124 Matrixology (Linear Algebra)—Practice exam #3
University of Vermont, Fall Semester

Name: ____________________________

➤ Total points: 24 (3 points per question); Time allowed: 75 minutes.
➤ Current brains only: No pensieves, calculators, or similar gadgets allowed.
➤ For full points, *please show all working clearly*.

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1. (a) Write down the formulae for the Gram-Schmidt process for determining a new orthogonal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ from a given basis $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

   $\vec{u}_1 =$

   $\vec{u}_2 =$

   $\vec{u}_3 =$

   (b) Given the following basis for a subspace S of R^4: $\vec{a}_1 = [2, 0, 3, 1]^{\mathrm{T}}$ and $\vec{a}_2 = [-1, 1, -4, 0]^{\mathrm{T}}$, determine $\vec{u}_1$ and $\vec{u}_2$ using the Gram-Schmidt process.

2. Given two $n \times n$ matrices A and B with $|A| = 5$ and $|B| = 3$, determine the following:

   (a) $\det(AB)$

   (b) $\det(B^4)$

   (c) $\det(A^{-1})$

   (d) $\det(B^{\mathrm{T}})$

   (e) $\det(C)$ where C is the same as A except the first row is doubled.

   (f) $\det(L)$ and $\det(U)$ if reduction of A to U requires three row swaps.

3. (a) Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}.$$

(b) Find the three matrices S, Λ, and S^{-1} that combine to give $A = S\Lambda S^{-1}$.

(c) Compute A^k for general k. (Your answer should be a 2 by 2 matrix.)

4. (a) For the following matrix, write down the minor matrices M_{12} and M_{32}, and compute the cofactors C_{12} and C_{32}:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

(b) Compute the determinant of the matrix in Q. 4a by using the cofactor method operating on the second column.

(c) Compute the determinant of the matrix given in Q. 4a by using row operations to first reduce the matrix to an upper triangular matrix. Please show all row operations clearly.

5. (a) Given the following basis for R^2,

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

write down the transformation matrix M that takes a vector's representation in the $\{\vec{v}_1, \vec{v}_2\}$ basis to the natural basis.

(b) Given $\vec{p} = [1, 5]^{\mathrm{T}}$ in the natural basis, find $\vec{p}$'s representation in the $\{\vec{v}_1, \vec{v}_2\}$ basis.

6. Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

7. Quick questions:

   (a) Given the factorization $A = QR$, write down a formula for R in terms of A and Q.

   (b) Write down the general formula connecting the pivots of an n by n matrix A with its determinant, $|A|$.

   (c) What's important/special/exciting about the eigenvectors of any n by n symmetric matrix A?

   (d) True or false: The geometric multiplicity of an eigenvalue can be zero:

   _________.

   (e) True or false: If A has eigenvalues 3, 3, and 1, it must be diagonalizable:

   _________.

   (f) True or false: If A has no inverse then it must have one or more eigenvalues equal to 0:

   _________.

8. (a) (2 pts) Consider a 3x3 matrix A with the following properties:

    i. $|A| = 12$,
    ii. $\mathrm{Tr}A = -1$ (where TrA means Trace of A),
    iii. and one eigenvalue of A is 3 (call this λ_1).

    Determine A's other eigenvalues.

   (b) (1 pt) Using the formula for $A^{-1} = \frac{1}{|A|}C^{\mathrm{T}}$, find $|C|$, the determinant of the cofactor matrix, in terms of $|A|$. Assume A is an n by n matrix.

Doodling space: