Chapter 3/4: Lecture 15

Linear Algebra
MATH 124, Fall, 2010

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Outline

Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- Main pieces:
  1. Big Picture of $A\vec{x} = \vec{b}$

     Must be able to draw the big picture!
  2. Projections and the normal equation
- As always, want 'doing' and 'understanding' and abilities.

Stuff to know/understand

Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.
- Concept of a basis.
- Basis = minimal spanning set.
- Concept of orthogonal complement.
- Various techniques for finding bases and orthogonal complements.

Stuff to know/understand:

Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix A.
- Symmetry of A and A^{T}.

- Column space $C(A) \subset R^m$.
- Left Nullspace $N(A^{\mathrm{T}}) \subset R^m$.
- $\dim C(A) + \dim N(A^{\mathrm{T}}) = r + (m - r) = m$
- Orthogonality: $C(A) \bigotimes N(A^{\mathrm{T}}) = R^m$

- Row space $C(A^{\mathrm{T}}) \subset R^n$.
- (Right) Nullspace $N(A) \subset R^n$.
- $\dim C(A^{\mathrm{T}}) + \dim N(A) = r + (n - r) = n$
- Orthogonality: $C(A^{\mathrm{T}}) \bigotimes N(A) = R^n$

Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce A to R.
- Identify pivot columns and free columns.
- Rank r of A = # pivot columns.
- Know that relationship between R's columns hold for A's columns.
- Warning: R's columns do not give a basis for $C(A)$
- But find pivot columns in R, and same columns in A form a basis for $C(A)$.

Stuff to know/understand:

More on bases for column and row space:

- Reduce $[A \,|\, \vec{b}\,]$ where $\vec{b}$ is general.
- Find conditions on $\vec{b}$'s elements for a solution to $A\vec{x} = \vec{b}$ to exist $\rightarrow$ obtain basis for $C(A)$.
- Basis for row space = non-zero rows in R (easy!)
- Alternate basis for column space = non-zero rows in reduced form of A^{T} (easy!)

Stuff to know/understand:

Bases for nullspaces, left and right:

- Basis for nullspace obtained by solving $A\vec{x} = \vec{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- # free variables = n - # pivot variables = $n - r$ = dim $N(A)$.
- Similarly find basis for $N(A^{\mathrm{T}})$ by solving $A^{\mathrm{T}}\vec{y} = \vec{0}$.

Stuff to know/understand:

Number of solutions to $A\vec{x} = \vec{b}$:

1. If $\vec{b} \notin C(A)$, there are no solutions.
2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.
    - Number of solutions now depends entirely on $N(A)$.
    - If $\dim N(A) = n - r > 0$, then there are infinitely many solutions.
    - If $\dim N(A) = n - r = 0$, then there is one solution.

Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b} = \vec{p} + \vec{e}$
- $\vec{p}$ = that part of $\vec{b}$ that lies in the line:

$$\vec{p} = \frac{\vec{a}^{\mathrm{T}}\vec{b}}{\vec{a}^{\mathrm{T}}\vec{a}}\vec{a} \left(= \frac{\vec{a}\vec{a}^{\mathrm{T}}}{\vec{a}^{\mathrm{T}}\vec{a}}\vec{b} \right)$$

- $\vec{e}$ = that part of $\vec{b}$ that is orthogonal to the line.
- Understand generalization to projection onto subspaces.
- Understand construction and use of subspace projection operator P:

$$\vec{P} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}},$$

where A's columns form a subspace basis.

Stuff to know/understand

Normal equation for $A\vec{x} = \vec{b}$:

- If $\vec{b} \notin C(A)$, project $\vec{b}$ onto $C(A)$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(A)$ so $\exists\ \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
- Error vector must be orthogonal to column space so $A^{\mathrm{T}}\vec{e} = A^{\mathrm{T}}(\vec{b} - \vec{p}) = \vec{0}$.
- Rearrange:

$$A^{\mathrm{T}}\vec{p} = A^{\mathrm{T}}\vec{b}$$

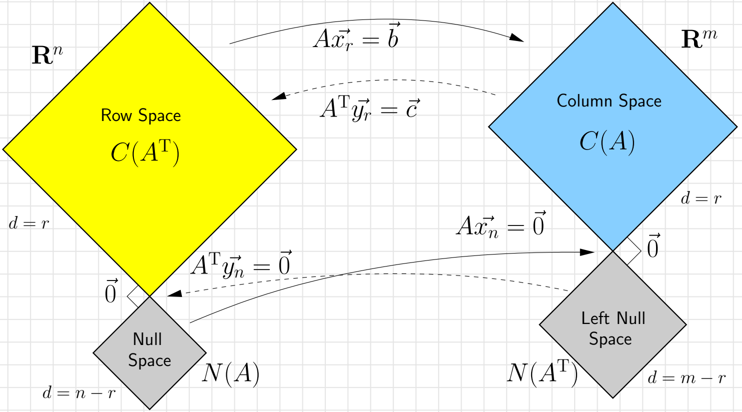
.

- Since $A\vec{x}_* = \vec{p}$, we end up with

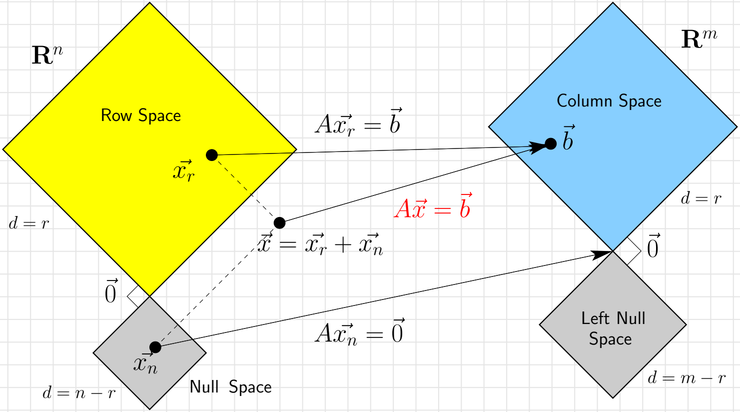
$$A^{\mathrm{T}}A\vec{x}_* = A^{\mathrm{T}}\vec{b}.$$

- This is linear algebra's normal equation;
  $\vec{x}_*$ is our best solution to $A\vec{x} = \vec{b}$.

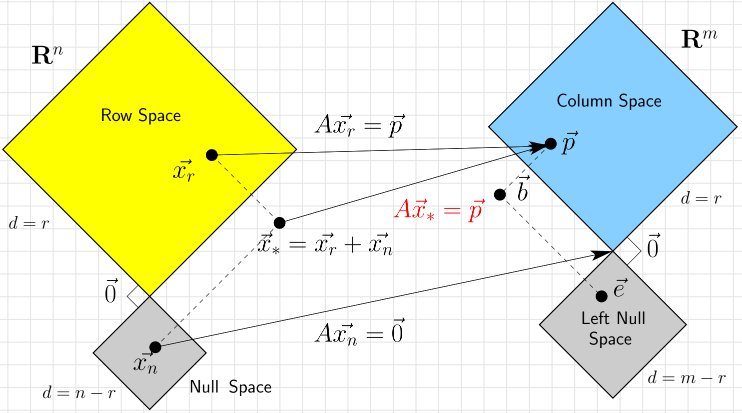
The symmetry of $A\vec{x} = \vec{b}$ and $A^{\mathrm{T}}\vec{y} = \vec{c}$:



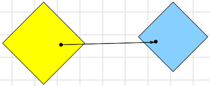
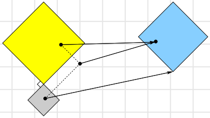
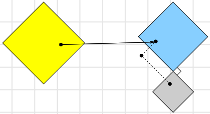
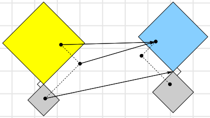
How $A\vec{x} = \vec{b}$ works:



Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:



The fourfold ways of $A\vec{x} = \vec{b}$:

case	example R	big picture	# solutions
$m = r$ $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		1 always
$m = r$, $n > r$	$\begin{bmatrix} 1 & 0 & ☕_1 \\ 0 & 1 & ☕_2 \end{bmatrix}$		∞ always
$m > r$, $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$		0 or 1
$m > r$, $n > r$	$\begin{bmatrix} 1 & 0 & 🚲_1 \\ 0 & 1 & 🚲_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		0 or ∞