Dispersed: Saturday, March 21, 2015.
Due: By start of lecture, Tuesday, April 7, 2015.
Sections covered: 4.4, 6.1, a smidgeon of 5.1.

Some useful reminders:
 Instructor: Prof. Peter Dodds
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 Course website: http://www.uvm.edu/~pdodds/teaching/courses/2015-01UVM-124

• All questions are worth 3 points unless marked otherwise.
• Please use a cover sheet and write your name on the back and the front of your assignment.
• You must show all your work clearly.
• You may use Matlab to check your answers for non-Matlab questions (usually Qs. 1–8).
• Please list the names of other students with whom you collaborated.

1. (Q 5, Section 4.4) Find two orthogonal vectors in the plane $x_1 + x_2 + 2x_3 = 0$. Make these vectors into an orthonormal basis.

   Do this by (a) finding a basis for the null space of the matrix $A = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$ and (b) using the Gram-Schmidt Process to generate an orthogonal basis.

2. First, please absorb this short video:

   **Help—Definition of Orthogonal matrices (6:36):**
   [http://www.youtube.com/v/hQG6w7Myqtw?rel=0](http://www.youtube.com/v/hQG6w7Myqtw?rel=0)

   Now show that any orthogonal matrix $Q$ preserves lengths and angles when it transforms vectors. Remember that orthogonal matrices have the special property that $Q^TQ = I$ (also: let’s allow $Q$ to be an $m$ by $n$ matrix where $m \geq n$—$Q$ need not be square for this question).

   (a) Lengths: Show that $Q\vec{x}$ has the same length (magnitude) as $\vec{x}$.

   **Hint**—Showing that the squares of the lengths match is sufficient.

   (b) Angles: Show that the cosine of the angle between two vectors $\vec{x}$ and $\vec{y}$ is the same as that for the angle between $Q\vec{x}$ and $Q\vec{y}$. 
**Hint**—We can show that the cosine of the angle between the $\vec{x}$ and $\vec{y}$ is the same as the cosine of the angle between $Q\vec{x}$ and $Q\vec{y}$. By rearranging the dot product formula,

$$\vec{x}^T\vec{y} = ||\vec{x}|| ||\vec{y}|| \cos \theta,$$

where $\theta$ is the angle between $\vec{x}$ and $\vec{y}$, we have

$$\cos \theta = \frac{\vec{x}^T\vec{y}}{||\vec{x}|| \, ||\vec{y}||}.$$

From part (a), we already have that the bottom of this fraction doesn’t change ($||Q\vec{x}|| = ||\vec{x}||$ and $||Q\vec{y}|| = ||\vec{y}||$), so we need to just show that the numerator remains the same.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 1 \end{bmatrix},$$

and its corresponding $Q$ (determined by Gram-Schmidt)

$$Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 0 & 1/3 \\ -1/\sqrt{2} & 2/3 \end{bmatrix},$$

find the upper triangular matrix $R$ so that $A = QR$. (Look up the formula for $R$).

4. *(Q 5, Section 6.1)* Find the eigenvalues and eigenvectors of $A$ and $A^2$:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

How are the eigenvectors and eigenvalues of $A$ and $A^2$ related?

5. Compute the eigenvalues (but not the eigenvectors) of $A^{-1}$ for the $A$ given above.

How are the eigenvalues of $A$ and $A^{-1}$ related?

6. Find the eigenvalues (but not the eigenvectors) of $A + 3I$ where for $A$ as given in question 4.

Now what’s the relationship between the eigenvalues of $A$ and $A + 3I$?

7. Building on your findings in the previous questions:

(a) Show that if $A^{-1}$ exists, then the eigenvalues of $A^{-1}$ are inverses of the eigenvalues of $A$, and that they share the same eigenvectors.

(b) Show that $\lambda^2$ is an eigenvalue of $A^2$ if $\lambda$ is an eigenvalue of $A$, and that they share the same eigenvectors.
(c) Show that $\lambda + k$ is an eigenvalue of $A + kI$ (where $k$ is any number and $I$ is the identity matrix) if $\lambda$ is an eigenvalue of $A$, and that they share the same eigenvectors.

Note: You are deriving these results for arbitrary $n$ by $n$ matrices.

**Hint**—please see this tweet:
https://twitter.com/matrixologyvox/status/582724638561693696

8. (Q 2, 5.1)

Watch the following video for help on this question and determinants in general.

**Help—Determinants from the ground up (19:23):**
http://www.youtube.com/v/CuFUXcXZrQ?rel=0

Given a 3 by 3 matrix $A$ has determinant equal to 5, find (a) $\det(1/2 \ A)$, (b) $\det(-A)$, (c) $\det(A^2)$, and (d) $\det(A^{-1})$.

Here are some things you are allowed to know (even if we haven’t covered them in class yet): $|tA| = t^n|A|$ and $|AB| = |A||B|$ and $|I| = 1$.

9. Matlab question:

Use Matlab’s eig command to find the eigenvalues and eigenvectors of the matrix we studied above:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$ 

Basic usage:

$[V,\text{Lambda}] = \text{eig}(A)$

The unit eigenvectors are the columns of $V$ and the eigenvalues are on the main diagonal of $\text{Lambda}$.

Confirm that you have a match with your pencil and paper calculations.

10. Matlab question:

In class, we talked about a random texter, lost on a network of paths connecting five locations labelled 1–5.

We found the following transition matrix:

$$A = \begin{bmatrix} 0 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/4 & 0 & 0 \\ 1/2 & 1/2 & 0 & 1 & 1 \\ 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 \end{bmatrix},$$

which connects the probability our magic rectangle user is at location $i$ at time $t$, $x_{i,t}$, via

$$x_{i,t+1} = A x_t,$$

where $[x_t]_i = x_{i,t}$.  

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(a) Find the eigenvalues and eigenvectors of $A$ using Matlab.

(b) Let's start our texter at location 3 at time $t = 0$:

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$  

Using Matlab to compute large powers, what does $\vec{x}_t$ tend toward as $t \to \infty$? Does it matter where our texter starts?

(c) Given $A$’s eigenvalues, explain why or why not $A$ is invertible.

11. (Bonus question, 1 point) How many players does each side have on the field in an Australian Rules Football match, and how long is a typical ground (playing area)?