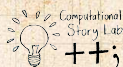


Review

Matrixology (Linear Algebra)—Episode 7/26 MATH 124, Spring, 2015

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$$\begin{bmatrix} I \\ N(A^T) \end{bmatrix}$$

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$



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Sealie & Lambie
Productions



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Outline

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Review

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Review for Challenge Level 1

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$

Material covered on first midterm:

- ▶ Episodes 1–7, online [here](#) .
- ▶ Chapter 1 and Chapter 2 (Sections 2.1–2.7).
- ▶ Chapter 2 is our focus.
- ▶ Knowledge of Chapter 1 as needed for Chapter 2 = solving $A\vec{x} = \vec{b}$.
- ▶ Want 'understanding' and 'doing' abilities.

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Row, Column, & Matrix Pictures of Linear Systems ($A\vec{x} = \vec{b}$)

- ▶ What dimensions of A mean:
 - ▶ m = number of equations.
 - ▶ n = number of unknowns ($x_1, x_2, \dots$).
- ▶ How to draw the row and column pictures.
- ▶ Be able to identify row picture (e.g., as representing 2 planes in 3-d).
- ▶ How to convert between the three pictures.

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Solving $A\vec{x} = \vec{b}$ by elimination

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Solve four equivalent ways:

1. Simultaneous equations (snore).
2. Row operations on augmented matrix:
 - ▶ Systematically transform $A\vec{x} = \vec{b}$ into $U\vec{x} = \vec{c}$ (and soon into $R\vec{x} = \vec{d}$).
 - ▶ Solve by back substitution.
3. Row operations with E_{ij} and P_{ij} matrices.
4. Factor A as $A = LU$:
 - ▶ Solve two triangular systems by forward and back substitution
 - ▶ First $L\vec{c} = \vec{b}$ then $U\vec{x} = \vec{c}$.
 - ▶ More generally, $PA = LU$ and sometimes $PA = LDU$ where D is the pivot matrix.

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$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$

Understand number of solutions business:

- ▶ 0, 1, or ∞ : why, when, ...



The Gaussian Eliminator 9000:

$$\left[\begin{array}{ccc|c} \overbrace{2 \quad -3 \quad 0}^A & \underbrace{3}_{\vec{b}} \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array} \right] \leftarrow \begin{array}{l} \text{Augmented} \\ \text{Matrix} \\ \text{for} \\ A\vec{x} = \vec{b} \end{array}$$

$$R_2' = R_2 - \left(\frac{4}{2} \right) R_1$$

multiplier l_{21}

big tilde $\rightarrow$

$\leftarrow D_1$

equivalent to

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -3 & 5 \end{array} \right]$$

$$R_3' = R_3 - \left(\frac{2}{2} \right) R_1$$

l_{31}

$\leftarrow D_1$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array} \right]$$

rowoperatorify

$$R_3' = R_3 - \left(\frac{2}{1} \right) R_2$$

l_{32}

$\leftarrow D_2$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array} \right]$$

#eliminated

echelon form

$$U\vec{x} = \vec{c}$$

easy to solve
with
back
substitution

$$\left[\begin{array}{c} I \heartsuit \\ A\vec{x} = \vec{b} \end{array} \right]$$

Stuff to know:

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Review

Review for
Challenge Level 1

More on $PA = LU$ and $PA = LDU$:

- ▶ If A is m by n then P is m by m , L is m by m , and U is m by n .
- ▶ Be able to find the pivots of A (they live in U and are hidden in A).
- ▶ Understand how elimination matrices (E_{ij} 's). are constructed from multipliers (l_{ij} 's).
- ▶ Understand how L is made up of inverses of elimination matrices
 - ▶ e.g.: $L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} A$.
- ▶ Understand how L is made up of the l_{ij} multipliers.
- ▶ Understand how inverses of elimination matrices are simply related to elimination matrices.

$$\begin{bmatrix} I & \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$

Matrix algebra:

- ▶ Understand basic matrix algebra (wrangling).
- ▶ Understand matrix multiplication.
- ▶ Understand multiplication order matters.
- ▶ Understand $AB = BA$ is rarely true.

Inverses:

- ▶ Understand identity matrix I .
- ▶ Understand $AA^{-1} = A^{-1}A = I$.
- ▶ Find A^{-1} with Gauss-Jordan elimination.
- ▶ Perform row reduction on augmented matrix $[A|I]$.
- ▶ Understand that finding A^{-1} solves $A\vec{x} = \vec{b}$ but is often prohibitively expensive to do.
- ▶ $(AB)^{-1} = B^{-1}A^{-1}$.

$$\begin{bmatrix} I & \heartsuit \\ A\vec{x} & = \vec{b} \end{bmatrix}$$

Advanced matrix wrangling 1/2:

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$$\underbrace{A \vec{x}}_{m \times n \times n \times 1} = \underbrace{A}_{m \times n} \underbrace{\vec{x}}_{n \times 1} \left\{ \begin{array}{l} m \text{ rows} \\ n \text{ columns} \end{array} \right\}$$

See this as the columns
of A being combined
with weights in vector $\vec{x}$;

$$A \vec{x} = \begin{bmatrix} | & | & | & | \\ \vec{a}_{*1} & \vec{a}_{*2} & \dots & \vec{a}_{*n} \\ | & | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{*=run over all indices}$$

column vectors inside A

$$= x_1 \begin{bmatrix} | \\ \vec{a}_{*1} \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \vec{a}_{*2} \\ | \end{bmatrix} + \dots + x_n \begin{bmatrix} | \\ \vec{a}_{*n} \\ | \end{bmatrix}$$

$$\text{ex } \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}_{2 \times 3} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}_{3 \times 1} = (-1) \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ -2 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}_{2 \times 1}$$

$$\underbrace{\vec{y}^T A}_{1 \times m \times m \times n} = \underbrace{\vec{y}^T}_{1 \times m} \underbrace{A}_{m \times n} \left\{ \begin{array}{l} \text{transpose} \\ n \times m \end{array} \right\}$$

n.b. $\vec{y}$ is $m \times 1$
column vector

row vectors inside A

$$\vec{y}^T A = y_1 \begin{bmatrix} - & \vec{a}_{1*} & - \end{bmatrix} + y_2 \begin{bmatrix} - & \vec{a}_{2*} & - \end{bmatrix} + \dots + y_m \begin{bmatrix} - & \vec{a}_{m*} & - \end{bmatrix}$$

See this as the rows of A
being combined with weights
in vector $\vec{y}^T$.

$$\text{ex } \begin{bmatrix} 3 & 0 & 2 \end{bmatrix}_{1 \times 3} \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} -3 & 4 \end{bmatrix}_{1 \times 2}$$

$$\begin{aligned} & (3) \begin{bmatrix} -1 & 0 \end{bmatrix} \\ & + (0) \begin{bmatrix} 2 & 1 \end{bmatrix} \\ & + (2) \begin{bmatrix} 0 & 2 \end{bmatrix} \end{aligned} = \begin{bmatrix} -3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} I \heartsuit \\ A \vec{x} = \vec{b} \end{bmatrix}$$

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Video explanation [here](#) .

Advanced matrix wrangling 2/2:

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$$C = A B \quad \begin{matrix} m \times k & m \times n & n \times k \\ \text{break into columns} \end{matrix} \rightarrow A \begin{bmatrix} | & | & \dots & | \\ \vec{b}_{*1} & \vec{b}_{*2} & \dots & \vec{b}_{*k} \\ | & | & \dots & | \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ (A\vec{b}_{*1}) & (A\vec{b}_{*2}) & \dots & (A\vec{b}_{*k}) \\ | & | & \dots & | \end{bmatrix}$$

$$\downarrow \begin{matrix} \text{break } A \\ \text{into rows} \end{matrix} \quad \text{2 views} \rightarrow$$

$$\begin{bmatrix} -\vec{a}_{1*} - \\ -\vec{a}_{2*} - \\ \vdots \\ -\vec{a}_{m*} - \end{bmatrix} B$$

$$= \begin{bmatrix} -(\vec{a}_{1*} B) - \\ -(\vec{a}_{2*} B) - \\ \vdots \\ -(\vec{a}_{m*} B) - \end{bmatrix}$$

$\begin{matrix} m \times n & n \times k \\ \text{break into rows} \end{matrix}$

$$\text{ex } A \quad B \quad \begin{matrix} 3 \times 3 & 3 \times 2 \end{matrix} = \begin{bmatrix} A & \vec{b}_{*1} \\ 3 & 0 & 2 & -1 \\ 1 & -2 & 2 & 2 \\ & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} A & \vec{b}_{*1} \\ 3 & 0 & 2 & -1 \\ 1 & -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\parallel \quad \begin{bmatrix} \vec{a}_{1*} & B \\ 3 & 0 & 2 & -1 & 0 \\ & 2 & 1 & 0 & 2 \end{bmatrix} \quad \parallel \quad \begin{bmatrix} -3 \\ -5 \end{bmatrix} \quad \parallel \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\parallel \quad \begin{bmatrix} \vec{a}_{2*} & B \\ 1 & -2 & 2 & -1 & 0 \\ & 2 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -5 & 2 \end{bmatrix}$$

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Video explanation [here](#) .

Stuff to know:

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Review for
Challenge Level 1

Transposes

- ▶ Definition: flip entries across main diagonal.
- ▶ $A = A^T$: A is **symmetric**.
- ▶ Important property: $(AB)^T = B^T A^T$.

Extra pieces:

- ▶ If $A\vec{x} = \vec{0}$ has a non-zero solution, A has no inverse.
- ▶ If $A\vec{x} = \vec{0}$ has a non-zero solution, then $A\vec{x} = \vec{b}$ always has infinitely many solutions.
- ▶ $(A^{-1})^T = (A^T)^{-1}$.

$$\begin{bmatrix} I \\ A\vec{x} = \vec{b} \end{bmatrix}$$

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$

$$\begin{bmatrix} I \heartsuit \\ A\vec{x} = \vec{b} \end{bmatrix}$$