Introduction

Matrixology (Linear Algebra)—Episode 1/26
MATH 124, Spring, 2015

Prof. Peter Dodds

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Vermont Advanced Computing Core | University of Vermont

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Outline

Basics:

- Instructor: Prof. Peter Dodds
- Lecture room and meeting times:
  Angell B112,
  Tuesday and Thursday, 1:00 pm to 2:15 pm
- Office: Farrell Hall, second floor, Trinity Campus
- E-mail: peter.dodds@uvm.edu
- Course website:
  http://www.uvm.edu/ pdodds/teaching/courses/2015-01UVM-124
- Textbook: "Introduction to Linear Algebra" (3rd or 4th editions) by Gilbert Strang (published by Wellesley-Cambridge Press).

Our Textbook of Excellence:

4th Edition ☑ 3rd Edition ☑ Unhelpful ☐

- "Introduction to Linear Algebra" by Gil Strang;
- Textbook website:
  http://math.mit.edu/linearalgebra/
- MIT Open Courseware site for 18.06 (=Linear Algebra):
  http://ocw.mit.edu/...linear-algebra-spring-2010/

Yesness:

Money quote from George Cobb's review of Strang's book:

Do you want a book written by a mathematician with a lifetime experience using linear algebra to understand important, authentic, applied problems, a former president of the Society for Industrial and Applied Mathematics, ...

or do you want a book shaped mainly by the [a]esthetics of pure mathematicians with only a weak, theoretical connection to how linear algebra is used in the natural and social sciences?

- George Cobb: Robert L. Rooke Professor of Mathematics and Statistics, Mount Holyoke College
- Full review here [amazon]

Gil Strang, Exalted Friend of the Matrix:

- Professor of Mathematics at MIT since 1962.

These are 121 cupcakes with my favorite -1, 2, -1 matrix. It was the day before Thanksgiving and two days before my birthday. A happy surprise.

- Many awards including MAA Haimo Award for Distinguished College or University Teaching of Mathematics
- Rhodes Scholar.
- Legend.

- More on Laplacian matrices, graphs, and other madnesses here.
- (Strang's Wikipedia page is here.

Admin:

Potential paper products:

1. Outline

Papers to read:

1. "The Fundamental Theorem of Linear Algebra" [2]
2. "Too Much Calculus" [3]

Office hours:

- 2 to 2:45 pm, Mondays; 3 to 3:45 pm Tuesdays; and 1 to 2:30 pm Wednesdays,
  Farrell Hall, second floor, Trinity Campus

Grading breakdown:

1. Levels (40%)
   - Ten one-week assignments.
   - Lowest assignment score will be dropped.
   - The last assignment cannot be dropped!
   - Each assignment will have a random bonus point question which has nothing to do with linear algebra.
2. Challenge Levels (35%)
   - Three 75 minutes tests distributed throughout the course, all of equal weighting.
3. Final Boss Level (25%)
   - $\le$ Three hours of joyful celebration.
   - Monday, May 4, 1:30 pm to 4:15 pm, in Angell B112.

Grading breakdown:

4. Homework (0%)—Problems assigned online from the textbook. Doing these exercises will be most beneficial and will increase happiness.
5. General existence—it is extremely desirable that students attend class, and class presence will be taken into account if a grade is borderline.

Questions are worth 3 points according to the following scale:

- 3 = correct or very nearly so.
- 2 = acceptable but needs some revisions.
- 1 = needs major revisions.
- 0 = way off.

Schedule: The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

Week # (dates)	Tuesday	Thursday
1 (1/13 and 1/15)	$\mathbb{A}\vec{x} = \vec{b}$	$\mathbb{A}\vec{x} = \vec{b}$ + Level 1
2 (1/20 and 1/22)	$\mathbb{A}\vec{x} = \vec{b}$	$\mathbb{A}\vec{x} = \vec{b}$ + Level 2
3 (1/27 and 1/29)	$\mathbb{A}\vec{x} = \vec{b}$	$\mathbb{A}\vec{x} = \vec{b}$ + Level 3
4 (2/3 and 2/5)	$\mathbb{A}\vec{x} = \vec{b}$ and review	*Challenge Level 1*
5 (2/10 and 2/12)	Big picture	Big picture + Level 4
6 (2/17 and 2/19)	Big picture	Big picture + Level 5
7 (2/24 and 2/26)	Big picture	Big picture + Level 6
– (3/3 and 3/5)	Spring recess	Spring recess
8 (3/10 and 3/12)	Big picture and review	*Challenge Level 2*
9 (3/17 and 3/19)	Eigenstuff	Eigenstuff + Level 7
10 (3/24 and 3/26)	Eigenstuff	Eigenstuff + Level 8
11 (3/31 and 4/2)	Eigenstuff	Eigenstuff + Level 9
12 (4/7 and 4/9)	Eigenstuff and review	*Challenge Level 3*
13 (4/14 and 4/16)	SVD + Level 10	SVD
14 (4/21 and 4/13)	SVD	SVD
15 (4/28)	SVD	—

Important dates:

1. Classes run from Monday, January 12 to Wednesday, April 29.
2. Add/Drop, Audit, Pass/No Pass deadline—Monday, January 26.
3. Last day to withdraw—Monday, March 27.
4. Reading and Exam period—Thursday, April 30 to Friday, May 8.

More stuff:

Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.

More stuff:

Being good people:

1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab).
2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.
3. Finally, as in all UVM classes, Academic honesty will be expected and departures will be dealt with appropriately. See `http://www.uvm.edu/cses/` for guidelines.

Even more stuff:

Late policy: Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

Computing: Approximately 2 out of 10 questions per assignment will be Matlab based.

Note: for assignment problems, written details of calculations will be required.

Why are we doing this?

Big deal: Linear Algebra is a body of mathematics that deals with discrete problems.

Many things are discrete:

- Information (0's & 1's, letters, words)
- People (sociology)
- Networks (the Web, people again, food webs, ...)
- Sounds (musical notes)

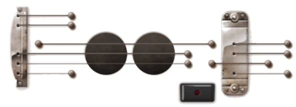
Even more:

If real data is continuous, we almost always discretize it (0's and 1's)

Why are we doing this?

Linear Algebra is used in many fields to solve problems:

- Engineering
- Computer Science
- Physics
- Biology
- Ecology
- Economics
- Science of the Sociotechnocene

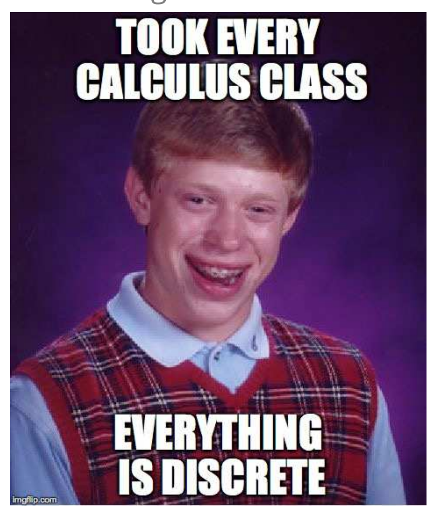


Big example: Google's Pagerank

Some truth:

- Linear Algebra is as important as Calculus...
- Calculus ≡ the blue pill...

Why are we doing this?



You are now choosing the red pill:

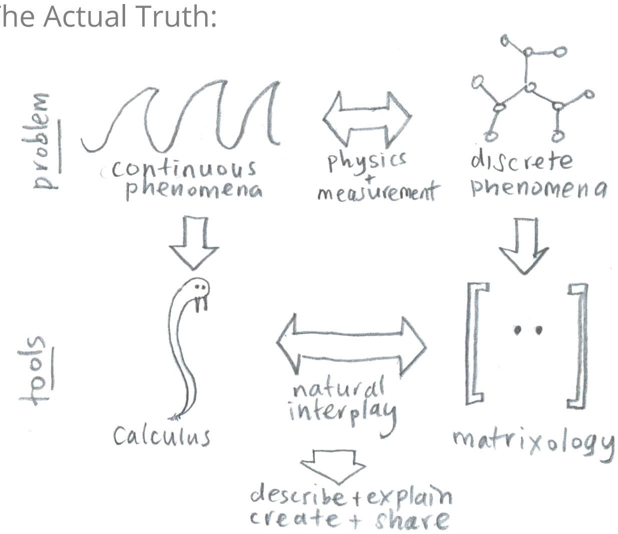
The Truth:

- Calculus is the Serpent's Mathematics.

The Platypus of Truth:

- Platypuses are masters of Linear Algebra.

The Actual Truth:



Matrices as gadgets:

A matrix A transforms a vector $\vec{x}$ into a new vector $\vec{x}'$ through matrix multiplication (whatever that is):

$$\vec{x}' = A\,\vec{x}$$

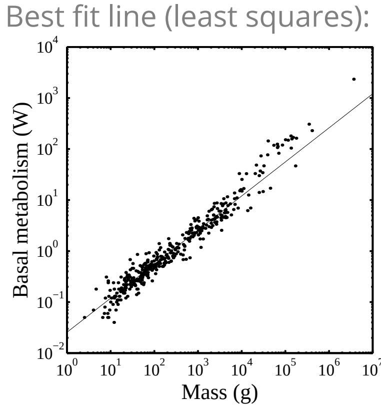
We can use matrices to:

- Grow vectors
- Shrink vectors
- Rotate vectors
- Flip vectors
- Do all these things in different directions
- Reveal the true ur-dystopian reality.

Digital photographs are matrices:

Usually three matrices: RGB color model.

Best fit line (least squares):

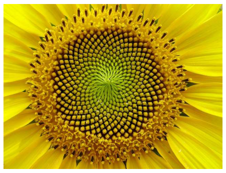


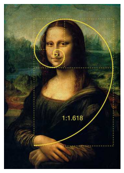
- Linear algebra does this beautifully;
- Calculus version is clunky. And evil.

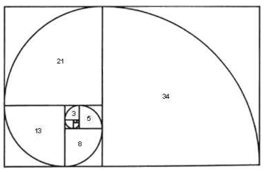
- From "Re-examination of the '3/4' law of metabolism" [1] Dodds, Rothman, and Weitz, Journal of Theoretical Biology, 209, 9–27, 2001

The many delights of Eigenthings:

Using Linear Algebra we'll somehow connect:

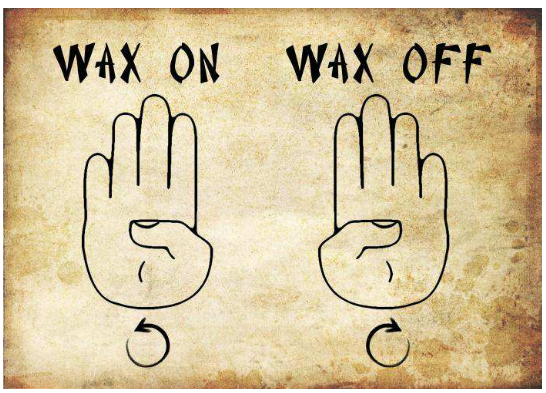






- Fibonacci Numbers,
- Golden Ratio,
- Spirals,
- Sunflowers, pine cones, ...
- Harvard Square.

This is a math course:



http://www.pimpartworks.com/artwork/randomsteveo/Wax-On-Wax-Off

- It's all connected. "More later."

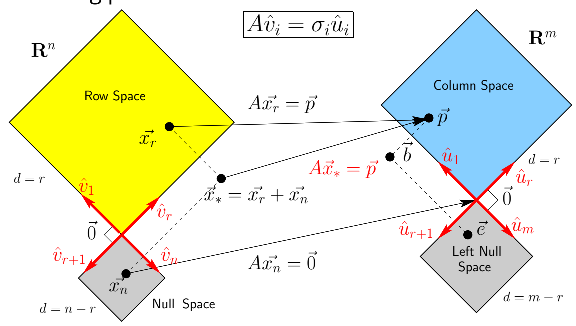
Three key problems of Linear Algebra

1. Given a matrix A and a vector $\vec{b}$, find $\vec{x}$ such that
$$A\vec{x} = \vec{b}.$$
2. Eigenvalue problem: Given A, find λ and $\vec{v}$ such that
$$A\vec{v} = \lambda\vec{v}.$$
3. Coupled linear differential equations:
$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = A\, y(t)$$

- Our focus will be largely on #1, partly on #2.

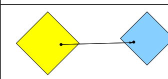
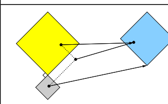
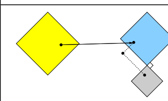
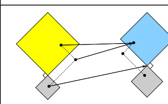
Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:



What is going on here? We have 26 episodes to find out...

The fourfold ways of $A\vec{x} = \vec{b}$:

case	example R	big picture	# solutions
$m = r$, $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		1 always
$m = r$, $n > r$	$\begin{bmatrix} 1 & 0 & \text{☕}_1 \\ 0 & 1 & \text{☕}_2 \end{bmatrix}$		∞ always
$m > r$, $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$		0 or 1
$m > r$, $n > r$	$\begin{bmatrix} 1 & 0 & \text{🚲}_1 \\ 0 & 1 & \text{🚲}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		0 or ∞

Our new BFF: $A\vec{x} = \vec{b}$

Broadly speaking, $A\vec{x} = \vec{b}$ translates as follows:

- $\vec{b}$ represents reality (e.g., music, structure)
- A contains building blocks (e.g., notes, shapes)
- $\vec{x}$ specifies how we combine our building blocks to make $\vec{b}$ (as best we can).

How can we disentangle an orchestra's sound?

- Radiolab's amazing piece: A 4-Track Mind

What about pictures, waves, signals, ...?

Is this your left nullspace?:

Linear Algebra compliments/putdowns:

- Wow, you have such a tiny/huge [delete as applicable] left nullspace!

- See also: The Dunning-Kruger effect.

Our friend $A\vec{x} = \vec{b}$

What does knowing $\vec{x}$ give us?

If we can represent reality as a superposition (or combination or sum) of simple elements, we can do many things:

- Compress information
- See how we can alter information (filtering)
- Find a system's simplest representation
- Find a system's most important elements
- See how to adjust a system in a principled way

Three ways to understand $A\vec{x} = \vec{b}$:

- Way 1: The Row Picture
- Way 2: The Column Picture
- Way 3: The Matrix Picture

Example:

$$\begin{array}{rcrcl} -x_1 & + & x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 4 \end{array}$$

- Call this a 2 by 2 system of equations.
- 2 equations with 2 unknowns.
- Standard method of simultaneous equations: solve above by adding and subtracting multiples of equations to each other = Row Picture.

Three ways to understand $A\vec{x} = \vec{b}$:

Row Picture—what we are doing:

- (a) Finding intersection of two lines
- (b) Finding the values of x_1 and x_2 for which both equations are satisfied (true/happy)
- A splendid and deep connection: (a) Geometry $\rightleftharpoons$ (b) Algebra

Three possible kinds of solution:

1. Lines intersect at one point —One, unique solution
2. Lines are parallel and disjoint —No solutions
3. Lines are the same —Infinitely many solutions

Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:

See

$$\begin{array}{rcrcl} -x_1 & + & x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 4 \end{array}$$

as

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

General problem

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$$

- Column vectors are our 'building blocks'
- Key idea: try to 'reach' $\vec{b}$ by combining (summing) multiples of column vectors $\vec{a}_1$ and $\vec{a}_2$.

Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:

- Intuitive.
- Generalizes easily to many dimensions.

Three possible kinds of solution:

1. $\vec{a}_1 \nparallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \nparallel \vec{b}$: No solutions
3. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: infinitely many solutions

(assuming neither $\vec{a}_1$ or $\vec{a}_1$ are $\vec{0}$)

Three ways to understand $A\vec{x} = \vec{b}$:

Difficulties:

- Do we give up if $A\vec{x} = \vec{b}$ has no solution?
- No! We can still find the $\vec{x}$ that gets us as close to $\vec{b}$ as possible.
- Method of approximation—very important!
- We may not have the right building blocks but we can do our best.

Three ways to understand $A\vec{x} = \vec{b}$:

The Matrix Picture:

Now see

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

as

$$A\vec{x} = \vec{b} : \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

A is now an operator:

- A transforms $\vec{x}$ into $\vec{b}$.
- Roughly speaking, A does two things to $\vec{x}$:
  1. Rotation/Flipping
  2. Dilation (stretching/contraction)

The Matrix Picture

Key idea in linear algebra:

- Decomposition or factorization of matrices.
- Matrices can often be written as products or sums of simpler matrices
- $A = LU$, $A = QR$, $A = U\Sigma V^T$, $A = \sum_i \lambda_i \vec{v}\vec{v}^T$, ...

More Truth about Mathematics:

The Colbert Report on Math (February 7, 2006)

"Equations are the Devil's sentences."

References I

[1] P. S. Dodds, D. H. Rothman, and J. S. Weitz.
Re-examination of the "3/4-law" of metabolism.
Journal of Theoretical Biology, 209:9–27, 2001. pdf

[2] G. Strang.
The fundamental theorem of linear algebra.
The American Mathematical Monthly, 100(9):848–855, 1993. pdf

[3] G. Strang.
Too much calculus, 2002.
SIAM Linear Algebra Activity Group Newsletter. pdf