits

### Observed for

- ▶ Fundamental constants (electron mass, charge, etc.)
- ▶ Utility bills
- ▶ Numbers on tax returns (ha!)
- ▶ Death rates
- ▶ Street addresses
- ▶ Numbers in newspapers
- ▶ Cited as [evidence of fraud](#) in the 2009 Iranian elections.

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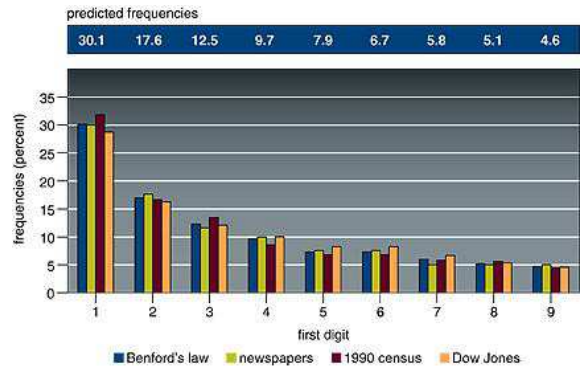
## Outline

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## Benford's Law—The Law of First Digits

### Real data:



From 'The First-Digit Phenomenon' by T. P. Hill (1998) [1]

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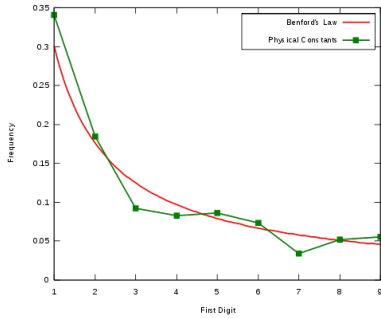
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# Benford's Law—The Law of First Digits

Physical constants of the universe:



Taken from [here](#).

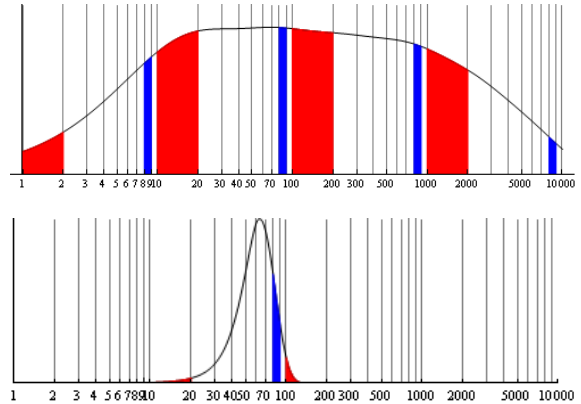
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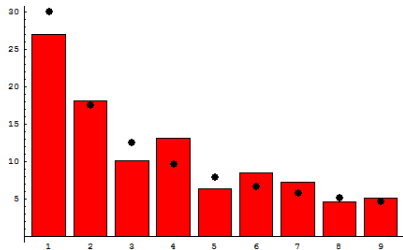
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# Benford's Law—The Law of First Digits

Population of countries:



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On counting and logarithms:



- ▶ Earlier: Listen to Radiolab's "Numbers."
- ▶ Now: Benford's Law.

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# Essential story

$$\begin{aligned}
 P(\text{first digit} = d) &\propto \log_b \left(1 + \frac{1}{d}\right) \\
 &\propto \log_b \left(\frac{d+1}{d}\right) \\
 &\propto \log_b (d+1) - \log_b (d)
 \end{aligned}$$

- ▶ Observe this distribution if numbers are distributed uniformly in log-space:

$$P(\ln x) d(\ln x) \propto 1 \cdot d(\ln x) = x^{-1} dx$$

- ▶ Power law distributions at work again...
- ▶ Extreme case of  $\gamma \approx 1$ .

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# References 1

- [1] T. P. Hill. The first-digit phenomenon. *American Scientist*, 86:358–, 1998.
- [2] S. Newcomb. Note on the frequency of use of the different digits in natural numbers. *American Journal of Mathematics*, 4:39–40, 1881. [pdf](#)

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