LINEAR ALGEBRA

Chapters 5&6. (6 with 5 squeezed in)

Ch 6
L 1

* Square Matrices! (mostly)

we'll return to $m \times n$ at the end of ch 6.

before lectures on chapter 5

* Think now of $\mathbb{A}$ as a gadget matrix that "transforms" $\vec{x}$ into a new vector

$$\mathbb{A}\vec{x}_1 = \vec{x}_2 \leftarrow \text{new } \vec{x}$$

$\mathbb{A}$ might rotate, flip, stretch, project

chop out pieces of $\vec{x}$.

better

$$\vec{a}_2 = \vec{c}_2 + \hat{q}_1 \hat{q}_1^T \vec{a}_2$$

$$\hat{q}_2^T \times \quad \hat{q}_2^T \vec{a}_2 = \hat{q}_2^T \vec{c}_2$$

$$= \hat{q}_2^T \hat{q}_2 \cdot \|\vec{c}_2\|$$

1

$\|\vec{c}_2\| \hat{q}_2$ ✓

$(\hat{q}_2^T \vec{a}_2)\, \hat{q}_2$ ✓

* Especially interested in what happens when we make repeated transformations:

What is $\mathbb{A}^n \vec{x}$?? (n large)

3 reasons why we care about $\mathbb{A}^n$:

Ch 6 L2

Linear Algebra (Square matrices only)!!

$\mathbb{A}^n$: why we care.

or

Reasons to love ~~arbitrary~~ abitrary powers of square matrices.

(1) $\vec{x}_{t+1} = \mathbb{A}\vec{x}_t$

system evolution may be expressed as matrix multiplicat.

eg. ~~probabil~~

$\vec{x}_t = \begin{bmatrix} x_{1,t} \\ \vdots \\ x_{3,t} \end{bmatrix}$ contains probability of being in state i at time t

$\mathbb{A}$ is a "transition" matrix containing conditional probabilities

$(\mathbb{A})_{ij} = P_{i,j}$ = prob of moving to state i given in state j

must have $\sum_{i=1}^{n} P_{i,j} = 1$ so columns add to 1

(Markov chain)

$\vec{x}_t = \mathbb{A}^t \vec{x}_0$

how this behaves governs system

(2) Solving linear diff eq's

$\frac{dy}{dt} = at \Rightarrow y = y_0 e^{at}$

$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= ax_1 + bx_2 \\ \frac{dx_2}{dt} &= cx_1 + dx_2 \end{aligned}\right\} \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{constants}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\frac{d}{dt}\vec{x} = \mathbb{A}\vec{x}$.

Solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\mathbb{A}t} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\mathbb{I} + \mathbb{A} + \frac{\mathbb{A}^2}{2!} + \frac{1}{3!}\mathbb{A}^3 + \dots + \frac{1}{n!}\mathbb{A}^n + \dots$$

need to be able to handle these powers

(3) Solving diff. eqs

e.g.

$F_{n+2} = F_{n+1} + F_n$

$\vec{x}_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$

$\vec{x}_{n+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}_n = \dots = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n \vec{x}_0$

$\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If we can compute $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then we have a way of solving for the n^{th} Fibonacci #.

(4) other reasons: for fun

: self-flagellation.

: etc.

To get to $\mathbb{A}^n$, we must first go through eigen-values. e-values also deepen our appreciation for $\mathbb{A}\vec{x} = \vec{b}$

~~Eigen = self or same~~ own.

Note change of focus:

Interested in how $\mathbb{A}$ transforms $\vec{x}$

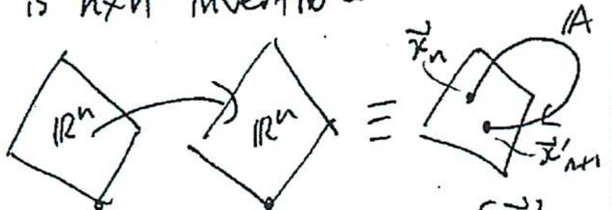
Before $\mathbb{A}\vec{x} = \vec{b}$ (known) task = find $\vec{x}$

Now $\mathbb{A}\vec{x}_n = \vec{x}_{n+1}$ or $\mathbb{A}\vec{x} = \vec{x}'$

$\mathbb{A}$ known understand how ... $\vec{x}$

Linear Algebra

A is $n \times n$ invertible



nullspace $= \{\vec{0}\}$
leftnullsp $= \{\vec{0}\}$.

An example:

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

observe

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (3/2)^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\vdots$

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (3/2)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda^n = e^{n(\ln \lambda)}$

at most n evectors

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \left(\frac{1}{2}\right)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

what about $A \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ ~~say?~~ for example

$$A \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

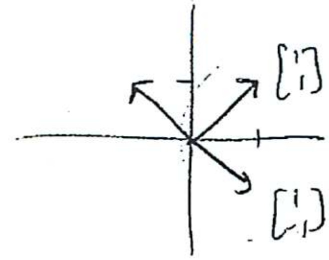
$$A^2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 2 \end{bmatrix} \leftarrow$$ not as clean

but $$A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \overset{\Leftrightarrow}{=} A \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow A \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A^2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \left(\frac{3}{2}\right)^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{\left(\frac{1}{2}\right)^2}_{\text{disappears}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Ch6 L3



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ~> natural basis for "A"

A stretches a vector in the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ direction

contracts a " " " $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ direct

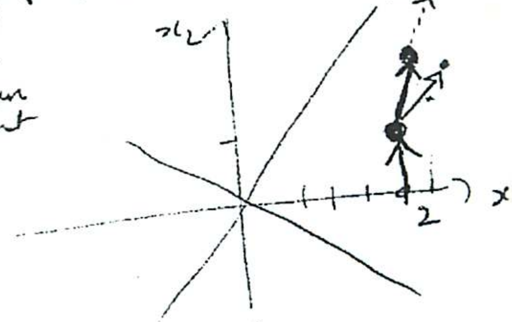
These directions are A's "own" directions

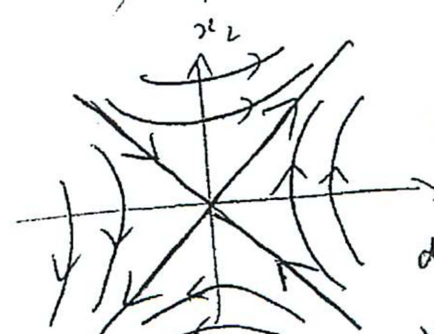
German for own: Eigen.
pwn: Pigen

Eigenvectors for A are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Eigenvalues are $3/2$, $1/2$

makes an excellent basis





directions A moves vectors in.

$\lambda > 1$ ~> vectors grows in exponentially direction of evector

$\lambda = 1$ ~> stay the same

$\lambda = 0$ → vanish straight away

$0 < \lambda < 1$ → shrink

$\lambda < 0$ ~> jump back & forth across origin

Linear Algebra

How do we find eigenvectors?

Solve

$$A\vec{x} = \lambda\vec{x}$$

scalar, $\lambda \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

key manoeuvres

$$(A - \lambda I)\vec{x} = \vec{0}$$

For $\vec{x}$ to be non-zero, λ must be such that $(A - \lambda I)$ has

1. rank $r < n$
2. $\Leftrightarrow$ have no inverse
3. be singular
4. have a non-trivial nullspace $n - r > 0$
5. have ~~zero~~ determinant $= 0$.

$\det(A - \lambda I)$ note notation

$$= |A - \lambda I|$$

$$= \begin{vmatrix} 1-\lambda & 1/2 \\ 1/2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - 1/4$$

$$= \lambda^2 - 2\lambda - 3/4$$

$$= (\lambda - 3/2)(\lambda - 1/2)$$

the Characteristic Equation

$\lambda_1 = 3/2,\ 1/2 = \lambda_2$

remember these?

Now find solution to $(A - 3/2\, I)\vec{x} = \vec{0}$; $-1/2$

~~3/6~~ Ch 6 L4

In other words, find basis vectors for nullspaces of $A - 3/2\, I$ & $A - 1/2\, I$

$$(A - \tfrac{3}{2}I) = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{array}\right] \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

These eigenvectors are the "natural basis" for A.

Special Matrices have special evalues & evectors

MISSED.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1, \lambda_2 = 1$

$P_{[1,0]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\lambda_1 = 1$ $\lambda_2 = 0$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$P_{[1,1]} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$R_{90°} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda_1 = +i$ — $\vec{v}_1 = [$; $\lambda_2 = -i$ — $\vec{v}_2$

flip $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda_{1,2} = \pm 1$ $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

if A is symmetric → evectors are orthogonal

A is singular → $\lambda = 0$ is an ~~evecto~~ evalue

Pieces that follow:

$\det \mathbb{A} = \pm \det \mathbb{U}$ as in $\mathbb{A} = \mathbb{L}\,\mathbb{U}$

$\pm$ depends on # row swaps

$\det \mathbb{D}$ or ∇ = product of diagonal elements

$\det \mathbb{A} = \pm$ product of $\mathbb{A}$'s pivots.

Reasons.

(a) if $\mathbb{A}$ has two rows the same the $\det(\mathbb{A}) = 0$ → parallel is missing a side!!

sign change $\begin{vmatrix} 3 & 4 \\ 3 & 4 \end{vmatrix} = 3\cdot 4 - 3\cdot 4 = 0$

$\begin{vmatrix} 3 & 6 & 6 \\ 3 & 6 & 6 \\ 2 & -2 & 1 \end{vmatrix} \cancel{=} - \begin{vmatrix} 3 & 6 & 6 \\ 3 & 6 & 6 \\ 2 & -2 & 1 \end{vmatrix} \stackrel{\therefore}{=} 0.$

(b) row reduction doesn't change val of det.

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - l \underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix}}_{=0 \text{ from (a)}}$

$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} - la_{11} & a_{22} - la_{12} \end{vmatrix}$

$\Rightarrow \det \mathbb{A} = \pm \det \mathbb{U}$

↑ 1 lec!

(c) if $\mathbb{A}$'s rows are lin dep, we can make a row of 0's with row reductn → use multilinearity

$\Rightarrow \det(\mathbb{A}) = 0$

now finding a volume of parallelipiped with one side of zero length!

(h) $|\mathbb{A}| = |\mathbb{A}^T|$

$\Downarrow$

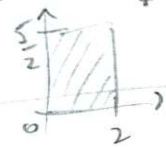
$|\mathbb{P}\mathbb{A}| = |\mathbb{L}\,\mathbb{U}|$ ← use row → columns ← multilinearity

(d) keep going past $\mathbb{U}$ to fully reduced form. Caution — do not divide by pivot!!

Ch5 2a

call this $\mathbb{R}^e_{\mathbb{A}}$ ← echelon form (n.b $\mathbb{D}\,\mathbb{R}_{\mathbb{A}} = \mathbb{R}^e_{\mathbb{A}}$; $\mathbb{D}$ ↑ diag w. pivots)

$$\begin{array}{ccc} \mathbb{A} & \sim & \mathbb{R}^e_{\mathbb{A}} \\ \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \underset{\text{equal}}{=} & \begin{vmatrix} 2 & 0 \\ 0 & 5/2 \end{vmatrix} \end{array} = \underbrace{2\cdot\tfrac{5}{2}}_{\text{pivots}} \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=1.} = 5$$



$\Rightarrow \det \mathbb{A} = \pm \det \mathbb{U}_{\mathbb{A}} = (\pm) \prod_{i=1}^{n} p_i^{\mathbb{A}}$ (sign changes; $\mathbb{A}$'s pivots)

→ turns out we need another way for evalue problem.. big deal.

(e) $|\mathbb{A}\mathbb{B}| = |\mathbb{A}|\,|\mathbb{B}|$ * see Ch5 2b

product of volumes.

n.b $|\mathbb{A} + \mathbb{B}| \neq |\mathbb{A}| + |\mathbb{B}|$ multilinear only!!!

~~Sneaky reason~~ $|\mathbb{B}| \neq 0$.

~~Show $\det\left(\frac{1}{|\mathbb{B}|}\mathbb{A}\mathbb{B}\right)$ satisfies $\mathbb{DP}$ 1, 2, & 3~~

~~(1) if $\mathbb{A} = \mathbb{I}$, both $\mathbb{I}$ & $\frac{1}{|\mathbb{B}|}\mathbb{I}\,|\mathbb{B}| = \mathbb{I}$ $\therefore$ have det 1. ✓~~

~~(2) if we swap two rows of $\mathbb{A}$, we swap the same two rows of $\mathbb{A}\mathbb{B}$. $\Rightarrow$ both dets change sign~~

~~(3) (i) if we multiply a row of $\mathbb{A}$ by t, we multiply the same row of $\mathbb{A}\mathbb{B}$ by t.~~

~~(ii) if we split a row of $\mathbb{A}$, we split the same row of $\mathbb{A}\mathbb{B}$ in same way $\Rightarrow$ dets add.~~

(f) (e) $\Rightarrow$ $|\mathbb{A}^{-1}| = \dfrac{1}{|\mathbb{A}|}$

n.b. if $|\mathbb{A}| = 0$, $|\mathbb{A}^{-1}| = \infty$ ouch!!

Ch5 2b

What can we say about

(e) $|\mathbb{A}\mathbb{B}|$?

know

$\mathbb{P}\mathbb{E}\mathbb{A} = \mathbb{R}^{echelon}_{\mathbb{A}}$

perms, eliminat

$|\mathbb{A}| = \underset{\text{row swaps}}{\pm} |\mathbb{R}^{e}_{\mathbb{A}}|$

$|\mathbb{P}\mathbb{E}(\mathbb{A}\mathbb{B})| = |\mathbb{R}^{e}_{\mathbb{A}}\mathbb{B}|$

$= \left(\prod_{i=1}^{N} P_i^{\mathbb{A}}\right)|\mathbb{I}\mathbb{B}|$

$= |\mathbb{A}||\mathbb{B}|$

* back to (f)

Example: ← used again later.

Use row ops to find determinant of $\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ *

$|\mathbb{A}| = 2.$

(g) det $D, \nabla = \prod$ entries on diagonal

$$\begin{bmatrix} 4 & -1 & 43 & 7 \\ 0 & -3 & 6 & 8 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & & 7 \end{bmatrix} = 4\times(-3)\times(2)\times(7)$$

(i) One or more pivots $= 0$

$\Leftrightarrow |\mathbb{A}| = 0.$

$|\mathbb{P}\mathbb{A}| = |\mathbb{L}\mathbb{U}|$

$|\mathbb{P}| = |\mathbb{P}^T| = \pm 1$

bc $\mathbb{P}^{-1} = \mathbb{P}^T$

$\therefore |\mathbb{P}^T\mathbb{P}| = 1 = \underbrace{|\mathbb{P}^T||\mathbb{P}|}_{\pm 1}$

t^u

Ch 5 3

see 3a

usual way for evalue probs

a

Linear Algebra

[The eigenvalue problem:
For given $\mathbb{A}$, find all $\lambda, \vec{v}$
$n \times n$ such that $\mathbb{A}\vec{v} = \lambda\vec{v}$

We turned this into a problem of algebra: ~~Solve~~ ~~Find~~ ← nth order polynomial in λ.

① find λ s.t $|\mathbb{A} - \lambda\mathbb{I}| = 0$ ← det

② find $\vec{v}$

$\left|\begin{matrix} u_{11} & 0 & 0 \\ & u & \end{matrix}\right| + \left|\begin{matrix} 0 & a_{12} & 0 \\ & u & \end{matrix}\right| + \left|\begin{matrix} 0 & 0 & a_{13} \\ & u & \end{matrix}\right|$,

Part ② is "e...

bigger piz ~~with~~ → (Find... solving ...

biggest piz → Really: finding basis for null space of $(\mathbb{A} - \lambda\mathbb{I})$

① is the tricky bit:

[We have to compute determinants of $n \times n$ matrices

So: Determinantarama

(Reason: if $|\mathbb{A} - \lambda\mathbb{I}| = 0$, ← we still have to prove this & we will!
$(\mathbb{A} - \lambda\mathbb{I})$ has no inverse
⟺ nullspace of $\mathbb{A} - \lambda\mathbb{I}$ is non-trivial
⟺ evectors exist!

2×2's $\left|\begin{matrix} a_{11} & \cancel{b}\,a_{12} \\ \cancel{c}\,a_{21} & d_{22} \end{matrix}\right| = \cancel{ad - bc}$ $a_{11}a_{22} - a_{21}a_{12}$

row ↑ ↖ column

recall $\mathbb{A}^{-1} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $\neq 0$ past

3×3's? 4×4's?

Warning:
Determinants have many curious properties & uses
you must concentrate

In general, determinants are computed recursively:

(First — recipes save for later *
later — understanding)

3×3's

$$\det \left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right|$$ ← use multilinearity

column swap

$$= a_{11}\left|\begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix}\right| - a_{12}\left|\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}\right| + a_{13}\left|\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}\right|$$

and   show piece of orig

$$= a_{31}\left|\begin{matrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{matrix}\right| - a_{32}\left|\quad\right| + a_{33}\left|\quad\right|$$

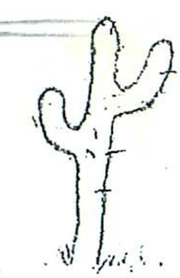
and

$$= \cancel{a_{13}}\left|\begin{matrix} \cancel{a_{21}} & a_{22} \\ a_{31} & \cancel{a_{32}} \end{matrix}\right| - a_{12}\left|\quad\right| + a_{22}\left|\quad\right| - a_{32}\left|\quad\right|$$

crazy!!

example

$$|\mathbb{A}| = \left|\begin{matrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{matrix}\right| = 4.$$



So 3×3 det's are built out of 2×2's (3 of them)

In general $n \times n$'s involve $(n-1) \times (n-1)$ matrices (n of them)
⇓
each of these involve $(n-2) \times (n-2)$'s ... etc.

Linear Algebra — Sudoko — Ch 5

Submatrices (defn) called ~~"corners"~~ minor matrices

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ has

a total of nine minor matrices (3×3) — n^2 for an $n\times n$.

$M_{13} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$

strike out row 1 & column 3

$M_{22} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

$M_{11} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

(defn) $\det(M_{ij})$ is called a minor (A has $n\times n$ minors)

(defn) $C_{ij} = (-1)^{i+j}\det(M_{ij})$

These #s are cofactors of A — checkerboard $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

(Theorem) Determinant of A is given by the ~~sum~~ dot product of A's cofactors with A's entries along any one row or column.

e.g. summing along row 1

$\det(A) = \sum_{j=1}^{n} \underbrace{(-1)^{1+j}\det(M_{1j})}_{C_{1j}} a_{1j}$

~~or~~ down ~~row~~ column 2

$\det(A) = \sum_{i=1}^{n} \underbrace{(-1)^{i+2}\det(M_{i2})}_{C_{i2}} a_{i2}$

*explain why...

---

* ~~Show 2x2 works~~ (*) improve section on determinants

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

$M_{11} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow C_{11} = (-1)^{1+1}\det(M_{11})$
$= (-1)^2(3\cdot2 - 1\cdot2)$
$= 4.$

$C_{12} = \ldots$

Cofactor Matrix

$A_{cofactor} = (C_{ij}) = \begin{bmatrix} 4 & 4 & -6 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$

even better $A^{-1} = \frac{1}{\det A} C^T$ — "adjoint of A" — later

$(a_{ij}C_{ij}) = \begin{bmatrix} 4 & 4 & -6 \\ 0 & 0 & 2 \\ -2 & -2 & 6 \end{bmatrix}$

check sum of each row & column

$\det(A) = 2$.

* [show general 2x2 works.

explain why... (3)

Sneaky methods:

B/c of theorem choose row or col with most 0's.

e.g. ~~Back to~~ $2\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix} = 7$

$\det\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -1 & 2 & 3 \end{bmatrix} = 2\cdot3\cdot3 = 18$ or $3\cdot6 = 18$ — show slow way

~~upper or lower triangul~~

for upper & lower triangul matrices, det is product of entries on leading diagonal

Cramer's Rule

determinants & $\mathbb{A}\vec{x}=\vec{b}$

$\mathbb{A}$: $n\times n$ only.

$$\left[\ \mathbb{A}\begin{bmatrix} \vert & 0 & 0\\ \vec{x} & 1 & 0\\ \vert & 0 & 1\end{bmatrix} = \begin{bmatrix}\underset{\vec{b}}{\mathbb{A}\vec{x}} & \mathbb{A}\begin{bmatrix}0\\1\\0\end{bmatrix} & \mathbb{A}\begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}\right.$$

(do + for 2nd & 3rd columns)

LHS

$$|\mathbb{A}|\begin{vmatrix}x_1&0&0\\x_2&1&0\\x_3&0&1\end{vmatrix} = |\mathbb{A}|\, x_1$$

RHS

$$|\mathbb{B}_1| = |\vec{b}\ \vec{a}_2\ \vec{a}_3|.$$ use col ops

$$x_1 = \frac{|\mathbb{B}_1|}{|\mathbb{A}|}.$$

Similarly $x_2 = \dfrac{|\mathbb{B}_2|}{|\mathbb{A}|}$, $x_3 = \dfrac{|\mathbb{B}_3|}{|\mathbb{A}|}$

A solution to $\mathbb{A}\vec{x}=\vec{b}$ with a formula ~ no need to solve numerically!! !!!.

But finding determinants is horribly slow!!

Utility → theoretical.

$n\times n$ $\quad x_i = \dfrac{|\mathbb{B}_i|}{|\mathbb{A}|}$ $\mathbb{A}$ with ith column replaced by $\vec{b}$.

A formula for $\mathbb{A}^{-1}$:

~~When $\vec{b}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ we find $\vec{x}_1$~~

~~s.t.~~ Solve all these problems

$$\left.\begin{aligned}\mathbb{A}\vec{x}_1 &= \begin{bmatrix}1\\0\\0\end{bmatrix}\\ \mathbb{A}\vec{x}_2 &= \begin{bmatrix}0\\1\\0\end{bmatrix}\\ \mathbb{A}\vec{x}_3 &= \begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned}\right\}\ \mathbb{A}\underbrace{[\vec{x}_1\, \vec{x}_2\, \vec{x}_3]}_{\mathbb{A}^{-1}\ !!!} = \mathbb{I}$$

$$\mathbb{A} = \begin{bmatrix}1&1&1\\0&3&2\\2&1&2\end{bmatrix}$$

prev example..

What are $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$ when $\vec{b} = \begin{bmatrix}1\\0\\0\end{bmatrix}$?

$$\underset{C_{11}}{\begin{bmatrix}1&1&1\\0&3&2\\0&1&2\end{bmatrix}},\ \underset{C_{12}}{\begin{bmatrix}1&1&1\\0&0&2\\2&0&2\end{bmatrix}},\ \underset{C_{13}.}{\begin{bmatrix}1&1&1\\0&3&0\\2&1&0\end{bmatrix}}$$

$$\vec{x}_1 = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{11}\\C_{12}\\C_{13}\end{bmatrix}$$

$$\Rightarrow \mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{11}&C_{21}&C_{31}\\C_{12}&C_{22}&C_{32}\\C_{13}&C_{23}&C_{33}\end{bmatrix}$$

$$= \frac{1}{|\mathbb{A}|}\mathbb{C}^T.$$ incroyable!

$n\times n$ $\quad$ exercise

Show we get

$$\mathbb{A}^{-1} = \frac{1}{2}\begin{bmatrix}4&-1&-1\\4&0&-2\\-6&1&3\end{bmatrix}$$

exercise check this works

Make a table

LINEAR ALGEBRA

Ch 6a 1

Back to eigenvalues & eigen vectors

$\boxed{A\vec{v} = \lambda\vec{v}}$

Again:

characteristic eq of A = polynomial in λ.

① Solve $|A - \lambda I| = 0$ for λ's — evalues

② plug λ's back into $(A - \lambda I)\vec{v} = \vec{0}$ to find eigenvectors (find basis for nullspace of $(A-\lambda I)$).

we are good at doing this!

Today: Some sneaky tricks and a (another) glorious factorization of A.

A small example

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \rightarrow \lambda_1 = 3/2,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda_2 = 1/2,\ \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ST1 $\prod_{i=1}^{n} \lambda_i = |A|$

The determinant of A is equal to the product of its eigenvalues

$\Rightarrow$ if $|A| = 0$, at least one evalue is 0. later.

Check $\lambda_1 \times \lambda_2 = 3/4$

$|A| = 1 - \frac{1}{2}\cdot\frac{1}{2} = 3/4$ ✓.

Why? ex.
$$|A - \lambda I| = (1-\lambda)(1-\lambda) - \tfrac{1}{4}.$$
$$= (\underset{\lambda_1}{\tfrac{3}{2}} - \lambda)(\underset{\lambda_2}{\tfrac{1}{2}} - \lambda)$$

equality true for all λ, so we can happily set $\lambda = 0$

$$|A - 0 \cdot I| = \frac{3}{2} \cdot \frac{1}{2} \quad !$$
$$= |A| = \lambda_1\lambda_2$$

In general
$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)\cdots(\lambda_n - \lambda)$$

set $\lambda = 0$: $|A| = \prod_{i=1}^{n} \lambda_i$

ST2 $\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$

Tr = "trace" = sum of the main diagonal elements

Our example:

$\mathrm{Tr}(A) = 1 + 1 = 2$
$\lambda_1 + \lambda_2 = 3/2 + 1/2 = 2$ oh boy!

General 2×2:

1×1 example know straight away.

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$
$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda - bc$$

General $n \times n$ $|A - \lambda I| = (-\lambda)^n + (\mathrm{Tr}A)(-\lambda)^{n-1} + \ldots = (-\lambda)^n + (\sum \lambda_i)(-\lambda)^{n-1} + \ldots$

* So when finding eigenvalues always check $\mathrm{Tr}(A) = \sum \lambda_i$ (easy)
& $|A| = \prod \lambda_i$ harder to calc but okay for 2×2's, 3×3's.

$$(-\lambda)^n + (\textstyle\sum \lambda_i)(-\lambda)^{n-1} + \ldots = |A - \lambda I|$$
$$(-\lambda)^2 + (\mathrm{Tr}\,A)(-\lambda)^{n-1} + \ldots ?$$

LINEAR ALGEBRA

Extra pieces from previous lectures & assignments

ST3 $|A| = \pm \prod d_i$ ← pivots

from $PA = LU$

$\prod \lambda_i$ $=$ $|A|$

~~do example~~ !

please note: pivots ≠ evalues !! little relationship apart from product.

ST4 if $A\vec{v} = \lambda\vec{v}$

$A^n\vec{v} = \lambda^n\vec{v}$

so λ^n is an evalue of A^n with evector $\vec{v}$.

ST5 if $A\vec{v} = \lambda\vec{v}$

then $(A + tI)$ has $\vec{v}$ as an evector with evalue $\lambda + t$.

ST6 if $A\vec{v} = \lambda\vec{v}$

then $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$ if A^{-1} exists.

↖ evalue of A^{-1}.

ST7 If A's evalues are all different (no repeats) then A's evectors are independent. (and form a basis for $\mathbb{R}^n$)

Reason;

Consider 2 evectors

If dependent $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ for some $c_1, c_2 \neq 0$

Ch 6, 2

ex $A = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1/2 \\ 0 & 3/4 \end{bmatrix}$

pivots : 1, 3/4 $\xrightarrow{\text{products}}$ $3/4 = |A|$

evalues : 1/2, 3/2.

① $A(c_1\vec{v}_1 + c_2\vec{v}_2) = A\vec{0}$

$\Rightarrow c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$

② $\times \lambda_2$

$c_1\lambda_2\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$

② − ① : $c_1(\lambda_2 - \lambda_1)\vec{v}_1 = \vec{0}$

$\Rightarrow c_1 = 0$ b/c $\lambda_2 \neq \lambda_1$

Can build up from here that evectors all together are independent.

$(A - \lambda_2)(A - \lambda_3)$

(λ_3)

$+(x_1\vec{v}_1)$

$+\lambda_2\vec{v}_2$

ST8 evalues & evectors of A & B generally not simply related to those of $A + B$ & AB.

if A & B share an evector $\vec{v}$ with evalues λ_A & λ_B

$AB\vec{v} = \lambda_A\lambda_B\vec{v}$

but this is a very special case.

SKIP → change probset

delay until after diag

But eigenvalues are not always happy

Some matrices are bad matrices

Definitions:

Algebraic Multiplicity of an evalue λ is the # times it is a root of $|A - \lambda I| = 0$

↑ the char eq of A

LINEAR ALGEBRA

Ch 6.3

ex

$$\mathbb{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda = 2$ has algeb. mult 1

$\lambda = 3$ " " 2.

Another term:

Geometric Multiplicity of evalue λ is the dimension of the nullspace of $\mathbb{A} - \lambda\mathbb{I}$. OR the # independent evectors corresponding to λ.

In above example

$\lambda_1 = 2 \Rightarrow \vec{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\lambda_2 = 3$, $\vec{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \qquad |\mathbb{A} - \lambda\mathbb{I}| = (1-\lambda)^3 = 0$$

$\lambda = \underline{1,1,1}$

$(\mathbb{A} - (1)\mathbb{I})\vec{v} \Rightarrow \vec{v} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ only.

$\lambda = 1$ has A.M of 3 but G.M of only 1

$N(\mathbb{A} - 1\mathbb{I})$ not big enough

in sum, $\mathbb{A}$ is a very naughty matrix

Let's just pretend this didn't happen & focus on good matrices (ones with $\underline{n}$ evectors)

Observation:

Diagonal matrices are highly appealing. They make us happy.

ex

$$\mathbb{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \mathbb{A}^n = \begin{bmatrix} 3^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 4^n \end{bmatrix} \text{ easy!}$$

$$\mathbb{A}\vec{x} = \begin{bmatrix} 3x_1 \\ 2x_2 \\ 4x_3 \end{bmatrix}$$

$\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_4 = 4$

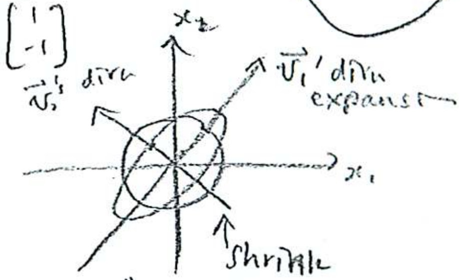
$\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

how $\mathbb{A}$ changes $\vec{x}$ is simple.

Recall $\mathbb{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

$\lambda_1 = 3/2$ $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$

$\lambda_2 = 1/2$ $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$



if we could rotate the axes, our $\mathbb{A}$'s "action" would be simple.

More generally - If we could change to our evector basis, we would be pretty happy

Well, we can $\underline{\text{if}}$ $\mathbb{A}$ has n independent evectors.

Let's Diagonalize $\mathbb{A}$:

know $\mathbb{A}\vec{v}_i = \lambda_i \vec{v}_i$ $i = 1 \ldots n$.

Create a new matrix with evectors as columns

$$\mathbb{S} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

Consider

$$\mathbb{A}\mathbb{S} = \begin{bmatrix} | & | & & | \\ \mathbb{A}\vec{v}_1 & \mathbb{A}\vec{v}_2 & \cdots & \mathbb{A}\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$\left[\mathbb{S}\begin{bmatrix}\lambda_1\\0\\0\end{bmatrix} \; \mathbb{S}\begin{bmatrix}0\\\lambda_2\\0\end{bmatrix} \cdots\right]$ sneaky observation

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

$$= \mathbb{S}\,\Lambda$$

capital λ lambda

LINEAR ALGEBRA

We are assuming the evectors are independent so S^{-1} exists:

$\Rightarrow \boxed{S^{-1}AS = \Lambda}$ ← similarity transformation

special example of two matrices being similar

or $\boxed{A = S\Lambda S^{-1}}$ ← fabulous factorization

second

second

How A works:

$A\vec{x} = S\Lambda S^{-1}\vec{x}$

- S: changes back to natural basis
- Λ: simple modification b/c Λ is diagonal
- S^{-1}: changes $\vec{x}$ to A's evector basis ← come back to this later

first

2nd

For $A = \begin{bmatrix}1 & 1/2\\ 1/2 & 1\end{bmatrix}$

$S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ $\quad \Lambda = \begin{bmatrix}3/2 & 0\\ 0 & 1/2\end{bmatrix}$

$S^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

more later: notice $S^{-1} = S^T$ & $S = Q$

So $A = S\Lambda S^{-1}$

$\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix} \equiv \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & +1\end{bmatrix}\begin{bmatrix}\frac{3}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & +1\end{bmatrix}$ remove

~~Example: $\vec{x} = \frac{2}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} + \frac{5}{\sqrt{2}}\begin{bmatrix}-1\\+1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-3\\+7\end{bmatrix}$~~ (evectors)

~~$S^{-1}\frac{1}{\sqrt{2}}\begin{bmatrix}-3\\7\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4\\10\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$~~ ← $\vec{x}$ in evector basis!

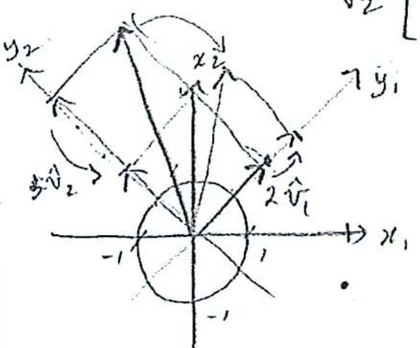
$\frac{1}{\sqrt{2}}$ too messy $\quad \vec{x} = 2\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix}$

$\Lambda S^{-1}\frac{1}{\sqrt{2}}\begin{bmatrix}7\\-3\end{bmatrix} = \begin{bmatrix}3\\5/2\end{bmatrix}$

Change basis back

$S\Lambda S^{-1}\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & +1\end{bmatrix}\begin{bmatrix}3\\5/2\end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix}11/2\\ -1/2\end{bmatrix}$



← big clear picture on board.

For many calculations, diagonalization brings happiness:

$A^2 = (S\Lambda \cancel{S^{-1}})(\cancel{S}\Lambda S^{-1})$ (I)

$= S\Lambda^2 S^{-1}$

$A^k = S\Lambda^k S^{-1}$ 1st

easy to calculate!

$\Lambda^k = \begin{bmatrix}\lambda_1^k & & 0\\ & \lambda_2^k & \\ 0 & & \ddots \; \lambda_n^k\end{bmatrix}$

So $\begin{bmatrix}1 & 1/2\\ 1/2 & 1\end{bmatrix}^{523}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}(3/2)^{523} & 0\\ 0 & (1/2)^{523}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ → large; → 0

$= \left(\frac{3}{2}\right)^{523}\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

Bonus
$A^0 = I$
A^{-1} works too
only if evectors are independent
$(0^{-1} = \infty \mid 0^0\ ?)$ oops

Next up:
Use all this to find Fibonacci numbers

gen fn

LINEAR ALGEBRA | Rabbits:

Fibonacci! 1, 1, 2, 3, 5 ↪ see Wikipedia

first known writing: over 2000 years ago in India patterns of speech and song

Fibonacci: 1200

(Crazy) model of rabbit breeding

F_n = # pairs of rabbits at time n ← month n

Rabbits can only breed after they reach 2 months of age (replacement)

So $F_n = F_{n-1} + F_{n-2}$

F_{n-1}: not yet breeding

F_{n-2}: replacement.

1 pair beget 1 pair.

Population explosion: Exponential... # spirals

= set (as before) $\vec{f_n} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ define $f_1 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$

then $\vec{f_n} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{f}_{n-1}$ redo!

What is the n^{th} Fibonacci Number?

$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \vec{f_1}$ just set up

So we need to diagonalize $\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

evalues:

$0 = |\mathbb{A} - \lambda \mathbb{I}| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ set $\lambda_1 = \frac{1+\sqrt{5}}{2} = \rho$ = golden ratio.

& $\lambda_2 = \frac{1-\sqrt{5}}{2}$

Note

$\lambda_1 + \lambda_2 = 1 = Tr(\mathbb{A})$

$\lambda_1 \lambda_2 = -1 = |\mathbb{A}|$

LINEAR ALGEBRA

(We are nearing greatness...)

Some key points

- A is diagonalizable only if A has n independent evectors
- if one or more λ's $= 0$, A^{-1} does not exist
  (evalues)
- A matrix with no inverse may or may not be diagonalizable
  (evalues, evectors)

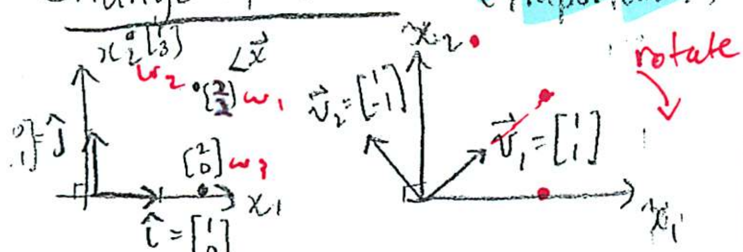
And if A can be diagonalized it may have no inverse

- S is always made up of eigenvectors.

Lecture 21

[~~Next~~ Today steps: More deeply see diagonalization as a change of basis and see what happens for symmetric matrices.]

Change of Basis (super important!)



$\vec{w}_1 = \begin{bmatrix}2\\2\end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix}1\\3\end{bmatrix}$, $\vec{w}_3 = \begin{bmatrix}0\\2\end{bmatrix}$

all in terms of our natural basis

$\vec{w}_1 = 2\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = 2\hat{i} + 2\hat{j}$ ← we hardly ever think about this

How do we systematically find $\vec{w}_1, \vec{w}_2, \vec{w}_3$ in terms of $\vec{v}_1$ & $\vec{v}_2$? ← new basis

By solving an $A\vec{x} = \vec{b}$ problem.. !! (sound of drooling)

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$\vec{w}_1 = \begin{bmatrix}2\\2\end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$

column picture

$\vec{w}_1$ in new basis

$= M\vec{w}_1'$

$\Rightarrow \vec{w}_1' = M^{-1}\vec{w}_1$ ← natural basis

new $\vec{v}$ basis

$= \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\2\end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$

draw pictures

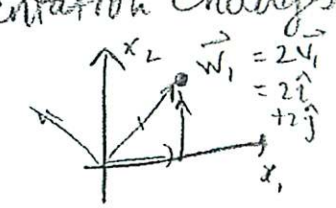
similarly $\vec{w}_2' = \begin{bmatrix}2\\1\end{bmatrix}$, $\vec{w}_3' = \begin{bmatrix}1\\1\end{bmatrix}$ ✓✓

[To change back: $\vec{w} = M\vec{w}'$
(Draw examples)]

natural basis $(\hat{i}, \hat{j})$; new $\vec{v}$ basis

[We say, In basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$
$\vec{w}_1$ is represented as $\begin{bmatrix}2\\2\end{bmatrix}$
In basis $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$,
$\vec{w}_1$ " " " $\begin{bmatrix}2\\0\end{bmatrix}$]

The point $\vec{w}_1$ never changes, just our representation changes



The big deal: We change basis to make action of A as simple as possible, then change back

e.g. diagonal

$A = S\Lambda S^{-1}$

change basis only — changes point

M — M^{-1}

already covered

LINEAR ALGEBRA

evector basis is super groovy

leave out

e.g. $A = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$

$\lambda_1 = 4,\ \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix};\ \lambda_2 = 2,\ \vec{v}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

$A =$

$$\underbrace{\begin{bmatrix}1 & -1\\ +1 & 1\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}}_{\Lambda}\underbrace{\tfrac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}_{S^{-1}}$$

draw picture (*) e.g. $\vec{x} = \begin{bmatrix}1\\3\end{bmatrix} = \vec{v}_1 + 2\vec{v}_2$

S: changes back

Λ: does the work: grow in $\begin{bmatrix}1\\1\end{bmatrix}$ direction by a factor of 4 ...... $\begin{bmatrix}-1\\1\end{bmatrix}$ ...... of 2.

S^{-1}: changes from natural basis to evector basis.

(We write M as S b/c evectors are so special)

$\boxed{\text{So diagonalizing is joyous}}$ & fun

~~but we must have n indep. evectors (and our matrix must be square!).~~

Crazily significant result:

Real Symmetric matrices ($A = A^T$) always have n independent eigenvectors (even if they have repeated evalues)

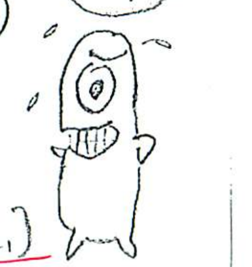
Not only that but

1. all ~~evectors~~ evalues are real
2. the evectors form an orthogonal basis for R^n (we can then make it orthonormal)

Mindblowing..

(These astounding results will matter hugely for our friend $A\vec{x} = \vec{b}$ ....)

Ch6 7a



~~Ex For $\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$ we had real evalues 4 & 2 and $\perp$ evectors $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}$~~

(*) Proof of 1

Assume $A = A^T$ & A is real (all entries are real)

Take $A\vec{v} = \lambda\vec{v}$

we wonder if λ is real or complex

Skip

top of board

Denote complex conjugate by over bar

$\overline{a+bi} = a - bi \qquad i = \sqrt{-1}$

$A\vec{v} = \lambda\vec{v}$

~~transpose~~

~~$\vec{v}^T A^T = \lambda\vec{v}^T$~~

~~sym~~

~~$\vec{v}^T A = \lambda\vec{v}^T$~~

$\longrightarrow \overline{A}\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$

real

$A\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$

transpose

$\overline{\vec{v}}^T A^T = \overline{\lambda}\,\overline{\vec{v}}^T$

symmetric

$\overline{\vec{v}}^T A = \overline{\lambda}\,\overline{\vec{v}}^T$

now pre & post multiply by $\overline{\vec{v}}$ & $\vec{v}$ respectively

$\overline{\vec{v}}^T A\vec{v} = \lambda\overline{\vec{v}}^T\vec{v} \quad | \quad \overline{\vec{v}}^T A\vec{v} = \overline{\lambda}\,\overline{\vec{v}}^T\vec{v}$

everything else matches so $\lambda = \overline{\lambda} \Rightarrow \lambda$ is real.

Proof of 2

Assume A is real and $A = A^T$

If A's evalues are all different (algebraic multiplicity = 1)

if $A\vec{v}_1 = \lambda_1\vec{v}_1$ & $A\vec{v}_2 = \lambda_2\vec{v}_2$

we want to show $\vec{v}_1^T\vec{v}_2 = 0$

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LINEAR ALGEBRA

$\vec{v}_1^T A \vec{v}_2$

More sneakiness

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T (A\vec{v}_2) = \lambda_2 \vec{v}_1^T \vec{v}_2$$

$$= (\vec{v}_1^T A)\vec{v}_2$$

$$= (A^T \vec{v}_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \vec{v}_2 = \lambda_1 \vec{v}_1^T \vec{v}_2$$

symmetric

only equal if $\vec{v}_1^T \vec{v}_2 = 0$ since $\lambda_1 \neq \lambda_2$

phew...

B/c evectors are orthogonal we now write Q instead of S

And even more, $Q^{-1} = Q^T$ as we have seen before

So for real, symmetric matrices:

$$A = Q \Lambda Q^T \quad \text{always}$$

next page

Spectral Theorem or principle axis theorem. + $A = \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T$

Strang: The best matrices have orthogonal evectors

For $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ choose $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Then

$$A = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}}_{\Lambda} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{Q}$$

calc.

$$+\ A = 4\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} + 2\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \end{bmatrix} \quad !!!$$

general truth.

Recall

Diagonalization fails when there are not enough evectors

This can only happen when a evalue is repeated. e.g. $|A_{3\times3} - \lambda I| = (\lambda-3)(\lambda-3)\lambda$

e.g. $\lambda = 3$ has algebraic multiplicity 1

defn

geometric multiplicity of λ = # indep evectors associated w. λ = dim $N(A - \lambda I)$

Must have a.m = g.m for all evalues for diag to be possible

Final observation:

(All $n\times n$ symmetric matrices have n orthogonal evectors) even if evalues are repeated ← very happy matrices

Reason

We know that if all evalues are different, & if $A = A^T$, evectors are orthogonal.

So take problem matrices and tweak them so their evalues separate.

e.g.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 1$

one evector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$:(

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 1$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

LINEAR ALGEBRA

"Tweaks"

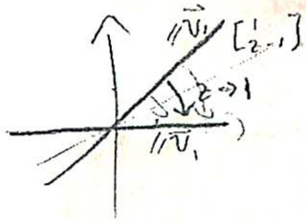
$A = \begin{bmatrix} 1 & 1 \\ 0 & z \end{bmatrix}$

$\lambda_1 = 1, \lambda_2 = z$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ z-1 \end{bmatrix}$

as $z \to 1$, evalues become the same and evectors collapse

Really: eigenspace collapses



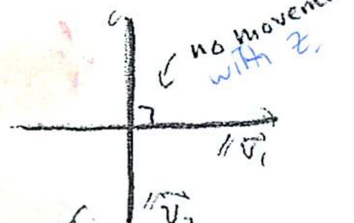
mighty

$A = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$

$\lambda_1 = 1, \lambda_2 = z$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

evectors remain independent as they are orthogonal $\forall z \neq 1$



can't collapse

Ch 6 9

Exam 3

1. $A = QR$ (4-4) Gram-Schmidt Process orthogonal base
2. Eigenthings 6.1, 6.2, 6.4, 7-3
3. Determinants 5.1, 5.2, 5.3

spectral theore

for symmetric matrices

*

$\overset{n \times n}{A} = Q \Lambda Q^T$

$= \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - \hat{v}_1^T - \\ - \hat{v}_2^T - \\ \vdots \\ - \hat{v}_n^T - \end{bmatrix}$

$= \begin{bmatrix} \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \hat{v}_1^T \\ \lambda_2 \hat{v}_2^T \\ \vdots \\ \lambda_n \hat{v}_n^T \end{bmatrix}$ ~ big dot product

$= \lambda_1 \hat{v}_1 \hat{v}_1^T + \lambda_2 \hat{v}_2 \hat{v}_2^T + \cdots + \lambda_n \hat{v}_n \hat{v}_n^T$

projection operators

$A\vec{x}$ if $A = A^T$, $A\vec{x}$ is very special!

✓

rotational matrices?

λ complex.

Trace stuff...