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A reformulation of Horton's laws for large river networks in terms of statistical self-similarity

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Abstract. The well-known Horton's laws are empirical observations on how the means of measurements for river networks and basins vary with Horton-Strahler order. It is now known that these laws are a consequence of an average-sense self-similarity in the bifurcation structure of river networks. In this paper we present a reformulation of Horton's laws which generalizes the familiar scaling of first moments, or means, to scaling of entire distributions. We also present extensive data analysis which supports this reformulation and show that this feature is also exhibited by Shreve's well-known random topology model.

1. Introduction

The Horton laws of drainage composition have been known in terms of statistical averages for more than half a century. We generalize Horton's laws by first reformulating them in terms of probability distributions. This reformulation implies that Horton-type relations are true not only for the means but for all higher-order statistical moments whenever these moments exist. The theoretical significance of this reformulation lies in the interpretation that it is a form of "statistical self-similarity" called "simple scaling." Self-similarity represents a symmetry of a system under scale change. It has attracted a great deal of attention within the last decade in a variety of hydrologic investigations [see, e.g., *Sposito*, 1998]. A mathematical formulation of the notion of simple scaling requires the specification of a natural "scale parameter," so that "topologic self-similarity" of a family of subnetworks at different scales of resolutions can be defined. In the context of a branching tree structure a scale parameter can be defined in a variety of different ways, and we give some examples. However, it appears that Horton-Straher (H-S) ordering is most natural as a scale parameter compared to other alternatives. For example, it is difficult to define topologic self-similarity of a family of networks at different scales of resolutions on the basis of link magnitude. Our reformulation is empirically tested here for large drainage networks which were extracted from digital elevation models (DEMs).

Horton's laws were formulated by R. E. Horton [*Horton*, 1945] on the basis of a numerical enumeration of drainage networks. This scheme was later modified by Strahler and is called H-S ordering. A large number of empirical studies [*Jarvis and Woldenberg*, 1984], analytical theories [*Shreve*, 1967; *Tokunaga*, 1966], and network evolution models [*Rodriguez-Iturbe and Rinaldo*, 1997] have illustrated that Hortonian relations generally hold for drainage networks. Horton's laws are typically stated in terms of averages or means of variables because spatial variability and randomness are present as dominant features in network branching patterns. A Horton law states that if $\langle X_\omega \rangle$ is the average measured value of some variable X for all complete H-S subbasins of order $\omega \in 1, 2, 3, \cdots$, then

$$\langle X_{\omega+1} \rangle / \langle X_\omega \rangle \sim R_X, \quad (1)$$

where the approximation sign indicates rapid convergence to the constant R_X, which is known as the Horton ratio. The convergence is often so rapid that the approximation sign can be replaced by an equals sign to good approximation when ω exceeds 2.

Horton's original work and subsequent investigations by others found that these relationships hold not only for the topologic and geometric variables, such as stream number, basin area, and stream length [*Schumm*, 1956], but also for the hydraulic-geometric variables, such as slope, width, depth, bankfull discharge, and velocity [*Leopold and Miller*, 1964]. Horton's laws seem to represent a fundamental mathematical property of how the geometry and the physics of drainage network patterns are spatially organized in the presence of systematic spatial variability and randomness. A lot of the recent literature has begun to shed light on this issue [see *Rodriguez-Iturbe and Rinaldo*, 1997, and references therein]. However, a precise theoretical understanding of Horton's laws cannot be achieved by limiting them to a statement about means rather than entire probability distributions.

Self-similarity (and self-affinity) of a system can be defined in terms of its geometry, statistics, dynamics, or some combination of these three, and it implies that some property remains similar across a wide range of scales. For instance, geometric self-similarity means that under magnification of scale the geometry of smaller features embedded within a large object is similar to that of the large object. *Mandelbrot*'s [1982] work drew a great deal of attention to geometric self-similarity because it was found to be present in a wide variety of natural objects such as coastlines, clouds, etc. Geometric self-similarity leads to nonintuitive mathematical attributes since these objects are characterized by fractional dimensions. They are called "fractals" following *Mandelbrot* [1982]. These ideas be-

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gan to be formally explored in the context of drainage network patterns almost a decade ago, and many of these ideas are reviewed by *Rodriguez-Iturbe and Rinaldo* [1997]. Intuitively, a drainage network is a natural candidate for a fractal because small subnetworks are nested within large networks. Indeed, *Tokunaga* [1966, 1978] introduced the idea of self-similarity in network topology within a H-S framework. This important body of work is acquiring a new significance as the ideas of self-similarity in channel networks are beginning to be explored widely [*Peckham*, 1995a; *Tarboton*, 1996].

Statistical self-similarity implies that probability distributions of a variable as measured over a continuous range of scales, or, in our case, at a particular discrete sequence of scales, can be viewed as rescaled versions of each other. This sequence of scales is mathematically represented by a scale parameter. As will be discussed in section 2, one can choose from a variety of scale parameters for branching trees, of which H-S ordering is only one. However, the H-S ordering seems most natural for this purpose, and it lends itself to formalizing the concept of topologic self-similarity in the sense of *Tokunaga* [1966, 1978].

Statistical simple scaling was first investigated in the context of river networks within a magnitude-based setting for link elevation drops [*Gupta and Waymire*, 1989]. Using a link-based formulation, it is possible to derive a distributional version of Horton's laws for the H-S stream lengths of different orders, as shown in this paper, within the context of the random topology model. However, such an approach is analytically complex for link drops because the magnitudes of links comprising a stream of a certain order are not statistically independent.

Section 2 gives a brief overview of H-S ordering as derived by *Melton* [1959] and explains why this ordering scheme is ideal for defining topologic self-similarity, whereas other ordering schemes are not. This is followed by a summary of some key results for deterministic self-similar trees and a review of some link- and magnitude-based distributional results, including statistical self-similarity. Section 3 gives a motivation for and the mathematical interpretation of the distributional version of Horton's laws in terms of statistical self-similarity. Two theoretical examples are also given there for stream lengths for the random topology model. Section 4 gives empirical tests of a distributional version of number of side tributaries and of Horton's laws for channel lengths, areas, and drops. Finally, we conclude with some comments for further research.

2. Background

In this section we begin with a discussion of two general stream ordering systems. This discussion is followed by a review of Horton-Strahler stream ordering, which has a number of unique properties that make it particularly useful as a scale parameter for river networks. Next, we give a brief review of the self-similar tree model, which is based on the H-S ordering concept. Finally, we describe some distributional results which are link- and magnitude-based.

2.1. General Stream Ordering Systems

Mathematically, a scale or similarity parameter can be introduced in the context of river networks by assigning a numerical value, or an "order," to every link in a branching tree. Such ordering schemes have a long history, which dates back to at least 1914 (see *Zavoianu* [1985, chapter 3] for a brief review). Once each link has been assigned a number (usually an integer), then "maximal" chains of links that have the same order can be identified as "streams" of that order, which can be compared both to other streams of the same order and to streams of other orders. That is, the branching tree can then be decomposed or partitioned into objects that consist of one or more links.

Most ordering schemes assign an order to a link on the basis of the orders of the two or more "child" links that flow into its upstream end. In this case, the order of a link is given by $\omega = f(i, j)$, where i and j are the orders of its child links and f is an arbitrary function. Two important special cases are magnitude, where $f(i, j) = i + j$, and Horton-Strahler order, where $f(i, j) = \max(i, j) + \delta_{ij}$. (In both cases, exterior links are assigned a value of 1.) The special properties of these two ordering schemes become apparent as soon as we try to generalize to other schemes. For example, we find that for some schemes it may happen that $f(i, i) = i$, which leads to streams that are treelike rather than single threaded. An interesting magnitude-based example that has some features in common with H-S order can be written as $\omega = \lfloor \log_4(6m - 2) \rfloor$, where m is the magnitude and the brackets indicate the smallest integer that is greater than or equal to the real-valued argument. Similarly, we find that if we want the streams to consist of multiple links (or to "persist"), then there must be (i, j) pairs such that $f(i, j) = \max(i, j)$. Hence, requiring streams to be single-threaded, multilink objects rules out a great number of schemes.

It is not until we turn to the issue of similarity that the unique properties of H-S ordering become apparent. Note that in the H-S ordering scheme, there is only one way for a Horton-Strahler stream to begin, and this occurs when $i = j$; that is, $f(i, i) = (i + 1)$. The fact that this equation holds for all values of i and is the only way for an H-S stream to begin is essentially what makes H-S ordering so useful as a scale parameter. Recent work *Peckham* [1995a] with the self-similar tree model has helped to clarify this issue.

2.2. Horton-Strahler Stream Ordering

Horton-Strahler (H-S) stream ordering is a numerical classification scheme for identifying and ranking the major and minor tributaries in a river network. The essential idea was introduced by *Horton* [1945] but was later modified by *Strahler* [1952, p. 1120, 1957] to make it more natural and easier to use. This scheme is usually presented via a recursive assignment rule, but this rule hides the underlying significance and uniqueness of H-S ordering with respect to changes in scale. An alternate derivation based on a pruning operation was suggested in a short paper by *Melton* [1959] and has recently been reemphasized by *Peckham* [1995a, b]. In Melton's scheme a reduction in scale is equivalent to a pruning of low-order streams. We will briefly describe Melton's method first, followed by the equivalent recursive assignment rule.

In both approaches the river network is viewed as a tree graph which is rooted at its outlet, and the exterior links, or leaves in the tree, are defined to have an H-S order of 1. Melton's idea was to look at the tree graph that results from removing all of these order 1 streams. We can imagine that this less-foliated tree graph has little scars where its order 1 streams were originally attached. Exterior links, or leaves, can again be identified for the pruned tree, and the multilink chains are the H-S streams of order 2. The ones that originally contained multiple links will have little scars. Repeating this pruning procedure allows H-S streams of order 3 and higher to be determined, until there is only one stream of the highest order remaining.

The recursive rule is (1) exterior links are assigned order 1, (2) any link with two or more child links (upstream) of the same order ω are assigned order $(\omega + 1)$, and (3) any other link is assigned the maximum order that any of its child links have. Consecutive links that have the same H-S order form chains, and maximal chains, which begin with the confluence of two links of order $(\omega - 1)$ and terminate at the downstream end in a link of order $(\omega + 1)$, are called complete H-S streams of order ω. This procedure assigns exactly the same H-S order to each link in a tree graph as the pruning method of the last paragraph.

2.3. Self-Similar Tree Model

Tokunaga [1966, 1978] introduced a general tree graph model for river networks which is based on the H-S ordering scheme of the last section. *Peckham* [1995a] extended the results of Tokunaga and showed that the model's defining equation,

$$T_{\omega,\omega-k} = T_k, \tag{2}$$

could be used as a general definition of self-similarity in the context of tree graphs. Here $T_{\omega,\omega-k}$ is the (average) number of tributaries of order $(\omega - k)$ that enter a stream of order ω from the sides. The log linearity on Horton plots is a mathematical consequence of the recursive formulas that all self-similar tree graph constructions must obey. One of these formulas gives the total number of streams of order ω (anywhere in the tree) as

$$N_\omega = 2N_{\omega+1} + \sum_{k=1}^{\Omega-\omega} T_k N_{\omega+k}. \tag{3}$$

It is immediately clear from this equation that Horton's laws do not follow as an automatic consequence of the H-S ordering scheme itself; this issue has been debated in the literature [*Kirchner*, 1993; *Troutman and Karlinger*, 1994]. While the first term on the right-hand side (which counts upstream tributaries) follows from the definition of H-S order, it is the term containing T_k that determines whether the ratio $N_\omega/N_{\omega+1}$ will converge to a number R_B, and if so, the value of $R_B \geq 2$. Generating functions can be used to compute R_B and other stream ratios from the sequence of T_k values, and except for a few special cases of self-similar trees, such as the "structurally Hortonian" trees where $T_k = 0$ for $k > 1$, strict log linearity is only achieved in an asymptotic sense [*Peckham*, 1995a]. This log linearity implies that (asymptotically) the sample average grows exponentially with Strahler order, namely,

$$\langle X_\omega \rangle \propto R_X^\omega. \tag{4}$$

Empirical laws of this generic form are known collectively as "Horton's laws," although some were discovered by other researchers after Horton's original work. The constant R_X, the log of which appears as the slope of the regression line on the Horton plot, is referred to as a stream ratio and typically varies from one basin or region to the next.

Data analysis by *Tokunaga* [1966, 1978], *Peckham* [1995a, b], and *Tarboton* [1996] has shown that the deterministic self-similar tree model does a very good job of capturing the average internal bifurcation structure in real river networks. The last two authors used DEM-derived data for large river networks.

2.4. Link- and Magnitude-Based Results

Several authors have obtained distributional results for link attributes within a magnitude-based framework. Perhaps the best known of these is a result concerning the tail probabilities of link magnitudes and areas. *Rodriguez-Iturbe et al.* [1992] and *Rigon et al.* [1993] showed that link magnitude and area distributions for both real and simulated river networks are described by the following formula:

$$P[M \geq m] \propto m^{-\alpha}. \tag{5}$$

$P[M > m]$ is the probability the network has magnitude greater than m. These and related results are summarized in a book by *Rodriguez-Iturbe and Rinaldo* [1997]. A similar formula, with $\alpha = 1/2$, was derived by *de Vries et al.* [1994] for the random topology model.

Peckham [1995b] showed how the methods of *de Vries et al.* [1994] could be extended to derive a similar formula that applied to any deterministic self-similar tree graph. The first step is to compute the probability that a randomly selected link has order ω (in a tree of order Ω) as

$$P[W = \omega] = N_\omega C_\omega \Big/ \sum_{k=1}^{\Omega} N_k C_k. \tag{6}$$

Here W is the order of the chosen link, C_ω is the number of links in any stream of order ω, and N_ω is the total number of streams of order ω. Next, one uses the fact that both N_ω and C_ω obey Horton-type laws (in the limit) to derive the approximation

$$P[W \geq \omega] = (R_C/R_B)^{\omega-1}, \tag{7}$$

where R_B and R_C ($<R_B$) are the bifurcation and number-of-links ratios, respectively. Finally, one makes the following critical observation for deterministic self-similar trees: a link's order exceeds ω if and only if its magnitude exceeds M_ω. Here M_ω is the magnitude of a complete subnetwork of order ω which obeys a recursive formula similar to (3) and a Horton-type law. Combining these facts leads to the result that

$$\alpha = (1 - \beta), \tag{8}$$

where β is the "topological" Hack exponent and is given by $\beta = \log(R_C)/\log(R_B)$. It is interesting that although this result is for links, the mathematical derivation of it is based on H-S order and both scaling exponents are functions of two key stream ratios.

A search for the correct statistical structure of link drops led *Gupta and Waymire* [1989] to the hypothesis of statistical similarity for the link drops. It says that the probability distributions of any two link drops $H(m_1)$ and $H(m_2)$ of links of magnitudes m_1 and m_2 in a drainage network are related as

$$H(m_1) \overset{d}{=} (m_1/m_2)^\theta H(m_2), \tag{9}$$

where θ is a statistical scaling exponent. Predictions of the empirical link concentration function (LCF) by the mean LCF using (9) showed very good agreement with the data. In contrast, three other sets of hypotheses regarding the distributions of link drops consistently gave much worse predictions of the empirical LCF by mean. *Gupta and Waymire* [1989] argued that by using empirically observed downstream hydraulic-geometric relations between the bankfull discharge and channel slopes one should expect a "power law" relationship between mean channel height and magnitude, as predicted by (9), and this reference should be consulted for further details regarding these issues. *Gupta and Waymire*

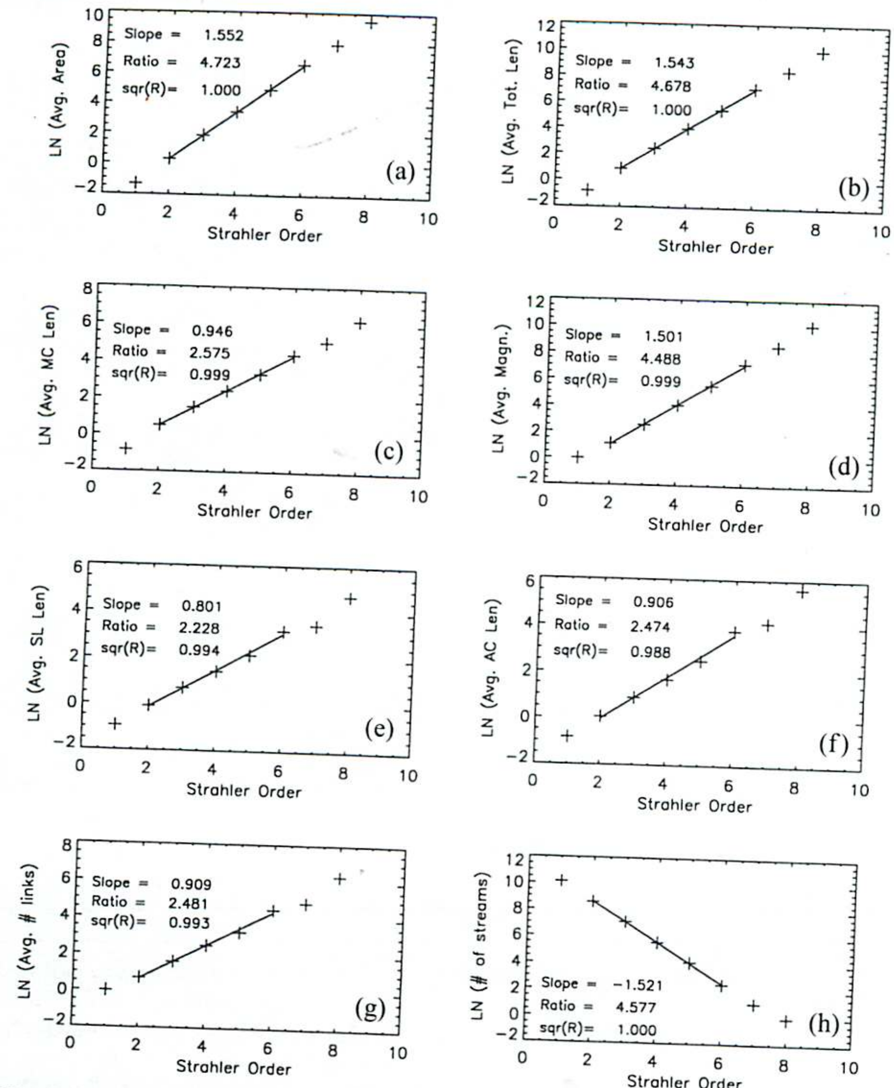

Figure 1. Horton plots for average (a) area, (b) total length, (c) longest channel length, (d) magnitude, (e) straight-line stream length, (f) along-channel stream length, (g) number of links per stream, and (h) number of streams.

[1989] also suggested that perhaps the simple scaling of (9) may not hold precisely and that other versions, such as multi-scaling, may be required. This suggestion was explored subsequently by *Tarboton et al.* [1989] and is discussed further by *Rodriguez-Iturbe and Rinaldo* [1997]. However, this issue is not the focus of the present paper.

3. Horton's Laws and Statistical Self-Similarity

3.1. What is Statistical Self-Similarity?

It is frequently useful to compare two random variables, say X and Y, and this is done by comparing their distribution functions. If they have the same cumulative distribution function (CDF), so that $F_X(x) = F_Y(x)$, then we write $X \stackrel{d}{=} Y$. For example, the distribution of link lengths in one river network might be the same as in another network. It can also happen that while the distributions of X and Y are different, the distributions of the normalized random variables $U = X/E(X)$ and $V = Y/E(Y)$ might be the same. Here $E(X)$ is the mean value of X, and it follows that $E(U) = E(V) = 1$. In this case, the random variables X and Y are said to be statistically similar.

The idea of statistical self-similarity is a further and significant generalization of statistical similarity that arises when the distributions of not just two random variables but a whole sequence of random variables, or even a whole continuum of random variables, are to be compared. Usually, these random variables are indexed by a scale parameter and refer to measurements that have been made at different scales within a single large system, hence the prefix "self." There is a growing literature which demonstrates the utility of the self-similarity concept in hydrology; for some recent references, see *Peckham* [1995b] *Sposito* [1998], *Rodriguez-Iturbe and Rinaldo* [1997], and *Foufoula-Georgiou and Tsonis* [1996]. Statistical self-similarity is also known as simple scaling. In this paper our scale parameter will be H-S order, and we will present data to show that measurements like H-S stream length are statistically self-similar within certain large river networks. However, we remind the reader that an ordering scheme is simply a "lense" which one uses to investigate and describe the internal bifurcation structure of a river network. The mathematical definition of self-similarity takes a somewhat different form depending on which ordering scheme one uses, as has been

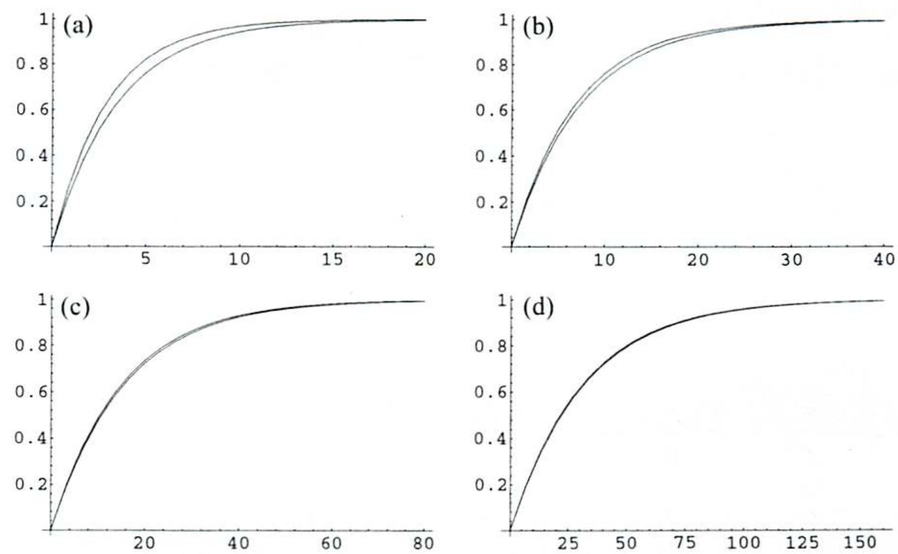

Figure 2. Comparison of $F_{L_{\omega+1}}(k)$ and $F_{L_\omega}(k/2)$ for the ω values (a) 3, (b) 4, (c) 5, and (d) 6. The curves are drawn as if k varies continuously, even though k is integer valued.

seen with previous magnitude-based analysis and the H-S ordering used here. However, the essential property that one is investigating remains the same. Self-similarity is either present or not and is not introduced by the ordering scheme itself.

3.2. A Reformulation of Horton's Laws in Terms of Statistical Self-Similarity

As explained in section 2, Strahler stream ordering is an ideal way to decompose a river tree or basin into a finite number of discrete scales. This decomposition groups subbasins of the same Strahler order into "ensembles" and puts Strahler order ω in the role of a scale parameter. In a traditional Hortonian analysis one measures some quantity X_ω (e.g., basin area, length of the highest-order stream, etc.) for all of the complete H-S subbasins of order ω in some large river basin. One then computes the sample average $\langle X_\omega \rangle$ for each ω and plots $\log\langle X_\omega \rangle$ versus ω. For almost any geometric or topologic quantity X_ω one finds that the data points can be fit quite well by a straight line. Figure 1 shows Horton plots for a variety of different measurements taken for the Kentucky River basin. The log linearity is seen to hold remarkably well.

Horton's laws can also be formulated mathematically by viewing X_ω as a random variable and $\langle X_\omega \rangle$ as an estimate of this random variable's expected value $E(X_\omega)$. In this context, Horton's laws are expressed in the form

$$E(X_\omega) = R_X^{\omega-k} E(X_k), \tag{10}$$

where ω and k are any two Strahler orders. This has been the accepted version of Horton's laws for half a century.

However, is this the strongest statement that one can make about how the measurement X_ω varies with the scale parameter ω? In section 4 we will present empirical evidence that the measurements X_ω are statistically similar to the measurements X_k, where ω and k are two different Strahler orders. We will also show that this feature is present in Shreve's well-known random model. A convenient mathematical formulation of this hypothesis is

$$\left(\frac{X_\omega}{E(X_\omega)}\right) \overset{d}{=} Z \qquad \forall\ \omega, \tag{11}$$

where Z is a random variable with mean 1 that does not depend on ω. Recall from section 3.1 that (11) states that if we normalize or rescale the random variable X_ω by its mean, which effectively nondimensionalizes the data, the distribution of the resulting random variable is independent of ω. We will refer to (11) as the hypothesis of weak statistical self-similarity (SSS). The prefix self indicates that (11) is hypothesized to hold for all values of ω present in some large basin.

A consequence of (11) is that

$$X_\omega \overset{d}{=} \left[\frac{E(X_\omega)}{E(X_k)}\right] X_k. \tag{12}$$

Combining (12) with a Horton law in the form (10) gives

$$X_\omega \overset{d}{=} R_X^{\omega-k} X_k, \tag{13}$$

where, again, ω and k are any two Strahler orders. Writing $R_X = e^c$, this type of statistical self-similarity can be shown to be the most general scaling property of the form

$$X_\omega \overset{d}{=} g(\omega - k) X_k, \tag{14}$$

where g is an arbitrary function. Similarly, the magnitude-based definition of self-similarity $X_m \overset{d}{=} (m/j)^\theta X_j$ is the most general scaling property of the form

$$X_m \overset{d}{=} g(m/j) X_j. \tag{15}$$

The functional forms for g follow from the definitions themselves using a recursive argument [see *Peckham*, 1995b]. We will refer to (13) as the hypothesis of strong SSS. Clearly, (13) is consistent with Horton's laws, since it was obtained by combining these laws with weak SSS: note that computing expected values on both sides of (13) returns us to (10). However, strong SSS is a significant generalization of Horton's laws from scaling of first moments, or means, to scaling of entire distributions. For example, assuming that the CDF of X_ω scales according to (13), it follows immediately that all of the moments must scale according to the formula

$$E(X_\omega^h) = R_X^{h(\omega-k)} E(X_k^h), \tag{16}$$

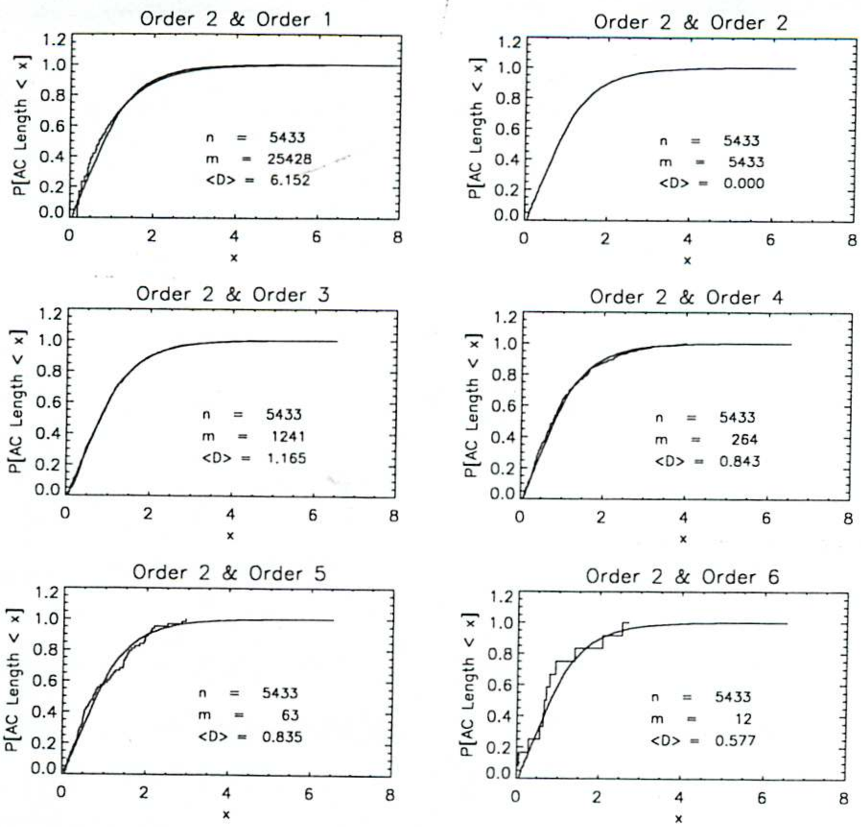

Figure 5. Testing the weak form of the statistical self-similarity hypothesis for along-channel stream lengths, using ECDFs.

distributions of these random variables. As discussed in section 2.4, it was shown by *Gupta and Waymire* [1989] that link drops cannot be assumed to be independent of link magnitude. Note, however, that link lengths are observed to be approximately independent of link magnitude for many large river networks [*Shreve*, 1969].

3.3.2. Example 2: Unit link lengths in Shreve's model. The results of the last section held exactly for any pair of Strahler orders ω and k and not just in an asymptotic sense. This relied, in part, on choosing the link lengths from an exponential distribution. However, we expect the statistical self-similarity to be primarily due to the geometric distribution of links in a Strahler stream, and hence it should appear in an asymptotic sense whenever the distribution used for the link lengths has a finite second moment. This conjecture can be motivated by the following heuristic argument.

Suppose that all of the links in G_m have unit length. This implies that the length of our randomly selected Strahler stream of order ω, denoted here as L_ω, is equal to the number of links in this stream, so that $L_\omega = C_\omega$. We now show that $L_{\omega+1} \overset{d}{=} 2L_\omega$, as ω tends to infinity. To do this, we will work backward through a sequence of equivalent statements until the postulated asymptotic equality can be verified. Hence we have

$$L_{\omega+1} \overset{d}{=} 2L_\omega$$

$$F_{L_{\omega+1}}(k) = F_{L_\omega}(k/2) \qquad \forall k$$

$$[1-(1-p_{\omega+1})^k] = [1-(1-p_\omega)^{k/2}] \qquad \forall k$$

$$(1-p_{\omega+1}) = (1-p_\omega)^{1/2} \qquad (28)$$

$$\left(1-\frac{1}{n}\right) = \left(1-\frac{2}{n}\right)^{1/2}$$

where $n \equiv 2^\omega$

$$\left(1-\frac{1}{n}\right)^n = \left(1-\frac{1}{(n/2)}\right)^{n/2}$$

However, for sufficiently large $n \equiv 2^\omega$ (or sufficiently large ω, note that n is not the magnitude) it is a well-known fact that both sides of the last equality approach the common value of e^{-1}. Actually, the limiting equality in the distribution is approached rather quickly, as can be seen from Figure 2. For $\omega = 3$ (Figure 2a) the curve for $F_{L_{\omega+1}}(k)$ is very close to the one for $F_{L_\omega}(k/2)$, and for $\omega = 6$ (Figure 2d) the two curves are indistinguishable.

4. Empirical Tests for Large Basins

4.1. Empirical Cumulative Distribution Functions (ECDFs)

Each of the formulations of statistical self-similarity that we have discussed involves the notion of "equality in distribution," as reviewed in section 3.1. Given a set of measurements (or samples) being viewed as realizations of a random variable, the ECDF can be estimated as follows. Let n be the number of samples, and let $\{x_1, x_2, \cdots, x_n\}$ be these samples sorted in ascending order. The ECDF $S_n(x)$ is defined as the step function that has height (k/n) on the interval $[x_{k-1}, x_k)$, where $k \geq 0$ and $x_0 = -\infty$. Note that $S_n(x)$ gives the proportion of the n sample values that are less than x. This function in-

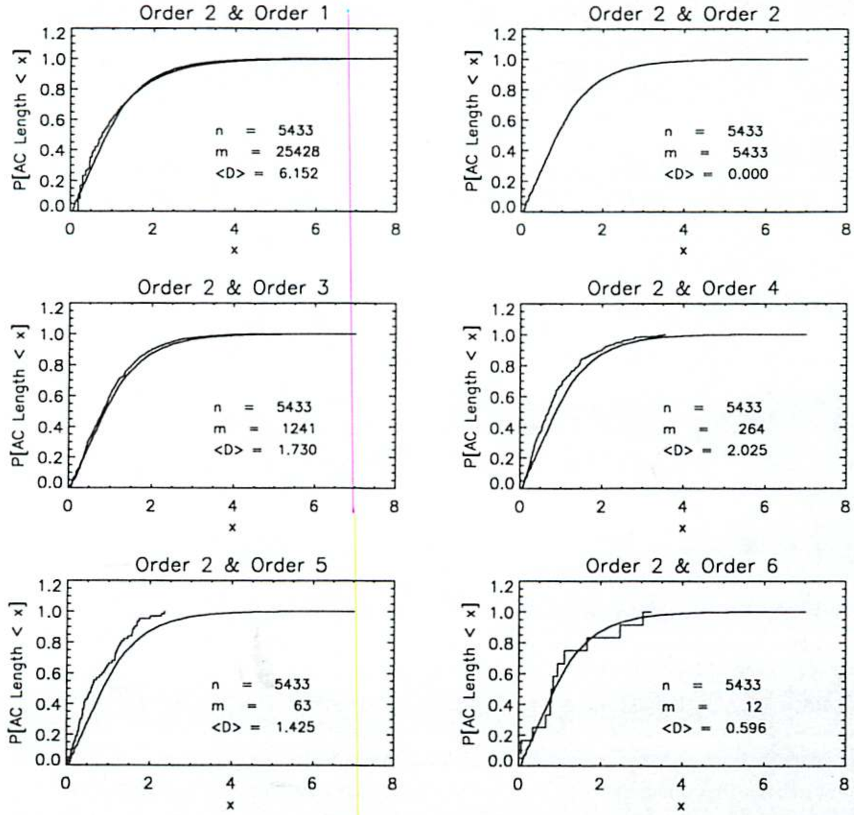

Figure 6. Testing the strong form of the statistical self-similarity hypothesis for along-channel stream lengths, using ECDFs.

creases from the value 0 up to the value 1 as it should, since this is a defining property of a CDF.

In this paper we are primarily interested in the stream-based formulation of statistical self-similarity. In sections 4.2–4.5 we will measure some quantity X_ω for all of the complete Strahler subbasins of order ω in a given large river basin. Then, these measurements will be used to compute an ECDF for X_ω. Repeating this for each value of ω, normalizing as necessary, and graphically comparing the resulting ECDFs will allow us to check the hypotheses (11) and (13) directly for many different basin measurements. In section 4.6 we will discuss a more quantitative method for comparing ECDFs called the Kolmogorov-Smirnov two-sample test.

The measurements that are analyzed here were made using the RiverTools software toolkit. RiverTools automates the extraction of river networks from very large DEMs and uses state-of-the-art algorithms that are similar to those used by the U.S. Geological Survey (USGS). These algorithms are continually evolving, and better methods for dealing with issues such as (1) source identification, (2) routing across flats, and (3) divergent flow at the hillslope scale are still an active area of research. For an in-depth discussion of these issues, see *Peckham* [1998, and references therein]. However, years of practical experience with these algorithms has demonstrated that they are robust and well suited to a variety of statistical analyses for large river networks. It is our judgement that the anticipated refinements to these procedures will not significantly affect the results we are presenting here.

4.2. ECDFs for Strahler Basin Areas

It turns out that both the weak and strong versions of the SSS hypothesis for basin areas hold remarkably well for many basins. Let A_ω denote the drainage area of a Strahler basin of order ω. The weak hypothesis for basin areas is then simply

$$\frac{A_\omega}{E(A_\omega)} \overset{d}{=} Z \qquad \forall\ \omega, \tag{29}$$

where Z is some random variable with mean 1. Figure 3 shows the ECDF for the variable $A_2/E(A_2)$ overlaid on the ECDF for $A_\omega/E(A_\omega)$, where ω is varied from 1 to 6. Hence (29) is being tested directly. A Strahler order of 2 was chosen as a "reference scale" because order 1 streams and basins are rather unique in that they have no streams entering their upstream end and they are positioned at the transition between hillslopes and channelized flow.

Not surprisingly, we have found that the statistics of first-order basins do not always conform to the same pattern as the other orders. Though the ECDFs are smoother for smaller values of ω (since the sample sizes are larger), all of the curves lie on top of one another. Though the agreement is visually quite good, a quantitative comparison using the Kolmogorov-Smirnov two-sample test also supports the weak SSS hypothesis. The strong SSS hypothesis,

$$A_\omega \overset{d}{=} R_A^{\omega-k} A_k, \tag{30}$$

can be tested in a similar manner, except that instead of comparing the ECDF for $A_\omega/E(A_\omega)$ to that of $A_2/E(A_2)$ one

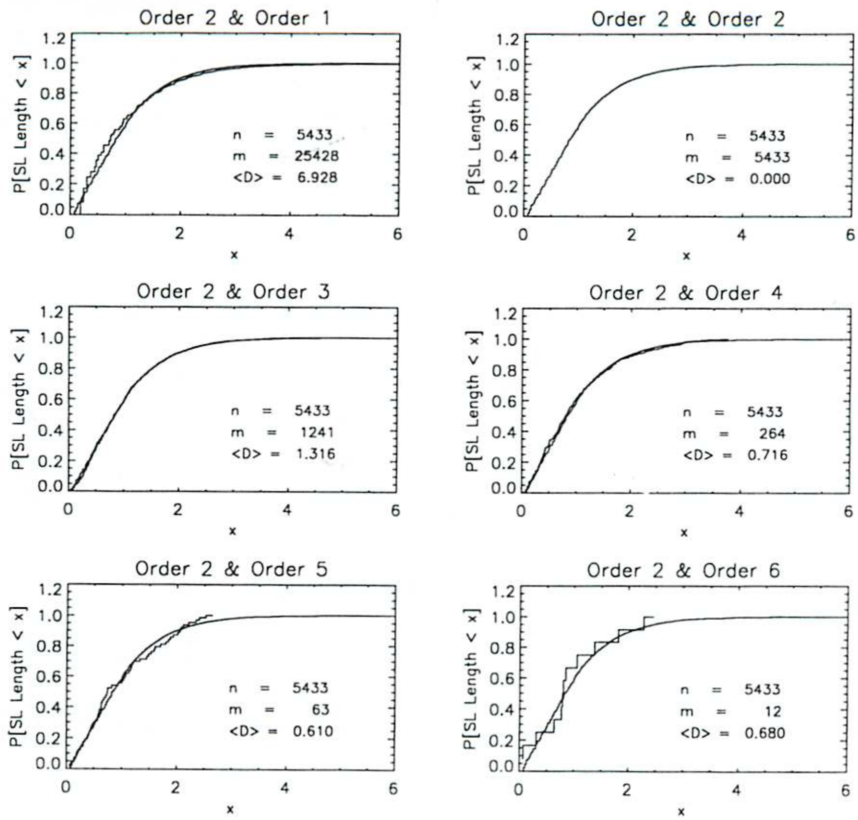

Figure 7. Testing the weak form of the statistical self-similarity hypothesis for straight-line stream lengths, using ECDFs.

compares the ECDF of $A_\omega/R_A^{\omega-2}$ to that of A_2, for each value of ω; see Figure 4. Again, the agreement is quite good. Similar results have been obtained for several other basins [*Peckham*, 1995b]. In fact, of all of the quantities to be examined in this paper, basin areas seem to satisfy the SSS hypotheses best. These results are consistent with the idea that basin shapes are statistically self-similar but do not necessarily imply that this is the case. See *Peckham* [1995b] for the construction of a simple and informative counterexample. Notice that the strong SSS hypothesis relies on rapid convergence to the stream area ratio R_A. Moreover, to test this hypothesis, we require an accurate estimate of this ratio. A Horton plot provides one method for estimating R_A; additional methods are discussed by *Peckham* [1995b]. Convergence is but one of many issues that make it difficult to obtain accurate estimates of stream ratios.

4.3. ECDFs for Strahler Stream Lengths

Let us now examine the weak and strong SSS hypotheses for the along-channel lengths of Strahler streams, namely,

$$\frac{L_\omega}{E(L_\omega)} \stackrel{d}{=} Z \qquad L_\omega \stackrel{d}{=} R_L^{\omega-1} L_1. \tag{31}$$

The hypotheses in (31) are tested through a graphical comparison of ECDFs in Figures 5 and 6. Both a graphical comparison and the more stringent Kolmogorov-Smirnov two-sample test support the weak SSS hypothesis for the Kentucky River basin and its subbasins, while the strong hypothesis appears to break down for Strahler basins of order 4 and 5; see Figure 6.

We can also check the weak and strong SSS hypotheses for the straight-line lengths L_ω^* of Strahler streams. Equations similar to (31) are checked graphically in Figures 7 and 8, and, again, the agreement is good. As argued by *Peckham* [1995b, pp. 152–156], straight-line stream lengths are thought to be largely determined (or constrained) by basin shape.

4.4. ECDFs for Strahler Stream Drops

For elevation drops across H-S streams H_ω the weak and strong SSS hypotheses take the form

$$H_\omega/E(H_\omega) \stackrel{d}{=} Z \qquad H_\omega \stackrel{d}{=} R_H^{\omega-1} H_1. \tag{32}$$

As in the previous sections, these hypotheses are tested graphically by comparing ECDFs in Figures 9 and 10.

The step-like character in the plots can be traced to the fact that 3-arc sec USGS DEMs are made by fitting a surface to the points obtained from digitizing a contour map. This causes contour elevations to be overrepresented in the resulting DEM, even if the width of the contour line is only one pixel wide. If the contour line is wider than a pixel, then "terraces" appear in the DEM. Notice, however, that stream drops of a size comparable to the contour interval are most affected. For streams with drops spanning several contour intervals, the problem begins to disappear. This explains why the tails of the stream-drop ECDFs are so much smoother. Considering this problem, the ECDFs are seen to match up fairly well for both versions of the SSS hypotheses. Clearly, additional analysis using DEMs that do not have the terracing problem will be needed to properly test these hypotheses. The terracing problem for stream drops also affects stream slopes but to a lesser extent because of the stabilizing effect of stream lengths.

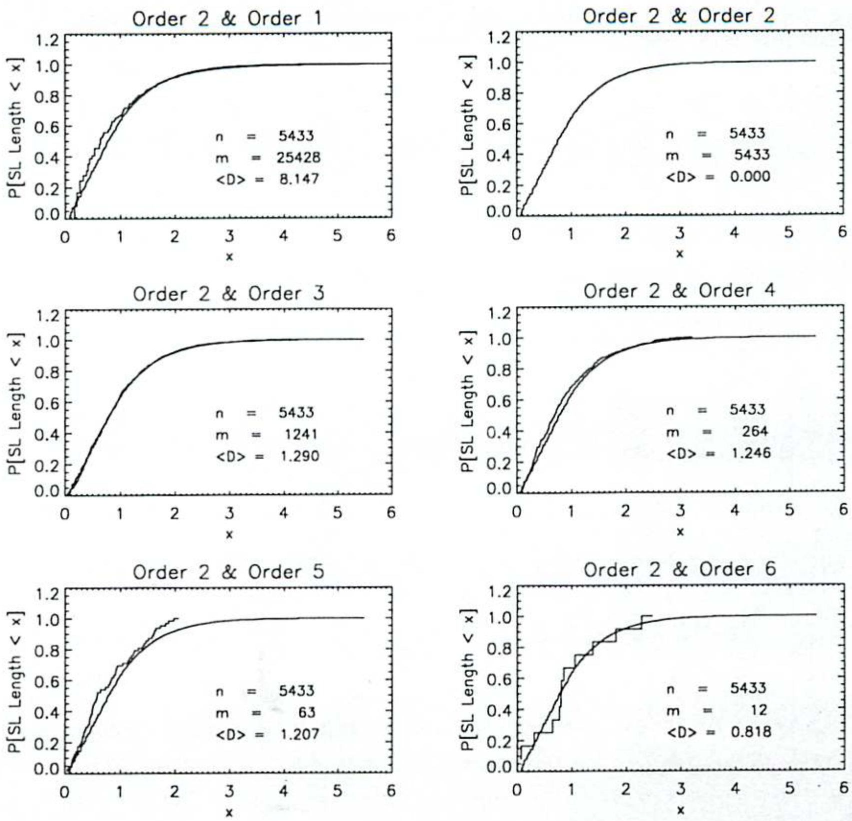

Figure 8. Testing the strong form of the statistical self-similarity hypothesis for straight-line stream lengths, using ECDFs.

4.5. Histograms for Number of Side Tributaries

We have seen that the hypothesis of statistical self-similarity seems to hold quite well for many of the geometric quantities in river basins. In this section we will give empirical evidence which supports the idea that real river networks also have statistically self-similar topologies as defined below.

Let $X_{\omega,\omega-k}$ be a random variable corresponding to the number of tributaries of order $(\omega - k)$ entering a given stream of order ω from the sides. *Peckham* [1995a] showed that the sample average $\langle X_{\omega,\omega-k}\rangle$ could be closely approximated by a constant T_k that depended on k but not ω, or $\langle X_{\omega\ \omega-k}\rangle \approx T_k$. This was evident from the fact that the generator matrix assembled from these averages had values that were roughly constant along diagonals. *Peckham* [1995b, p. 173] suggested that a natural extension of statistical self-similarity to network topology is the following

$$X_{\omega,\omega-k} \stackrel{d}{=} Z_k \qquad \forall\ \omega, \tag{33}$$

where Z_k is a discrete (i.e., integer valued) random variable that depends only on k, with mean $E(Z_k) = T_k$. Notice that this hypothesis differs somewhat from the weak and strong SSS hypotheses, in that it involves no ω-dependent rescaling.

Since (33) involves discrete (i.e., integer valued) distributions and no rescaling, it is more appropriately tested by comparing frequency histograms rather than ECDFs. Figure 11 shows such a comparison for the case $k = 1$, while Figure 12 shows the results for the case $k = 2$. The agreement seems quite good based on a visual comparison. In particular, the histograms for $\omega = 2$ and $\omega = 3$ are virtually identical. Unfortunately, the Kolmogorov-Smirnov test is not appropriate for discrete random variables like $X_{\omega,\omega-k}$, as discussed in section 4.6.

4.6. Kolmogorov-Smirnov Two-Sample Test

Throughout this paper we have been computing ECDFs from data and testing the weak and strong SSS hypotheses graphically. However, it is also possible to make this comparison quantitative and objective using a statistical test called the Kolmogorov-Smirnov two-sample test. The underlying idea behind this test is quite simple. Let $S_n(x)$ and $T_m(x)$ be two ECDFs, where n and m are the number of samples in the two different ensembles. Now define a test statistic as

$$D = \max_x |S_n(x) - T_m(x)|, \tag{34}$$

which is just the maximum difference between our two ECDFs. Note that if these two step functions overlap exactly, then we will have $D = 0$, otherwise D will be a measure of the discrepancy between the two functions. However, this measure is based on the worst case, or the largest gap between the curves. Other measures are also possible; for example, one could use the average of the mean square separation.

In practice, D can be computed by combining the measurements for the two ensembles and sorting these values to get a sequence $\{z_1, z_2, \cdots, z_{n+m}\}$. We then define r_k as the number of values in the subsequence $\{z_1, z_2, \cdots, z_k\}$ that came from the ensemble with n samples and s_k as the number that came from the one with m samples. Notice that $(r_k + s_k) = k$. It is not difficult to show that D can be computed from the formula

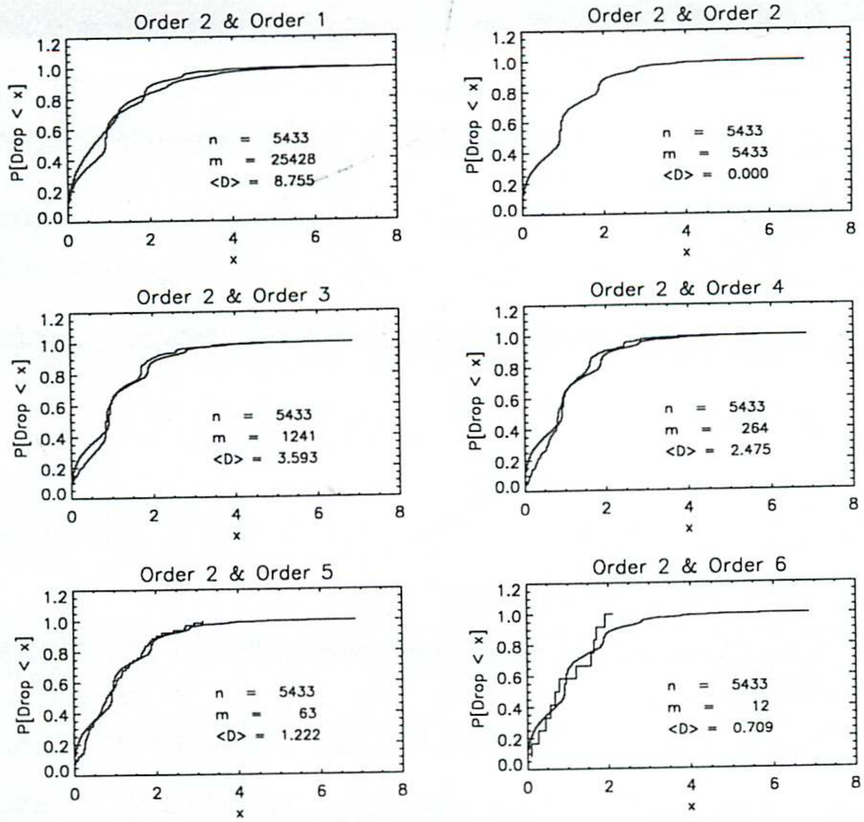

Figure 9. Testing the weak form of the statistical self-similarity hypothesis for stream elevation drops, using ECDFs.

$$D = \max_k \left| \frac{r_k}{n} - \frac{s_k}{m} \right| . \tag{35}$$

The remarkable thing about the test statistic D is that its distribution is independent of the distributions of the random variables we are measuring. Statistical tests with this property are called nonparametric or distribution-free tests. Unlike parametric tests, these tests require no a priori assumptions about the distribution of the data.

It is clear that the distribution of D must depend on the sample sizes n and m. However, as n and m become moderately large (say, greater than ~40), the distribution of the normalized test statistic $\hat{D} = D\sqrt{nm/(n + m)}$ approaches a limiting distribution that is independent of n and m. Tables for this limiting distribution are widely available and contain values for p and d_p, which are defined as

$$P(\hat{D} \geq d_p) = p. \tag{36}$$

Observe that d_p is the pth quantile of the random variable $\hat{D}$. Some benchmark (d_p, p) pairs are given by {(1.224, 0.10), (1.358, 0.05), (1.517, 0.02), (1.628, 0.01), (1.858, 0.002), (1.950, 0.001)}. Hence, if our hypothesis is true, then the odds of D being >1.224 are 1 in 10, while the odds of it being >1.950 are only 1 in 1000. This means that if D is <1.224, we can accept the hypothesis with a confidence of 90%. The statistic $\hat{D}$ is included as an inset in each of the ECDF plots in this paper. As is typical for statistical tests, the emphasis is on whether we should reject the hypothesis. While a failure to be rejected is reassuring, it does not prove that our hypothesis is correct.

In closing, we should point out that this test assumes that the measured quantities vary continuously and that there are no ties. Therefore this test is suitable for measurements like length and area but not for discrete measurements like magnitude and number of links per stream. It should be kept in mind, however, that even our length and area measurements are likely to contain some ties because of the finite pixel size of the DEMs. It is also implicitly assumed that we are drawing our samples at random from an infinite population. For more details on ECDFs and the Kolmogorov-Smirnov test, see *Bradley* [1968, 288–295]. A one-sided test based on this same idea can also be constructed by removing the absolute value signs from the previous equations for D. It is also possible to generalize from a two-sample test to an n-sample test, which in some ways would be more natural for the current problem. However, the two-sample test seems adequate for our purposes.

5. Final Remarks

We have presented a theoretical formulation of statistical self-similarity in the geometry of river basins involving H-S stream lengths and areas, in the hydraulic-geometry involving H-S stream drops, and in the topological or branching structure of river networks in terms of the statistical distribution of side tributaries as defined by (33). We have shown that a mathematical reformulation of Horton's laws in terms of probability distributions is in excellent agreement with available data. The traditional Horton's laws in terms of statistical averages are a very special case of this reformulation. The far-reaching implication is that certain information learned from studying the low-order basins in a region can be scaled up in a

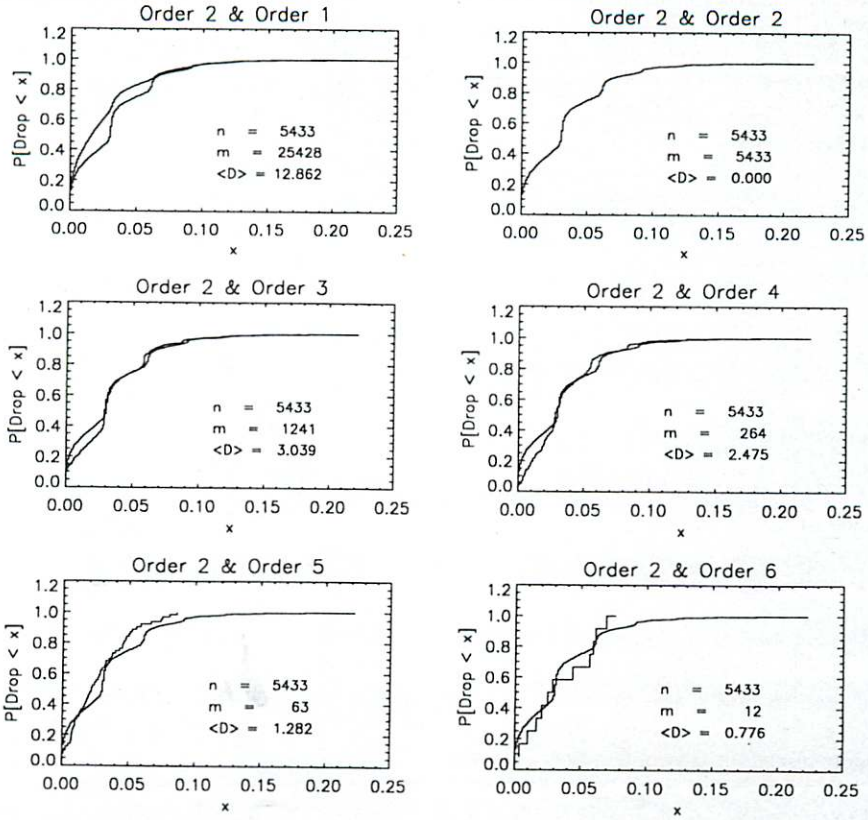

Figure 10. Testing the strong form of the statistical self-similarity hypothesis for stream elevation drops, using ECDFs.

straightforward manner to get statistical information about much larger basins.

These results in conjunction with a recent result by G. A. Burd et al. (unpublished manuscript, 1999) suggest that there is a critical need for developing a new class of stochastic self-similar models for river networks. This class must be able to accommodate the observed deviations between data and the predictions of the random model and the deterministic self-similar class of *Tokunaga* [1966, 1978] should be a special case of this reformulation. Moreover, statistical self-similarity in Horton's laws as reported here should also follow from such a class of models. Some new theoretical results in this line of research have been obtained by S. Veitzer and V. Gupta (unpublished manuscript, 1999). Previous analytical results by *Wang and Waymire* [1991] for fluctuations around average stream numbers are specific to the random topology model.

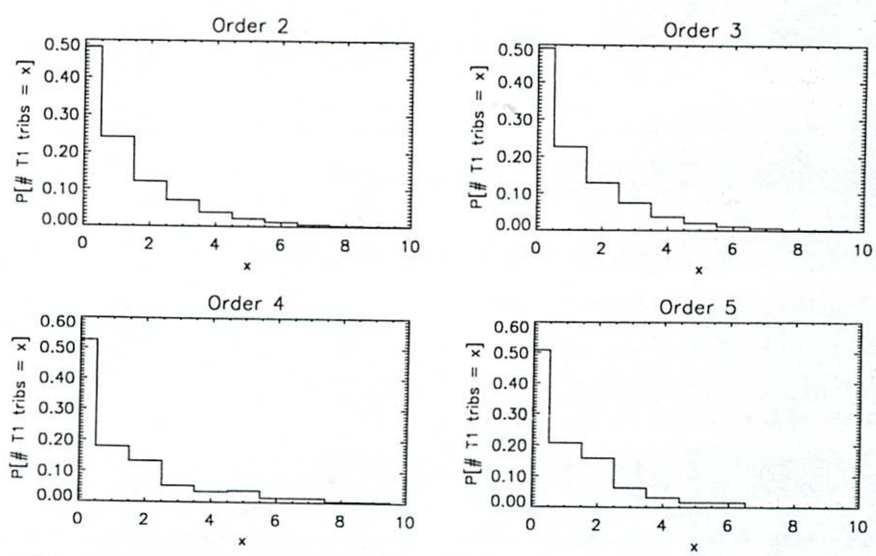

Figure 11. Testing the statistical self-similarity hypothesis for $T_{\omega,\omega-1}$ for orders 2–5, using simple histograms with no scaling.

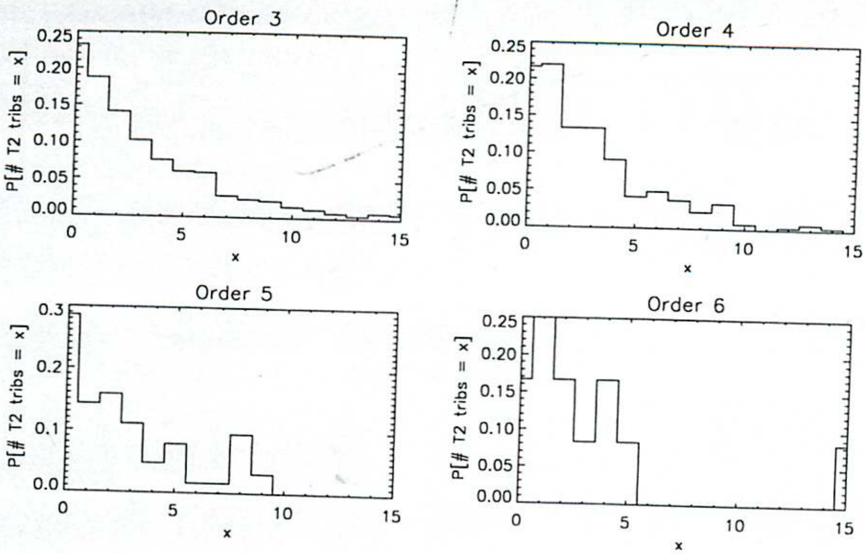

Figure 12. Testing the statistical self-similarity hypothesis for $T_{\omega,\omega-2}$ for orders 3–6, using simple histograms with no scaling.

However, they can be used to devise a statistical test of hypotheses for the observed deviations between data and the predictions of the random model for stream numbers reported by *Peckham* [1995a].

How basins evolve to a statistical self-similar state as described here is a newly developing theme of research in theoretical geomorphology. For example, in a companion paper (S. D. Peckham et al., manuscript in preparation, 1999) we have demonstrated the dynamical significance of Horton ratios by starting from first principles involving conservation laws and a postulate of "dynamic self-similarity." Although this companion paper has thus far considered only averages rather than entire probability distributions, it leads to a new theory of downstream hydraulic geometry and to theoretical predictions of numerous hydraulic-geometric exponents that are all in good agreement with empirical observations for drainage networks [*Leopold et al.*, 1964]. Other new results involving scaling symmetries and conservation laws are given by *Peckham* [1995b, 1999]. We expect that a certain degree of statistical homogeneity in geology and climate is required for basins to reach a state of statistical self-similarity.

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