

Review

Matrixology (Linear Algebra)—Episode 7/26 MATH 124, Spring, 2015

Prof. Peter Dodds

Dept. of Mathematics & Statistics | Vermont Complex Systems Center
Vermont Advanced Computing Core | University of Vermont



Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



1 of 13

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



2 of 13

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



3 of 13

Basics:

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



4 of 13

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



5 of 13

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



6 of 13

Material covered on first midterm:

- ▶ Episodes 1–7, online [here](#).
- ▶ Chapter 1 and Chapter 2 (Sections 2.1–2.7).
- ▶ Chapter 2 is our focus.
- ▶ Knowledge of Chapter 1 as needed for Chapter 2 = solving $A\vec{x} = \vec{b}$.
- ▶ Want ‘understanding’ and ‘doing’ abilities.

These slides are brought to you by:



Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



2 of 13

Stuff to know:

Row, Column, & Matrix Pictures of Linear Systems ($A\vec{x} = \vec{b}$)

- ▶ What dimensions of A mean:
 - ▶ m = number of equations.
 - ▶ n = number of unknowns (x_1, x_2, \dots).
- ▶ How to draw the row and column pictures.
- ▶ Be able to identify row picture (e.g., as representing 2 planes in 3-d).
- ▶ How to convert between the three pictures.

Outline

Review for Challenge Level 1

Solving $A\vec{x} = \vec{b}$ by elimination

Solve four equivalent ways:

1. Simultaneous equations (snore).
2. Row operations on augmented matrix:
 - ▶ Systematically transform $A\vec{x} = \vec{b}$ into $U\vec{x} = \vec{c}$ (and soon into $R\vec{x} = \vec{d}$).
 - ▶ Solve by back substitution.
3. Row operations with E_{ij} and P_{ij} matrices.
4. Factor A as $A = LU$:
 - ▶ Solve two triangular systems by forward and back substitution
 - ▶ First $L\vec{c} = \vec{b}$ then $U\vec{x} = \vec{c}$.
 - ▶ More generally, $PA = LU$ and sometimes $PA = LDU$ where D is the pivot matrix.

Understand number of solutions business:

- ▶ 0, 1, or ∞ : why, when, ...

The Gaussian Eliminator 9000:

Augmented Matrix for $A\vec{x} = \vec{b}$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array} \right]$$

row 2 $\leftarrow R_2 - 2R_1$ (multiplier $l_{21} = 2$, sig hide)

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -3 & 5 \end{array} \right]$$

row 3 $\leftarrow R_3 - R_1$ (multiplier $l_{31} = 1$)

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array} \right]$$

row 3 $\leftarrow R_3 - 2R_2$ (multiplier $l_{32} = 2$)

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array} \right]$$

#eliminated echelon form

easy to solve with back substitution

row operatorify

$U\vec{x} = \vec{c}$

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I & \vec{0} \\ A & \vec{x} = \vec{b} \end{bmatrix}$$

UNIVERSITY
VERMONT

7 of 13

Advanced matrix wrangling 1/2:

See this as the columns of A being combined with weights in vector \vec{x} :

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + \dots + x_n a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \dots + x_n a_{2n} \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \dots + x_n a_{mn} \end{bmatrix}$$

See this as the rows of A being combined with weights in vector \vec{x} :

$$\vec{x}^T A = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} x_1 a_{11} + x_2 a_{21} + \dots + x_n a_{n1} & x_1 a_{12} + x_2 a_{22} + \dots + x_n a_{n2} & \dots & x_1 a_{1n} + x_2 a_{2n} + \dots + x_n a_{nn} \end{bmatrix}$$

Video explanation [here](#)

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I & \vec{0} \\ A & \vec{x} = \vec{b} \end{bmatrix}$$

UNIVERSITY
VERMONT

10 of 13

Stuff to know:

More on $PA = LU$ and $PA = LDU$:

- ▶ If A is m by n then P is m by m , L is m by m , and U is m by n .
- ▶ Be able to find the pivots of A (they live in U and are hidden in A).
- ▶ Understand how elimination matrices (E_{ij} 's) are constructed from multipliers (l_{ij} 's).
- ▶ Understand how L is made up of inverses of elimination matrices
 - ▶ e.g.: $L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} A$.
- ▶ Understand how L is made up of the l_{ij} multipliers.
- ▶ Understand how inverses of elimination matrices are simply related to elimination matrices.

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I & \vec{0} \\ A & \vec{x} = \vec{b} \end{bmatrix}$$

UNIVERSITY
VERMONT

8 of 13

Advanced matrix wrangling 2/2:

break A into columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} & \dots & \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{bmatrix}$$

break B into rows

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \end{bmatrix} \\ \begin{bmatrix} b_{21} & b_{22} & \dots & b_{2n} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \end{bmatrix}$$

Video explanation [here](#)

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I & \vec{0} \\ A & \vec{x} = \vec{b} \end{bmatrix}$$

UNIVERSITY
VERMONT

11 of 13

Matrix algebra:

- ▶ Understand basic matrix algebra (wrangling).
- ▶ Understand matrix multiplication.
- ▶ Understand multiplication order matters.
- ▶ Understand $AB = BA$ is rarely true.

Inverses:

- ▶ Understand identity matrix I .
- ▶ Understand $AA^{-1} = A^{-1}A = I$.
- ▶ Find A^{-1} with Gauss-Jordan elimination.
- ▶ Perform row reduction on augmented matrix $[A | I]$.
- ▶ Understand that finding A^{-1} solves $A\vec{x} = \vec{b}$ but is often prohibitively expensive to do.
- ▶ $(AB)^{-1} = B^{-1}A^{-1}$.

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I & \vec{0} \\ A & \vec{x} = \vec{b} \end{bmatrix}$$

UNIVERSITY
VERMONT

9 of 13

Stuff to know:

Transposes

- ▶ Definition: flip entries across main diagonal.
- ▶ $A = A^T$: A is **symmetric**.
- ▶ Important property: $(AB)^T = B^T A^T$.

Extra pieces:

- ▶ If $A\vec{x} = \vec{0}$ has a non-zero solution, A has no inverse.
- ▶ If $A\vec{x} = \vec{0}$ has a non-zero solution, then $A\vec{x} = \vec{b}$ always has infinitely many solutions.
- ▶ $(A^{-1})^T = (A^T)^{-1}$.

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I & \vec{0} \\ A & \vec{x} = \vec{b} \end{bmatrix}$$

UNIVERSITY
VERMONT

12 of 13

Episode 7/26:
Review

Review for
Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$

