TAMMES'S PROBLEM

EDITH MOOERS

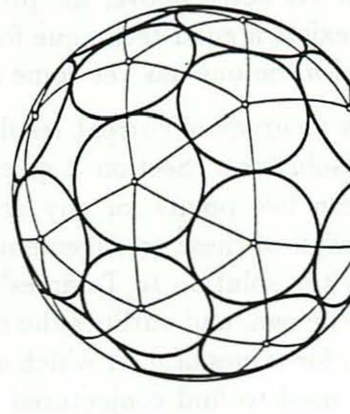

FIG. 0.1. *The solution to Tammes's Problem for 24 exit places, from [16].*

1. Introduction. In 1930 the Dutch botanist R. M. L. Tammes [12] noticed that certain numbers and arrangements of exit places on the surface of spherical pollen grains are favored over others. An exit place is an area on the surface of the pollen grain from which a pollen tube emerges during the process of fertilization. The exit places on the pollen grains of a given plant can be assumed to be nonoverlapping circular 'caps' of the same area on the surface, so that the area needed for one exit place puts an upper limit on the number of exit places that can appear on a pollen grain. This number is different for different species, since the area needed for an exit place and the size of pollen grains can vary greatly from plant to plant. Tammes conjectured that the arrangement of places of exit for a given plant maximizes the possible number of exit places given the area needed for an exit place and the size of the pollen grain.

Assuming that each exit place on a pollen grain takes up a certain circular area on the surface, Tammes's Problem is equivalent to asking how many spherical caps of a certian radius can be places on a unit sphere, without overlapping. There are several other ways to formulate this problem. Another way is to find the arrangement of n points on a unit sphere that maximizes the minimum distance between any two points. This is seen to be equivalent to the original problem by thinking of the n points as the centers of spherical caps of equal maximum radius. This is the form of the problem that I will be using for most of this paper.

Tammes's Problem has been solved on the 2-sphere in the cases $1 \leq n \leq 12$ and $n = 24$. Solutions that are conjectured to be optimal have been found for $n \leq 60$ and for isolated higher values. Values for conjectured solutions up to $n = 60$ are given in figure 1.1. In the graph, height represents the ratio of the surface area summed over all spherical caps of maximum radius to 4π, the surface area of the unit sphere. In this paper only the case of the 2-sphere is considered. For a discussion of Tammes's Problem on the d-sphere with $d > 2$, check [6] and [7].

This problem is quite a bit harder than it would appear to be at first glance. Induction seems out of the question since, in general, the best set of points for n bears no relation to the best set of points for $(n+1)$. Although good upper and lower bounds exist for the distances between points, actual solutions vary greatly as to how close they come to reaching these bounds. As noted above, the problem has been solved only for very low values of n. If there exists a good technique for showing that a given vertex set is optimal for large values of n, no one has yet come across it.

This paper is intended as a survey of current results for Tammes's Problem and techniques used to arrive at solutions. Section 2 gives an upper limit for the maximum minimum distance between two points for any arrangement of n points on the unit sphere and describes a graph of these arrangements that is useful in proving some results. Section 3 describes the solution to Tammes's Problem for values of n for which optimal arrangements are known, and outlines the methods of proof used in these cases. Section 4 describes results for values of n for which optimal arrangements are not known and describes techniques used to find conjectured solutions. Section 5 reviews an interesting open problem concerning the solution to Tammes's Problem on the 2-sphere.

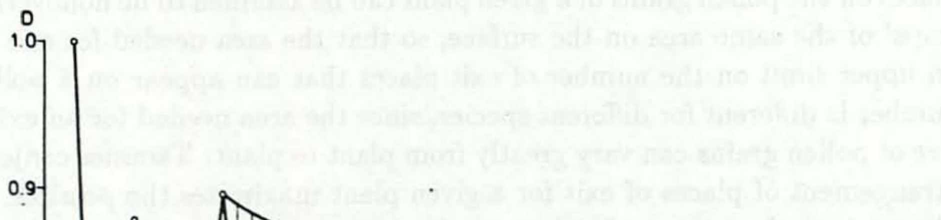

FIG. 1.1. *Conjectured solutions to Tammes's Problem for* $n \leq 60$*, from [16]. The height of the graph is the ratio of the combined area of all sphercial caps to the area of the sphere.*

2. Basic Results. Tammes's Problem can be stated as follows: What arrangement of n points on the surface of a sphere will maximize the minimum distance between any two points? For points $P_i, 1 \leq i \leq n$ on the unit sphere, let

$$d_n = max(min\{P_iP_j : 1 \leq i < j \leq n\})$$

where P_iP_j is the distance between the points P_i and P_j and the maximum is taken over all possible arrangements of n points on the sphere. It is immaterial whether this is measured as spherical or Euclidean distance. This problem is equivalent to the problem of how to put n non-overlapping spherical caps of maximum equal radius on the unit sphere.

2.1. An Upper Limit. The following inequality of L. Fejes-Tóth [4] gives a useful upper bound on d_n:

THEOREM 2.1. *Given $n > 2$ points on the surface of the unit sphere, there always exist two having spherical distance*

$$\alpha \leq cos^{-1}\{\frac{cot^2 w - 1}{2}\} \quad where \quad w = \frac{n}{n-2}\frac{\pi}{6}$$

The right hand side of the above inequality is the side length of an equilateral spherical triangle of area $\frac{4\pi}{(2n-4)}$.

If $n = 3$, a spherical triangle with vertices lying on any great circle has area 2π and the vertices of this triangle give the best arrangement for d_3. If $n \geq 4$ the n points may be assumed not to lie on any great circle. The convex hull of these points is a convex polyhedron containing the origin which may be assumed to have only triangular faces, since the faces may be made into triangles by adding extra lines. Projecting the edges of this polyhedron from the origin onto the unit sphere gives a triangulation of the sphere. Euler's formula shows that this triangulation divides the surface into $2n-4$ triangular faces.

LEMMA 2.2. *If the area of a spherical triangle ABC is less than the area of the equilateral spherical triangle ABD drawn on the shortest side AB of ABC, then the spherical radius of the circle circumscribed to ABC is greater than AB.*

Using Lemma 2.2, Fejes-Tóth shows that if all edge lengths in the triangulation of a system of n points are greater than γ, where γ is the side length of an equilateral triangle of area $\frac{4\pi}{(2n-4)}$, an additional point can be added to the system at a distance greater than γ from the n original points. Since the number of points that can be placed on a sphere having mutual distance greater than γ is bounded, this gives a contradiction.

2.2. The Graph. Given a set of n points on the unit sphere with minimum distance α between pairs of points, form the graph G consisting of the n points and all segments (arcs of great circles) of length α which join pairs of points. The minimum distance α can be increased only by increasing the length of all edges of the graph G simultaneously. If $a(\alpha)$ is the interior angle of an equilateral triangle of sidelength α, $\cos\alpha = \frac{\cos a(\alpha)}{1-\cos a(\alpha)}$. $a(\alpha)$ is increasing as a function of α with $180° \geq a(\alpha) \geq 60°$.

The *valence* of a point P of a graph G is the number of arcs in G of length α connecting it to other points of G. An *isolated* point P of G is a point with valence zero. Two vertices are *adjacent* if they are connected by an edge of length α. A set of n points for which $P_iP_j \geq \alpha$ for all points P_i, P_j of G, where $i \neq j$, will be called *extremal* if $\alpha = d_n$ and no point of G can be isolated by moving one vertex. A graph G will be called *irreducible* if no improvement can be made to the graph, in terms of increasing the edge length α, by altering a single vertex.

This graph has been useful in finding solutions and conjectured solutions to Tammes's Problem. When the number n of points is relatively low, it is possible to analyze the irreducible graphs that can occur and, by computing the minimum distance α between any two vertices of the graph, find the arrangement for which α is largest. Such exaustive methods are not feasible for large values of n. Danzer [2] mentions an attempt made to use computers to find extremal graphs for $n = 10$ by shifting vertices of existing graphs, but even for such a small value the program tended to get stuck at a relatively maximum irreducible graph that did not correspond to a global maximum.

Figures accompanying the text show simplified stereographic projections of these graphs onto the plane.

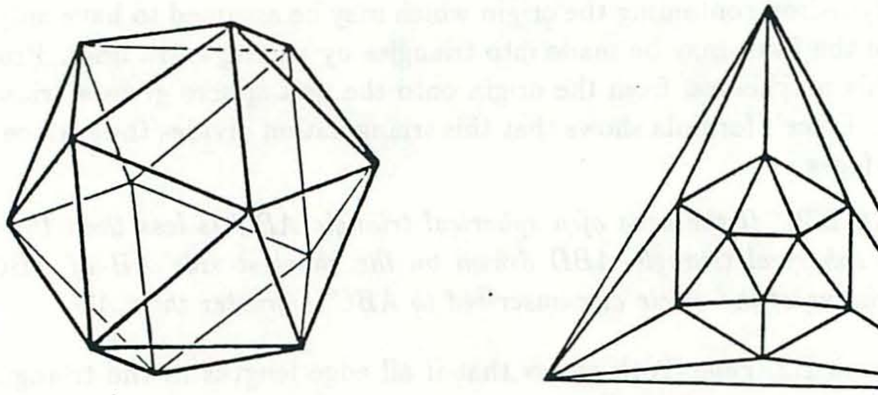

FIG. 2.3. *The solution to Tammes's Problem for $n = 12$ and its graph, from [3].*

The following two theorems give a feel for the basic properties of these graphs and will be useful later on in discussing the solutions for particular values of n.

THEOREM 2.4. *Let G be the graph of an extremal point set on $n > 6$ points. Then the valence of any non-isolated vertex of G is at most 5 and at least 3.*

Proof. Tammes [12] noticed that each point could have at most 5 nearest neighbors. Let m be the number of points of a graph G that are at a distance α from a point P of G, where α is the minimum distance between any two vertices of G. These points are all connected to P by an edge of G, and they must all also be at a distance no less than α from each other. The number m of points connected to P by an edge of G is largest when all the polyhedra meeting at P are right spherical triangles, therefore each angle at P is greater than $60°$, the lower bound for the angle of an equilateral spherical triangle. Therefore $60°m \leq 360°$ and it follows that $m \leq 5$.

By Theorem 2.1, if $n > 6, d_n \leq \alpha < 90°$ for any set of n points on the sphere. If all arcs of G issuing from P lie within an angle of $180°$, P can be shifted slightly and isolated. Therefore we can assume that the vertices of any extremal graph G either have valence at least 3 or are isolated. □

Triangle inequalities show that no two edges of the graph can intersect, so the edges of G divide the sphere into strictly convex polygons, all angles being less than $180°$. All triangles of G are equilateral since all their edge lengths must be equal.

THEOREM 2.5. *An irreducible graph G on $6 < n \leq 12$ points contains only equilateral triangels, quadrangles and pentagons. If there exists a configuration on n points for which $\alpha \geq 72°$, then the graph G contains only equilateral triangles and quadrangles and no isolated points.*

Proof. The regular icosahedron gives an arrangement of 12 points on the sphere with $\alpha > 60°$, therefore there exist extremal arrangements of n points on the sphere with $\alpha > 60°$ for all $n \leq 12$. If m is the number of edges a spherical polygon in an extremal graph G with $6 < n \leq 12$ vertices then, since all polygons of G are strictly convex, $m\alpha < 360° \Rightarrow 60°m < 360° \Rightarrow m \leq 5$.

When $\alpha \geq 72°$, $m < 5$ and G contains only triangles or quadrilaterals. Furthermore, since each triangle or quadrangle of side length α is totally covered by circles of radius α placed on its vertices, G can contain no isolated points. □

3. Some Solutions. Using theorems already presented, the number of possible vertex sets for low values of n can be narrowed down quite a bit. I have divided the values for which a solution is known into three groups, according to the type of proof used to show that the given point sets represent the solution to Tammes's Problem. The first group is found as a simple consequence of Theorem 2.1. The second group is found using more complicated but still elegant area estimates on the polygons meeting at the vertices of extremal graphs. The last group consists of extremal sets whose proof involves painful analysis of the vertex sets possible for irreducible graphs with the given values of n. The fact that $n \leq 12$ implies that the graph contains no polygons with more than five sides is central to these proofs. No one seems to have been so foolish as to try to apply the technique when $n > 12$, in which case one must consider hexagons as well. It is interesting to notice that for values of n for which the proof is easy or elegant, the vertex sets correspond to very nice regular polyhedra.

3.1. n = 3,4,5,6 or 12. When $n = 3,4,6$ or 12, configurations of n points exist such that each face of their triangulation is an equilateral triangle, therefore the inequality of Theorem 2.1 cannot be improved. The resulting best configurations are a triangle inscribed in a great circle, a regular tetrahedron, a regular octahedron and a regular icosahedron, respectively.

Tammes showed that $d_5 = d_6 = \frac{\pi}{2}$. A best configuration for 5 points is obtained by omitting one point from the best configuration for 6 points, but it is not necessarly true that the addition of one point to a optimal configuration for 5 points give an optimal configuration for 6. Since $d_5 = 90°$, if two of the five points are placed on opposite poles of the sphere then the remaining three can can be placed anywhere on the sphere's equator as long as they are mutually seperated by a distance of at least 90°. Therefore there are infinitely many best arrangements of points for $n = 5$, and most of these arrangements preclude the addition of a sixth point. The proof of this fact as presented in [5] is easy and elegant and you should try it yourself.

3.2. n = 8 or 24. The square anti-prism gives a tessellation of the sphere with 8 vertices for which $\alpha > 74°$, so by Theorem 2.5 an extremal graph with 8 vertices can contain only equilateral triangles and quadrangles and no isolated points.

K. Schütte and B. L. van der Waerden in [10] prove that d_8 is attained for a square anti-prism, but the proof in [5] is a better choice for anglophiles. The following theorem is used to solve Tammes's Problem for $n = 8$.

THEOREM 3.1. *Let G be a graph of sidelength α containing only triangles and quadrangles. If G contains at most one quadrangle at each vertex, then*

$$(2n-4)\Delta + \frac{n}{4}(Q(\beta) - 2\Delta) \leq 4\pi$$

where Δ is the area of an equilateral triangle of side length α and $Q(\beta)$ is the area of a quadrangle with side length α and interior angle β.

When this inequality is an equality, $\alpha = d_n$ and the extremal configuration is unique up to rotation.

The proof can be outlined as follows: Let T be the sum of the areas of all quadrangles meeting at a given vertex of the graph G. Then T takes on its mimum value when the interior angles of all quadrangles, with the exception of at most one, become equal to $a(\alpha)$ or $2a(\alpha)$ and the quadrangles decompose into triangles. Therefore if q quadrangles meet at a given vertex, $T \geq 2(q-1)\Delta + Q(\beta)$. Since $a(\alpha) \leq \beta \leq 2a(\alpha)$ and $\beta = 2\pi - ka(\alpha)$ for some integer k, β is the same for each vertex of G and the quadrangle, if it exists, is a square. Summing over all the vertices of G gives the inequality in Theorem 3.1.

When $n = 8$, the inequality of Theorem 3.1 becomes

$$2\Delta + \frac{1}{2}Q(\beta) \leq \pi$$

$$\textit{hence} \quad 90^\circ = a(d_6) \geq a(d_8) > a(d_{12}) = 72^\circ \quad \textit{and} \quad \beta = 360^\circ - 3a(\alpha).$$

Plugging in the appropriate expresions for Δ and $Q(\beta)$ gives a function of α which is strictly increasing in the given range and leads to an upper bound that is attained when the vertices correspond to the vertices of the square anti-prism {3,3,3,4} which has 3 equilateral triangles and a square meeting at each vertex. Since the total area covered by the polygons meeting at the vertices of a graph with this edge length is minimized by this arrangement of vertices, the extremal configuration on 8 vertices is unique.

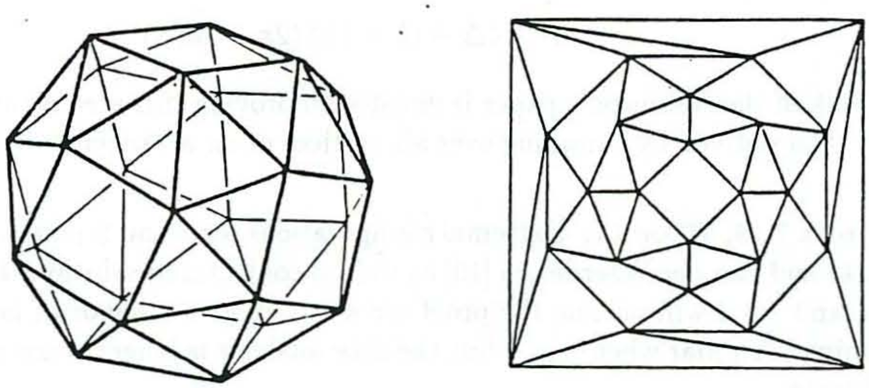

FIG. 3.2. *The solution to Tammes's Problem for n = 24 and its graph, from [3].*

If extremal graphs G on $n = 24$ vertices are assumed to be made up entirely of triangles and quadrangles, Theorem 3.1 gives a unique extremal configuration corresponding to the Archimedean semiregular polyhedron {3,3,3,3,4}, which has four equilateral triangles and a square meeting at each vertex. Since it cannot be assumed that the graph of the extremal configuration for $n = 24$ points does not contain polygons with more than 4 sides, this area estimate is not enough to show that the arrangement is optimal. Although Schütte and van der Warden conjecture the Archimedean semiregular polyhedron as the correct solution of Tammes's Problem for $n = 24$, they do not prove it.

Robinson [8] shows that {3,3,3,3,4} is the best solution by improving the inequality of Theorem 3.1 for $n > 12$, using estimates on the area of the triangles meeting at a given vertex in the triangulation of n points. The inequality gives a more general area estimate because the analysis is not restricted to graphs containing only triangles or quadrangles.

THEOREM 3.3. *Let G be a graph of side length at least α. Suppose that $60° < a(\alpha) < 72°$, Then*

$$2n\Delta + (n-6)\Delta(2\pi - 4a(\alpha)) \leq 6\pi.$$

where $\Delta(\rho)$ is the area of a triangle with two sides of length α meeting at an angle ρ.

Equality is possible only if $4a(\alpha) + \beta = 2\pi$ and the given points are the vertices of a semiregular polyhedron with four equilateral triangles and a square meeting at each vertex. The only arrangement of 24 points on the sphere whose distances are at least equal to the distance α corresponding to the condition $4a(\alpha) + \beta = 2\pi$ is one in which the distance α actually ocurs, so α is the critical distance for an arrangement of 24 points.

Theorem 3.3 is an easy consequence of the fact that, if the number of triangles meeting at some vertex is k, and if T is their total combined area,

$$T \geq 4\Delta + (k-4)\Delta(2\pi - 4a(\alpha)).$$

The bulk of the Robinson's paper is devoted to proving this area estimate and Theorem 3.3 is then derived by summing over all vertices of an arrangement.

3.3. n = 7, 9, 10 or 11. Extremal configurations for 7 and 9 points were presented by Schütte and van der Warden in [10] as well as conjectured solutions for $n = 10, 11, 13-16, 24$ and 32. I will outline the proof for $n = 7$ to give a flavor of how it is done. The reasoning is similar when $n = 9$ but the case analysis is longer since pentagons must be considered.

There exists a configuartion of 7 points on the sphere such that $a(\alpha) = 80°$ so $\alpha > 77°$ and by Theorem 2.5, an extremal graph G on 7 points can contain only triangles and quadrangles and no isolated points. Furthermore, each vertex has valence 3 or 4, since the sum of interior angles of polygons meeting at a vertex must be at most 360° and every angle is at least as large as $a(\alpha) > 80°$.

The graph G cannot consist solely of vertices of valence 3, since then the number of edges of G would not be integral. Let P be a point of G of valence 4 connected to points

A, B, C, and D. There remain two points Q and R, both with valence at least 3. They must each be adjacent to at most two of the points A,B,C or D, with the condition that they cannot be adjacent to any two vertices if the angle between the edges connecting those vertices to P is bisected by another edge. Furthermore, Q and R cannot be adjacent to the same vertex without creating a pentagon in the final graph. Therefore it can be assumed that Q, A, P and B form the vertices of a convex quadrilateral, as do R, C, P and D, and the Q and R are adjacent. Since each vertex has valence at least 3, A is adjacent to D and B is adjacent to C.

By taking into account the fact that all edges of the graph must be of equal length α, it can be shown that when α is maximum, three of the angles at P must be equal to $a(\alpha)$, therefore either A and B or C and D must be adjacent. This describes the graph of the maximum configuration for 7 points.

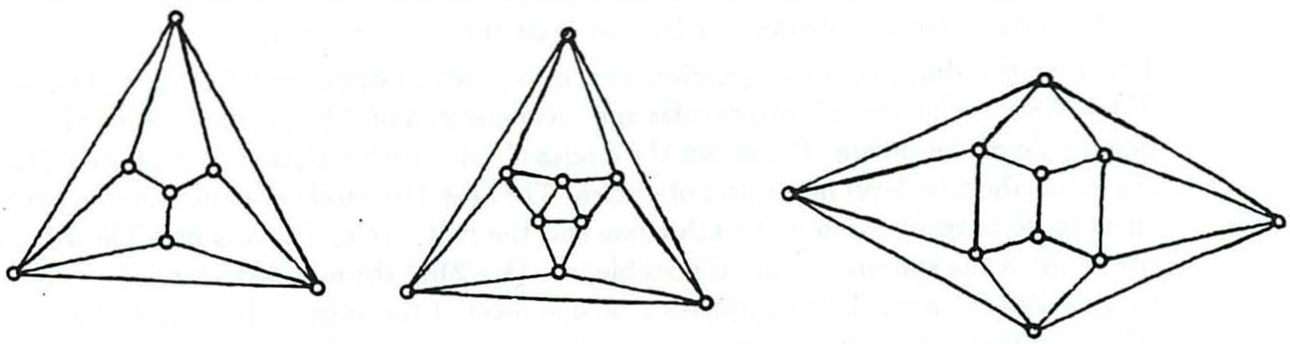

FIG. 3.4. *Graphs of extremal arrangements for* $n = 7, 9$ *and* 10, *from [5].*

Danzer [2] uses similar methods to find the best solution for the cases $n = 10$ and $n = 11$. Danzer shows quite a bit more than just the solution to Tammes's Problem in his Habilitationsschrift, and the alternative proof for the case $n = 11$ given by K. Böröczky [1] is much shorter and more readable. I will not attempt to show a proof for either of these cases here.

When $n = 11$ the best arrangement of points is obtained by removing one vertex of the regular icosahedron, the best arrangement for $n = 12$, so $d_{11} = d_{12}$. This is similar to the case of $n = 5$, with the important difference that for $n = 11$ the best arrangement of points is unique up to rotation and therefore the best arrangement for $n = 12$ can be found by adding one vertex to the best arrangement for $n = 11$.

4. Conjectured Solutions. For large values of n a vertex-by-vertex analysis of possible graphs would on a practical level be impossible, and would arguably be a waste of time. Several aproaches have been used to find 'good' solutions for Tammes's problem for values of n that I have not already considered. I use the term 'good' here in a very loose sense, since it seems hard to tell whether or not an arrangement is near extremal. One definition for the relative 'goodness' of an arrangement is how high the *density* is, where the density is defined as the ratio of the area covered by n spherical caps of radius α on the unit sphere to the surface area 4π. It can be seen from figure 1.1 that the density of an arrangement does not necessarily increase with n, whereas $d_n \leq d_{n+1}$ for all n. The arrangements presented here have relatively high density and the distinction of being the best solution that anyone has come up with to date.

4.1. Spiral Packings. E. Székely [11] presents what he calls "the method of spiral construction" for obtaining good packings of n points on a sphere for certain values of n. This method of construction produces sets of vertices such that all but at most two of the vertices are at a distance α from at least three other vertices.

Let d be the diameter of a spherical cap in a given system, and let $O_1, ..., O_k, (k = 3, 4, 5, 6, 7)$ be the vertices of a regular spherical polygon of side length d centered at the north pole of the sphere. Construct the circles C_i with center O_i and diameter d. These represent the first level in a spiral of circles. The $(k+1)$st circle of diameter d is placed so as to be tangent to both the kth circle and the first circle, if this is possible without overlapping the spherical caps. If possible, the $(k+2)$nd through $2k$th circles are added in a similar manner. This completes a second level of the spiral. Rings of k circles are added until there is not enough room to place a new one without overlapping.

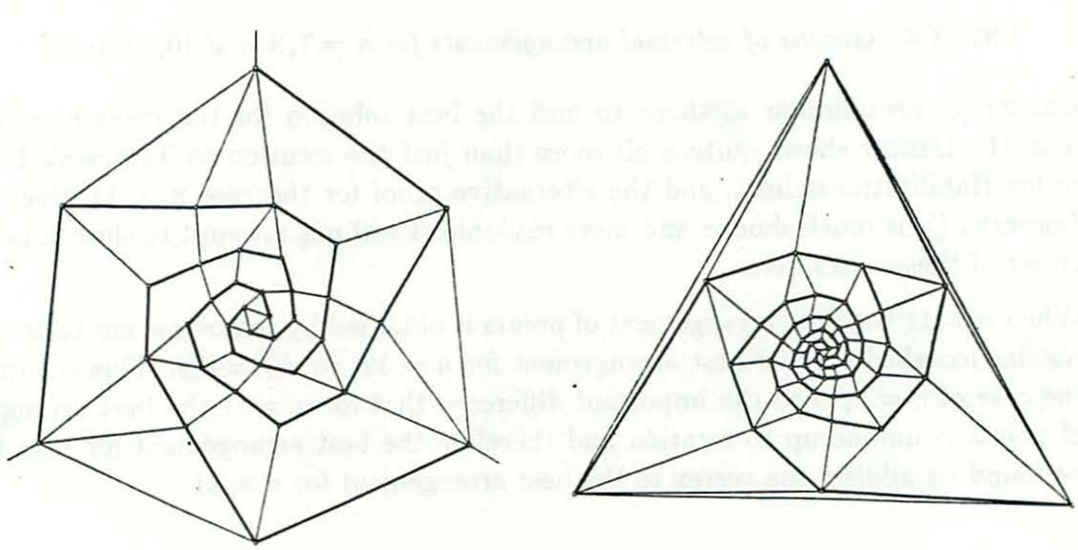

FIG. 4.1. *Arrangements given by Székely for $n = 37$ and $n = 60$, from [11].*

If the diameter d is chosen so that the last k circles are tangent, then the arrangement of circles is supposed optimal for $n \equiv 0(mod\ k)$ for $k = 5$. If $k \geq 6$ then by Theorem 2.4 there is enough room at the north and south poles to place two additional circles which are not tangent to any other circles in the construction. This gives a good arrangement when $n \equiv 2(mod\ k)$. Similarly, if $k \leq 5$ and the diameter d is chosen so that the distance of the last set of circles from the south pole is d, one more circle may be added and the method gives a good arrangement when $n \equiv 1(mod\ k)$.

Another version of the method of spiral construction places a circle of radius d at the north pole and arranges k circles about this first one. This leads to good arrangements of $n \equiv 1(mod\ k)$ and $n \equiv 2(mod\ k)$ points that are in general different from those obtained by the first version.

This "method of spiral construction" can be used to find good arrangements for $12 < n \leq 60$ points, except when $n = 14, 17, 19, 20, 22, 33, 46 - 48$. This method gives very pleasantly symmetric solutions, several of which have been improved by later techniques.

4.2. Symmetric packings. The packings of $4, 6, 12$ and 24 points on the sphere turn out to be very good packings in the sense that they have high density. It seems reasonable to look for packings for larger values of n that posses the symmetries of the regular polyhedra associated with these solutions in the hopes that they too will have high density. It is tempting to assume that very symmetric arrangements will be optimal. This is not always the case, but if not they at least provide a starting place.

Tarnai presents in [15] a method for constructing new packings with rotational symmetries of these regular polyhedra based on the regular tetrahedron, regular octahedron and regular icosahedron. These polyhedral tessellations can be augmented with smaller tessellations of equilateral triangles overlayed on each of the faces of the original polyhedron so that the later's vertex set contains the former's. The polyhedral tessellation is then blown up onto the sphere. In general not all edge lengths are equal, but some will be, and these can be chosen as the edges of a graph G.

The construction of smaller tessellation is described in the following way: Consider the regular tessellation $\{3, 6\}$ which fills and covers the Euclidian plane and consists of equilateral triangles six at a vertex. Proceed from a vertex A along one edge, continuing in the same direction until b edges have been traversed, then change direction by $60°$ and proceed straight along c edges to a new vertex B. The starting point A and the end point B determine a line segment that can be considered the edge of a larger equilateral triangle. The larger triangles are used as the faces of the regular polyhedra and the smaller triangles give the augmented tessellation.

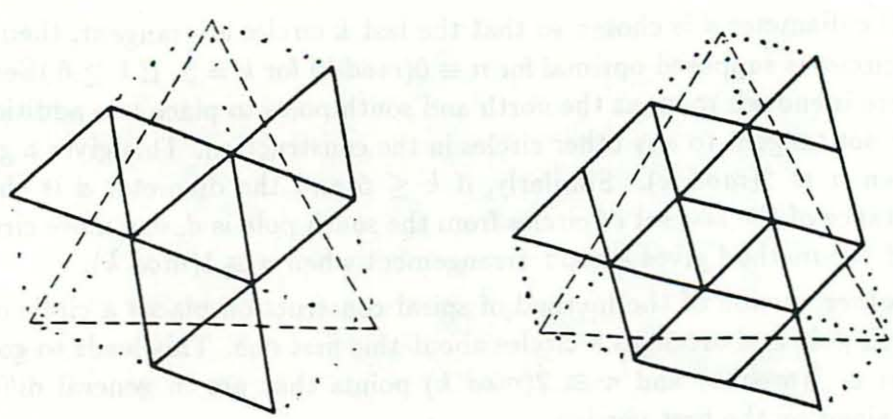

FIG. 4.2. *Augmented tessellations with $b = 2$, $c = 1$ and $b = 3$, $c = 1$, from [15].*

Tarnai notes that the packings for $n = 12, 24, 48, 60$ and 120 presented by Robinson in [9] can all be obtained in this way. Tarnai creates arrangements for new values of n by adding the vertices of the original polyhedron to the vertex sets used in Robinson's arrangements. For example, Robinson constructs an arrangement of 48 points by tiling each face of the regular octahedron with triangles meeting at six vertices per face. Tarnai's method adds six more vertices for an arrangement of 54 points.

This method can be used to create arrangements for $n = 16, 28, 30, 54, 72$ and 132. In the cases $n = 16, 28$ and 30 the arrangements are not the best that have been found and the arrangement for $n = 54$ is the same as the one obtained by Székely. The arrangements obtained for $n = 54, 72$ and 132 are good and in each case $2n - 4 < e$, so the packings probably cannot be improved. In each case the arrangements are derived from tilings of the regular icosahedron. In [14] Tarnai presents a packing for $n = 180$ with icosahedral symmetry as well as improved packings for $n = 32$. In both cases the density is quite good.

4.3. Improving existing configurations. In [2] Danzer introduces the idea of a graph G having a *degree of freedom* if the edges can rotate freely around the vertices and the edge lengths of the graph can be varied freely and continuously but simultaneously for all edges and in equal proportion. Danzer show that in many cases, if the graph has a degree of freedom in this sense, the arrangement is improvable until the degree of freedom is removed by the appearance of an additional edge which prohibits the graph from admitting motions other than isometries. This idea has been used by Danzer and several others to improve existing packings on the sphere.

Tarnai and Gáspár apply the theory of bar structures to discover which graphs have a degree of freedom and use this knowledge to improve those arrangements. In [13], they present improved solutions for $n = 18, 27, 34, 35$ and 40 and indicate other values of n for which application of there methods should result in improved vertex sets.

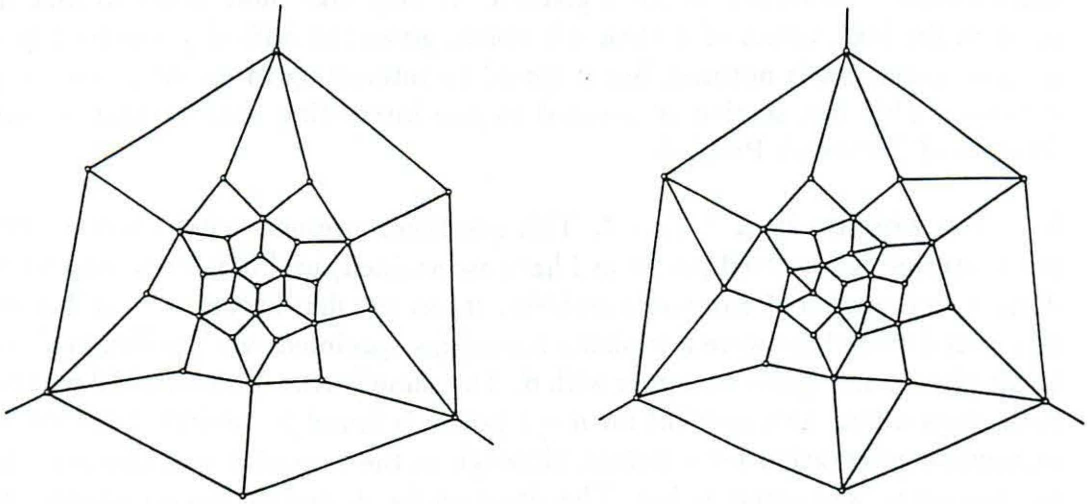

FIG. 4.3. *An arrangement for $n = 32$ given by Schütte and van de Warden, and an improvement of this arrangement given by Tarnai and Gáspár, from [13].*

The graph of an arrangement is modeled as a series of straight bars and frictionless pin joints lying on the surface of a sphere of fixed diameter. Bars are considered to be under the influence of some external force, such as change of temperature, causing their lengths to expand. The structure is characterized by its compatibility matrix containing e rows and $2n - 3$ columns, where e is the number of bars (edges) and n is the number of joints (vertices). Through analysis of the compatability matrix Tarnai and Gáspár show that an arrangement is generally improvable if $e \leq 2n - 3$, or when the graph G contains no triangles. This method is particularly intresting since it takes an engineering approach to a math problem.

5. What's left. Tammes's Problem remains largely unsolved if ones goal isto find the aboslute best vertex set for a given n. It may take more effort to find the exact solution for high values of n than it's worth, given the difficulty involved in deciding if an arrangement is optimal, but it would be interesting to see what sort of patterns develop. This last section is devoted to one interesting pattern that occurs in the solution of Tammes's Problem.

5.1. For which n is $d_n = d_{n+1}$?. This question represents what I would consider the most interseting unsolved (as far as I have ascertained) problem remaining for the three dimensional version of Tammes's problem. It too was first brought up by Tammes when he noticed that there were few plants among his specimens with pollen grains favoring 5 exit places, and quite a number with 6. The same is true for 11 and 12 exit places. In both cases a best arrangement for $n - 1$ points is found by omitting one vertex of the maximal configuration for n points, although in the case of $n = 5$ this arrangement is not unique as was noted earlier. The situation for d_6 and d_5 has an interesting analog in higher dimensions. Using induction on the dimension k it can easily be shown that on a $(k-1)$-sphere with $k \geq 3$, $d_i = d_{2k}$ for $k + 2 \leq i < 2k$.

Robinson in [9] shows that the only configurations of n points on the sphere with the property that their graph remains irreducible after any one point is removed occur when $n = 6, 12, 24, 48, 60$ or 120. For any such graph with $n > 6$, edges must issue from any vertex in such a way that the sum of any two consecutive angles is less than 180°. This implies that each vertex of the graph has valence 5.

Robinson had hoped to show that these were the only cases for which $d_{n-1} = d_n$; several objections present themselves. First the graphs that satisfy the condition that all vertices have 5 nearest neighbors do not necessarly give arrangements of n points with maximum minimum distance between any two points. In fact, for the case $n = 60$, Székely gives a better arrangement than the one in which each vertex has valence 5 found by Robinson. There may also be cases for which $d_{n-1} = d_n$, but the arrangement for $(n-1)$ points is not obtained by omitting one vertex from the arrangement for n points, or where it is obtained by omitting a point, but not just any point. For example, there may exist an extremal graph containing an isolated point for which the graph would still be irreducible if this point were removed, or an extremal graph with a vertex whose adjacent vertices are all of valence 5 which may be removed without affecting the irreduciblitiy of the graph.

At the end of his paper on spiral packings Székely states that, according to J. Molnár, there are indeed numbers other than the ones mentioned by Robinson for which $d_{n-1} = d_n$. Székely gives no references or proof for this statement and I have been unable to dig any up or find any examples. This hardly seems surprising since actual extremal arrangements are known in so few cases. He also conjectures that $n = 48$ might be the first case for which $d_{n-2} = d_n$, but again offers no evidence to support this statement.

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