

# Chapter 3/4: Lecture 15

Linear Algebra  
MATH 124, Fall, 2010

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## Stuff to know/understand

### Vector Spaces:

- ▶ Vector space concept and definition.
- ▶ Subspace definition (three conditions).
- ▶ Concept of a **spanning set** of vectors.
- ▶ Concept of a **basis**.
- ▶ Basis = minimal spanning set.
- ▶ Concept of **orthogonal complement**.
- ▶ Various techniques for finding bases and orthogonal complements.

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## Outline

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## Stuff to know/understand:

### Fundamental Theorem of Linear Algebra:

- ▶ Applies to any  $m \times n$  matrix  $A$ .
- ▶ Symmetry of  $A$  and  $A^T$ .
- ▶ Column space  $C(A) \subset R^m$ .
- ▶ Left Nullspace  $N(A^T) \subset R^m$ .
- ▶  $\dim C(A) + \dim N(A^T) = r + (m - r) = m$
- ▶ Orthogonality:  $C(A) \otimes N(A^T) = R^m$
- ▶ Row space  $C(A^T) \subset R^n$ .
- ▶ (Right) Nullspace  $N(A) \subset R^n$ .
- ▶  $\dim C(A^T) + \dim N(A) = r + (n - r) = n$
- ▶ Orthogonality:  $C(A^T) \otimes N(A) = R^n$

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## Basics:

### Sections covered on second midterm:

- ▶ Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- ▶ **Main pieces:**
  1. Big Picture of  $A\vec{x} = \vec{b}$   
**Must be able to draw the big picture!**
  2. Projections and the normal equation
- ▶ As always, want 'doing' and 'understanding' and abilities.

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## Stuff to know/understand:

### Finding four fundamental subspaces:

- ▶ Enough to find bases for subspaces.
- ▶ Be able to **reduce  $A$  to  $R$** .
- ▶ Identify pivot columns and free columns.
- ▶ **Rank  $r$  of  $A = \#$  pivot columns.**
- ▶ Know that relationship between  $R$ 's columns hold for  $A$ 's columns.
- ▶ **Warning:**  $R$ 's columns do not give a basis for  $C(A)$
- ▶ But find pivot columns in  $R$ , and same columns in  $A$  form a basis for  $C(A)$ .

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### More on bases for column and row space:

- ▶ Reduce  $[A | \vec{b}]$  where  $\vec{b}$  is general.
- ▶ Find conditions on  $\vec{b}$ 's elements for a solution to  $A\vec{x} = \vec{b}$  to exist  $\rightarrow$  obtain basis for  $C(A)$ .
- ▶ **Basis for row space** = non-zero rows in  $R$  (easy!)
- ▶ Alternate **basis for column space** = non-zero rows in reduced form of  $A^T$  (easy!)



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## Projections:

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- ▶ Understand how to project a vector  $\vec{b}$  onto a line in direction of  $\vec{a}$ .
- ▶  $\vec{b} = \vec{p} + \vec{e}$
- ▶  $\vec{p}$  = that part of  $\vec{b}$  that lies in the line:

$$\vec{p} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \vec{a} \left( = \frac{\vec{a} \vec{a}^T}{\vec{a}^T \vec{a}} \vec{b} \right)$$

- ▶  $\vec{e}$  = that part of  $\vec{b}$  that is orthogonal to the line.
- ▶ Understand generalization to projection onto subspaces.
- ▶ Understand construction and use of subspace projection operator  $P$ :

$$\vec{P} = A(A^T A)^{-1} A^T,$$

where  $A$ 's columns form a subspace basis.



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### Bases for nullspaces, left and right:

- ▶ Basis for nullspace obtained by solving  $A\vec{x} = \vec{0}$
- ▶ Always **express pivot variables in terms of free variables**.
- ▶ Free variables are unconstrained (can be any real number)
- ▶ # free variables =  $n$  - # pivot variables =  $n - r = \dim N(A)$ .
- ▶ Similarly find basis for  $N(A^T)$  by solving  $A^T \vec{y} = \vec{0}$ .



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### Normal equation for $A\vec{x} = \vec{b}$ :

- ▶ If  $\vec{b} \notin C(A)$ , project  $\vec{b}$  onto  $C(A)$ .
- ▶ Write projection of  $\vec{b}$  as  $\vec{p}$ .
- ▶ Know  $\vec{p} \in C(A)$  so  $\exists \vec{x}_*$  such that  $A\vec{x}_* = \vec{p}$ .
- ▶ Error vector must be orthogonal to column space so  $A^T \vec{e} = A^T (\vec{b} - \vec{p}) = \vec{0}$ .
- ▶ Rearrange:

$$A^T \vec{p} = A^T \vec{b}$$

- ▶ Since  $A\vec{x}_* = \vec{p}$ , we end up with

$$A^T A \vec{x}_* = A^T \vec{b}.$$

- ▶ This is linear algebra's **normal equation**;  $\vec{x}_*$  is our best solution to  $A\vec{x} = \vec{b}$ .



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### Number of solutions to $A\vec{x} = \vec{b}$ :

1. If  $\vec{b} \notin C(A)$ , there are **no solutions**.
2. If  $\vec{b} \in C(A)$  there is either one unique solution or infinitely many solutions.
  - ▶ Number of solutions now depends entirely on  $N(A)$ .
  - ▶ If  $\dim N(A) = n - r > 0$ , then there are **infinitely many solutions**.
  - ▶ If  $\dim N(A) = n - r = 0$ , then there is one solution.



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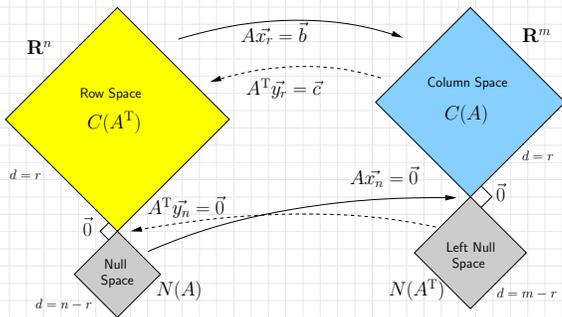
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### The symmetry of $A\vec{x} = \vec{b}$ and $A^T\vec{y} = \vec{c}$ :

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### The fourfold ways of $A\vec{x} = \vec{b}$ :

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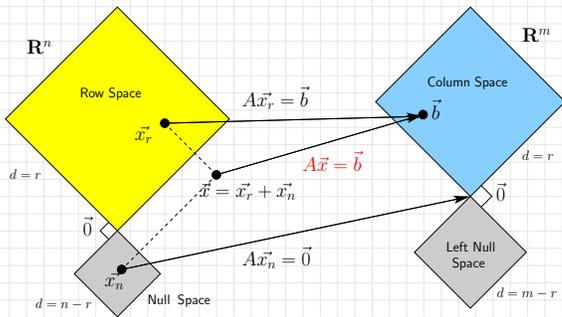
case	example $R$	big picture	# solutions
$m = r$ $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		1 always
$m = r$ , $n > r$	$\begin{bmatrix} 1 & 0 & \text{⚡}_1 \\ 0 & 1 & \text{⚡}_2 \end{bmatrix}$		$\infty$ always
$m > r$ , $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$		0 or 1
$m > r$ , $n > r$	$\begin{bmatrix} 1 & 0 & \text{⚡}_1 \\ 0 & 1 & \text{⚡}_2 \\ 0 & 0 & 0 \end{bmatrix}$		0 or $\infty$

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### How $A\vec{x} = \vec{b}$ works:

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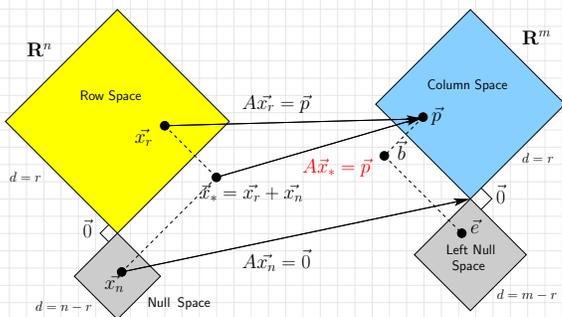


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### Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$ :

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