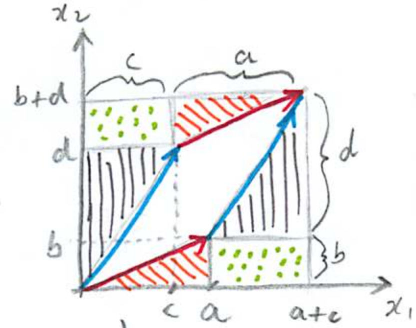
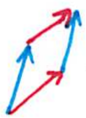
Determinants from the ground up:

$\mathbb{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

idea: consider area of parallelogram formed by row vectors of $\mathbb{A}$: $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$



Area of  $= (a+c)(b+d)$

$-a \cdot b \quad -dc \quad -2bc$

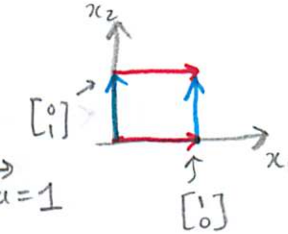
$= ab + ad + cb + cd$
$\quad - ab - dc - 2bc$

$= ad - bc$ → call this $\mathbb{A}$'s determinant: $\det(\mathbb{A})$ or $|\mathbb{A}|$

Three Observations about this determinant thing for 2×2s:

① $|\mathbb{I}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

anchor



② If we swap $\mathbb{A}$'s rows, we flip the sign of the determinant

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ but $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$
$= -(ad - bc)$

straight lines for determinant

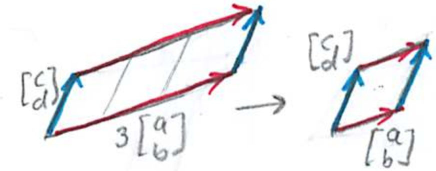
−ve area indicates ordering of vectors.

③ $|\mathbb{A}| = ad - bc$ is **multilinear** in the rows of $\mathbb{A}$

Two pieces

3.1 Area scales:

ex $\begin{vmatrix} 3a & 3b \\ c & d \end{vmatrix} = 3 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$



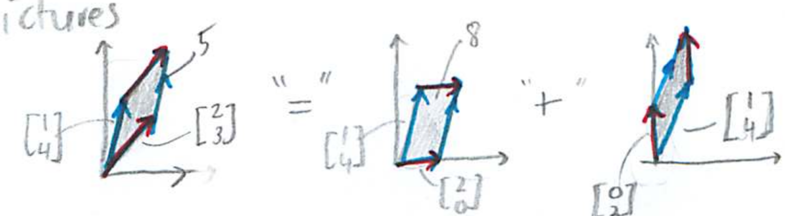
ex. $\begin{vmatrix} 2a & 2b \\ 4c & 4d \end{vmatrix} = 2 \cdot 4 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3.2 Areas add when single rows add:

ex $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}$

formula: $\underbrace{2 \cdot 4 - 3 \cdot 1}_{5} = (\underbrace{2 \cdot 4 - 1 \cdot 0}_{8}) + (\underbrace{0 \cdot 4 - 1 \cdot 3}_{-3})$ ✓

pictures



ex $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -7 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 8 & -4 \end{vmatrix}$

5 $\quad$ 37 $\quad$ −32 ✓

Determinants from the ground up:

The plan: [We Assert Three Properties for Determinants of $n \times n$ matrices]

[
1. $|\mathbb{I}| = 1$ ← unit $n \times n$ hypercube

[Determinant = ± volume of parallelipiped created by row vectors of A]

2. Swapping any two rows of A changes the sign of the determinant.

3. Determinants are multilinear in their rows.
]

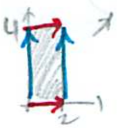
[Can now connect $|A|$ to $|\mathbb{I}| = 1$ and many good things will follow]

Let's fully connect $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$ to $|\mathbb{I}|$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}$$ ← linear in row 1 ← (3)

need to introduce 0's to get to $\mathbb{I}$

$$= \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix}$$



linear in row 2 ← (3)

$$= 2 \cdot 4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 3 \cdot 4 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

0 volume

row swap

0 volume

$$= 8 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 3(-1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (8-3) \underbrace{|\mathbb{I}|}_{=1} = 5. \quad |\mathbb{I}| = 5$$

Next: Show how our standard row operations lead to many results ← $PA = LU$
including $|A| = \pm |U|$
& $|AB| = |A||B|$

#excitement