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INVARIANCE AND SCALING PROPERTIES IN THE DISTRIBUTIONS OF CONTRIBUTING AREA AND ENERGY IN DRAINAGE BASINS

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ABSTRACT

The cumulative probability distributions for stream order, stream length, contributing area, and energy dissipation per unit length of channel are derived, for an ordered drainage system, from Horton's laws of network composition. It is shown how these distributions can be related to the fractal nature of single rivers and river networks. Finally, it is shown that the structure proposed here for these probability distributions is able to fit the observed frequency distributions, and their deviations from straight lines in a log–log plot.

KEY WORDS Geomorphology Fractals Drainage basins

INTRODUCTION

Mandelbrot (1982) suggests that dissipative systems with many spatial degrees of freedom, such as river networks, will follow power law distributions of mass and energy. The cumulative probability distribution of these characteristics can therefore be written as

$$P[X > x] \propto x^{-\beta} \tag{1}$$

where $\propto$ indicates the presence of a proportionality factor. Rodriguez-Iturbe *et al.* (1992a) describe the invariance properties of this probability distribution for mass and energy in river basins under wide changes of spatial scales. Assuming the total cumulative area draining into a given site as a surrogate variable for discharge, the values of the exponent β are explained on the basis (1) of the empirical relationship proposed by Gray (1961), that is

$$L \propto A^{\alpha} \tag{2}$$

which, with $\alpha = 0{\cdot}568$, provides a very good fit between main stream length L, and basin area, A, and (2) assuming that the Euclidean length from the outlet to the most distant point in the boundary of the basin can be described as the first collision time of two fractal trails (Mandelbrot, 1982; Feder, 1988; Takayasu *et al.*, 1988). From this, a value of β approximately equal to 0·45 and 0·90 is predicted for mass and energy, respectively, in good agreement with the observed values measured in five basins in North America. In fact, the exponent β in the power law distribution of mass is 'statistically indistinguishable among the different basins and approximately equal to 0·43', Rodriguez-Iturbe *et al.* (1992a).

In this paper the values assumed in Equation (1) for both the exponent β and the proportionality factor

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are explained on the basis of a quantitative analysis of river networks by means of Horton's laws of network composition (Horton, 1932; 1945; Strahler, 1952; Shumm, 1956). The structures proposed here for the cumulative probability distributions for mass and energy are able to fit the observed distributions and their deviation from straight lines at large values of area. It is also shown that the value of β can be linked to the fractal structure of single rivers, with fractal dimension d, and river networks, with fractal dimension D. These measures are derived, for an ordered drainage system, from Horton's laws of drainage composition, as proposed and discussed in Mandelbrot (1982), La Barbera and Rosso (1987; 1989; 1990), Tarboton *et al.* (1988; 1990), and Rosso *et al.* (1991). Finally, with reference to the exponent fitted in the empirical Equation (2), it is shown that the expected value of β is 0·432.

HORTON'S LAWS AND THE FRACTAL STRUCTURE OF RIVERS AND RIVER NETWORKS

Horton's laws of network composition are stated here in terms of Strahler's ordering scheme. The structure of a river network is therefore described as a system of streams which recognizes, through stream order, a hierarchy among the different branches. Strahler's ordering scheme postulates that: (1) source streams are of order 1; (2) when two streams of equal order join, a stream of one order higher is formed; and (3) when two streams of different order join, the continuing stream retains the order of the higher order stream. The empirical laws of stream numbers and stream lengths (Horton, 1945) state that the bifurcation ratio, R_b, and the stream length ratio, R_l, are constant within a catchment; the empirical laws of stream areas and stream slopes (Shumm, 1956; Strahler, 1952) states that the stream area ratio, R_a, and the stream slope ratio, R_s, are also constant. Denoting with ω the order of a stream segment, these ratios are defined as

$$R_b = \frac{n_\omega}{n_{\omega+1}} \tag{3}$$

$$R_l = \frac{l_{\omega+1}}{l_\omega} \tag{4}$$

$$R_a = \frac{A_{\omega+1}}{A_\omega} \tag{5}$$

$$R_s = \frac{S_{\omega+1}}{S_\omega} \tag{6}$$

where n_ω is the number of streams of order ω; l_ω is the mean length of streams of order ω; A_ω is the mean tributary area of streams of order ω; and S_ω is the mean slope of streams of order ω.

Estimates of R_b, R_l, R_a and R_s for a river network can be obtained from the slopes of the straight lines resulting from plots of the logarithmically transformed values of n_ω, l_ω, A_ω and S_ω versus order ω, for ω ranging from 1 to Ω, the order of the basin.

Horton's laws are geometric scaling relationships which yield the self-similarity of the catchment stream system within a certain range of scales (Nikora, 1989); the following derivations should therefore be regarded as a mathematical descriptions which applies at the range of scales associated with Horton's laws. Unfortunately, this range is not yet well assessed; a possible route in this direction could be from the description of the erosional development of drainage networks, on the basis of the results obtained from field observations (Montgomery and Dietrich, 1988; 1989), laboratory experiments (Sawai *et al.*, 1986; Shumm *et al.*, 1987), mathematical models (Roth *et al.*, 1989; Willgoose *et al.*, 1991) and stability analysis (Smith and Bretherton, 1972; Loewenherz, 1991).

On the basis of the self-similarity described by laws of stream lengths and stream areas, Rosso *et al.* (1991) reported that rivers are fractal with a fractal dimension

$$d = \max\left(1, 2\frac{\log R_l}{\log R_a}\right) \tag{7}$$

The estimates of d obtained from Equation (7) fit the measured values satisfactorily and are close to the value of 1·136 hypothesized by Mandelbrot (1982) under the assumption that

$$d = 2\alpha \tag{8}$$

where α is the fitted exponent in Equation (2) between L and A.

On the basis of the self-similarity described by the laws of stream numbers and stream lengths, La Barbera and Rosso (1987) reported that the fractal dimension of river networks is given by

$$D = \min\left[2, \max\left(1, \frac{\log R_b}{\log R_l}\right)\right] \tag{9}$$

The fractal dimension of a stream network, as estimated from Equation (9), can take values from two to unity for the combined ranges of R_b and R_l values observed in nature. Although it has been observed that river networks display varying values of fractal dimension D, this generally lies between 1·5 and 2, with an average of approximately 1·7.

The scaling properties of the river network as a whole can be viewed as the product of the structural composition of the drainage system, reflected by D, and the fractal nature of river length, described by d. By introducing this source of fractal behaviour of individual streams in Equation (9), Tarboton *et al.* (1990) obtain the fractal measure

$$\mathfrak{D} = Dd \tag{10}$$

and, by combining Equations (7), (9) and (10), Rosso *et al.* (1991) obtain

$$\mathfrak{D} = \min\left(2, 2\frac{\log R_b}{\log R_a}\right) \tag{11}$$

At scales greater then the fundamental length scale given by the drainage density, it can be assumed that the network drains the whole river basin. This constrains it to be space filling with a fractal dimension $\mathfrak{D} = 2$ (Mandelbrot, 1982; Tarboton *et al.*, 1990). Assuming $d = 2\alpha = 1{\cdot}136$, the value $D = \mathfrak{D}/d = 1{\cdot}761$ is predicted from Equation (10).

DISTRIBUTION OF CONTRIBUTING AREA

The cumulative probability distribution for contributing area (i.e. mass) is here derived from a general approach based on a knowledge of the distribution of stream order. This is obtained from the cumulative probability distribution of a site having an upstream total stream length larger than a given value. The total length Z_ω of streams in a single subnetwork of order $\omega \leq \Omega$ is given by Eagleson (1970) as

$$Z_\omega = l_1 R_b^{\omega-1} \frac{\left(\frac{R_l}{R_b}\right)^\omega - 1}{\left(\frac{R_l}{R_b}\right) - 1} \tag{12}$$

in which l_1 is the mean length of first order streams. With $\omega = \Omega$, Equation (12) gives the total length of streams for the whole catchment. The total length of streams in the subnetworks of order ω within a basin of order Ω is therefore obtained as

$$Z_\omega^t = Z_\omega R_b^{\Omega-\omega} \tag{13}$$

The cumulative probability distribution of a site having an upstream total stream length Z^t larger than $Z^t_{\omega_*}$ can be consequently written as

$$P[Z^t \geq Z^t_{\omega_*}] = 1 - \frac{Z^t_{\omega_*}}{Z_\Omega} = 1 - \frac{\left(\frac{R_l}{R_b}\right)^{\omega_*} - 1}{\left(\frac{R_l}{R_b}\right)^\Omega - 1} = \frac{\left(\frac{R_l}{R_b}\right)^\Omega}{1 - \left(\frac{R_l}{R_b}\right)^\Omega}\left[\left(\frac{R_l}{R_b}\right)^{-(\Omega-\omega_*)} - 1\right] \tag{14}$$

The total length Z^t_ω of streams in the subnetworks of a given order within a catchment of order Ω is related, throughout Equations (12) and (13), to the order ω taken into consideration and to the contributing area, A_ω. From this, it follows that the cumulative probabilities of a site having an order ω larger than ω_*, or a mass larger than A_{ω_*}, can be characterized by the same distribution, that is

$$P[Z^t \geq Z^t_{\omega_*}] = P[\omega \geq \omega_*] = P[A \geq A_{\omega_*}] \tag{15}$$

It must be pointed out that the knowledge of $P[\omega \geq \omega_*]$ could be viewed as the basis for the derivation of the distributions of different characteristics for river basins, e.g. contributing area, energy dissipation, stream gradient and mean basin altitude. Here it is used to obtain the distribution for contributing area and energy.

From the definition of the area ratio, R_a, and assuming the total area of the basin, A_Ω, as a reference value for normalization, the cumulative probability distribution of a site having a relative mass larger than $R_a^{-(\Omega-\omega_*)}$ is

$$P[A \geq A_{\omega_*}] = P\left[\frac{A}{A_\Omega} \geq \frac{A_{\omega_*}}{A_\Omega} = R_a^{-(\Omega-\omega_*)}\right] = \frac{\left(\frac{R_l}{R_b}\right)^\Omega}{1-\left(\frac{R_l}{R_b}\right)^\Omega}\left[\left(\frac{R_l}{R_b}\right)^{-(\Omega-\omega_*)}-1\right] \tag{16}$$

Finally, from the estimations of the fractal dimension of rivers [Equation (7)] and river networks [Equation (9)], as obtained on the basis of self-similarity described by Horton's laws, and with the mathematical derivations reported in the Appendix, we obtain

$$P\left[\frac{A}{A_\Omega} \geq \frac{A_{\omega_*}}{A_\Omega}\right] = \frac{\left(\frac{R_l}{R_b}\right)^\Omega}{1-\left(\frac{R_l}{R_b}\right)^\Omega}\left[\left(\frac{A_{\omega_*}}{A_\Omega}\right)^{-\frac{d}{2}(D-1)}-1\right] \tag{17}$$

and, assuming $d = 2\alpha = 1{\cdot}136$ and $D = \mathfrak{D}/d = 1{\cdot}761$

$$P\left[\frac{A}{A_\Omega} \geq \frac{A_{\omega_*}}{A_\Omega}\right] \propto \left[\left(\frac{A_{\omega_*}}{A_\Omega}\right)^{-0{\cdot}432}-1\right] \tag{18}$$

It must be pointed out that the use of Equation (17) is not limited to fixed values of the contributing area to a given site, A_{ω_*}, obtained with reference to integer values of ω_*. In fact, contributing area in natural river basins is a real continuous variable of which A_{ω_*}, with integer ω_*, is the expected value for a given stream order, i.e. an integer parameter used to order and scan the network. In this mathematical framework, it can be assumed that a corresponding real value of ω_* can be associated with a given value of the contributing area. In other words, the continuous variable area, reduced to some expected values by the ordering procedure, is expanded again with reference to hypothetical real values of the order. Obviously, this assumption cannot be applied to deterministic networks, such as the Peano fractal network; in fact, these are defined only with reference to some fixed spatial scales, and not in a continuum of scales, such as natural river networks. Equation (17) can be applied in deterministic networks only with reference to the scales at which the network is defined.

The analysis of the drainage network, as obtained from the interpretation of digital elevation maps (DEMs) for different river basins, shows that the cumulative probability distribution for mass follows a straight line for nearly three scales in a log–log plot with a deviation at large values of area, i.e. when the values of the areas approach the total area of the basin. This behaviour is predicted by Equation (17). For the range in which the cumulative probability distributions of mass follows straight lines, i.e. for the range in which $A_{\omega_*} \ll A_\Omega$, we obtain

$$\left(\frac{A_{\omega_*}}{A_\Omega}\right)^{-\frac{d}{2}(D-1)} \gg 1 \tag{19}$$

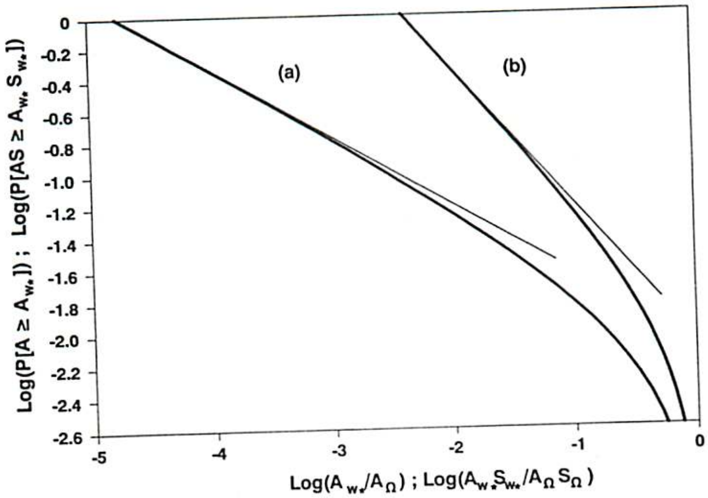

Figure 1. Cumulative probability distributions for contributing area (a) and energy (b) $d = 1{\cdot}136$; $D = 1{\cdot}761$; $\mathfrak{D} = 2{\cdot}0$; and $\Theta = 2{\cdot}0$.

and Equation (17) can be approximated, in a log–log plot, with a straight line with slope equal to

$$-\frac{d}{2}(D-1) \approx -0{\cdot}432 \tag{20}$$

The cumulative probability distribution for area can be estimated from Equation (17) on the basis of the knowledge of the values assumed by Ω, R_b, R_l and R_a, and, consequently, by d and D. The expected shape of this function is presented in Figure 1, from which we can observe that the distribution is completely defined, in a quantitative form, by Equation (17) and that a deviation from a straight line, at large values of areas, is predicted. Moreover, a single basin can be described with reference to different spatial scales, i.e. at different values of channel maintenance area. This case is presented in Figure 2, in which basin area is

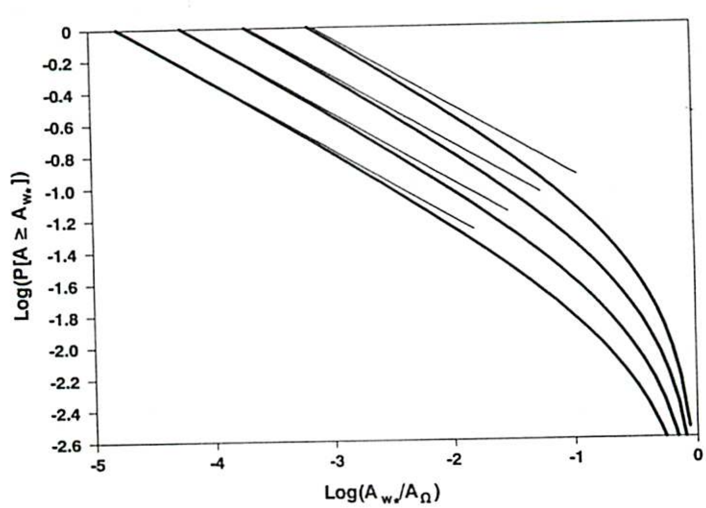

Fig. 2. Cumulative probability distribution for contributing area. The same basin is described with reference to different values of channel maintenance area. $d = 1{\cdot}136$; $D = 1{\cdot}761$; and $\mathfrak{D} = 2{\cdot}0$

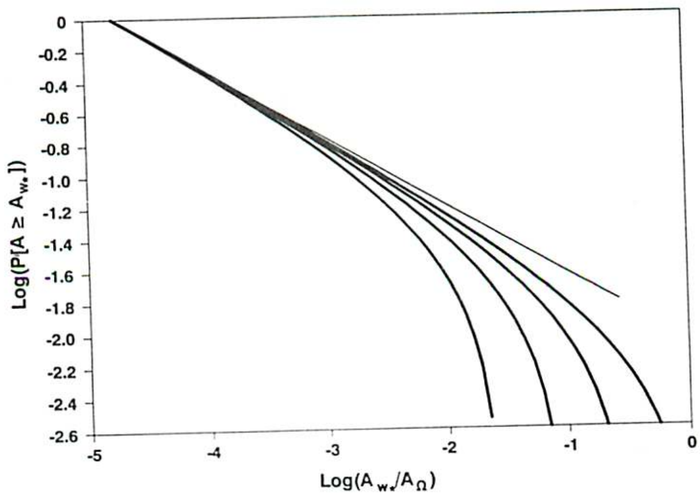

Fig. 3. Cumulative probability distribution for contributing area. Different basins are described with reference to the same value of channel maintenance area. $d = 1{\cdot}136$; $D = 1{\cdot}761$; and $\mathfrak{D} = 2{\cdot}0$

constant and the fractal dimensions are assumed to be insensitive to scale within the proposed range of scales. Finally, different basins, with different catchment areas, can be described with reference to the same fundamental scale, i.e. at the same value of channel maintenance area. This case is presented in Figure 3, in which it is assumed that the different basins are characterized by the same set of fractal dimensions and the reference area for normalization is the maximum basin area within the proposed set.

DISTRIBUTION OF ENERGY

The rate of energy expenditure per unit length of channel at any point in the network is proportional to the product of discharge and slope. Assuming contributing area as a surrogate variable for discharge, we can imagine the energy expenditure to be proportional to the product of contributing area and slope. From the definition of the area and slope ratios, R_a and R_s, and assuming $A_\Omega S_\Omega$ as a reference value for normalization, the cumulative probability distribution for energy can be therefore written as

$$P[AS \geq A_{\omega_*} S_{\omega_*}] = P\left[\frac{AS}{A_\Omega S_\Omega} \geq \frac{A_{\omega_*} S_{\omega_*}}{A_\Omega S_\Omega} = (R_a R_s)^{-(\Omega - \omega_*)}\right] \tag{21}$$

In the same framework used to derive the distribution of contributing areas, i.e. from the estimations of the fractal dimension of rivers and river networks as obtained on the basis of self-similarity described by Horton's laws, introducing

$$\Theta = -\frac{\log R_a}{\log R_s} \tag{22}$$

and with reference to the mathematical derivations reported in the Appendix, we obtain

$$P\left[\frac{AS}{A_\Omega S_\Omega} \geq \frac{A_{\omega_*} S_{\omega_*}}{A_\Omega S_\Omega}\right] = \frac{\left(\frac{R_1}{R_b}\right)^{\Omega}}{1 - \left(\frac{R_1}{R_b}\right)^{\Omega}} \left[\left(\frac{A_{\omega_*} S_{\omega_*}}{A_\Omega S_\Omega}\right)^{-\frac{d}{2}(D-1)\frac{\Theta}{\Theta-1}} - 1\right] \tag{23}$$

The shape of this function is presented in Figure 1. For the range in which the cumulative probability distributions of energy follows straight lines, the function can be approximated, in a log–log plot, with a

Table I. Characteristics of the basins under analysis as evaluated from a digital elevation model. Areas are expressed in Km^2; β and β_e are the calculated exponents in the cumulative probability distributions for contributing area and energy, respectively

River	Area	Ω	R_b	R_l	R_a	R_s	d	D	Θ	β	β_e
Magra	1600	8	3·97	2·10	4·25	0·47	1·03	1·86	1·92	−0·44	−0·92
Entella	400	7	3·88	2·11	4·15	0·49	1·05	1·82	2·03	−0·43	−0·84

straight line with slope equal to

$$-\frac{d}{2}(D-1)\frac{\Theta}{\Theta-1} \approx -0{\cdot}864 \tag{24}$$

in which the value −0·864 is predicted assuming $\Theta = 2$. Flint (1974) first derived Θ on the basis of power law relationships used to describe the scaling of hydraulic and geometric variables, such as slope, contributing area and discharge. In the work of Flint (1974) the estimates of Θ, in the form introduced in the present paper, take values in the range 1·2–2·7 with an average of 1·7. A relationship between slope and discharge (i.e contributing area) was also proposed and discussed by Rodriguez-Iturbe *et al.* (1992b) as a scaling implication of the principles of energy expenditure in drainage networks. The value 2, proposed by Rodriguez-Iturbe *et al.* (1992b), is here assumed as a first estimation of Θ. This assumption is confirmed by the Θ values obtained from the analysis of natural river systems reported in Table I.

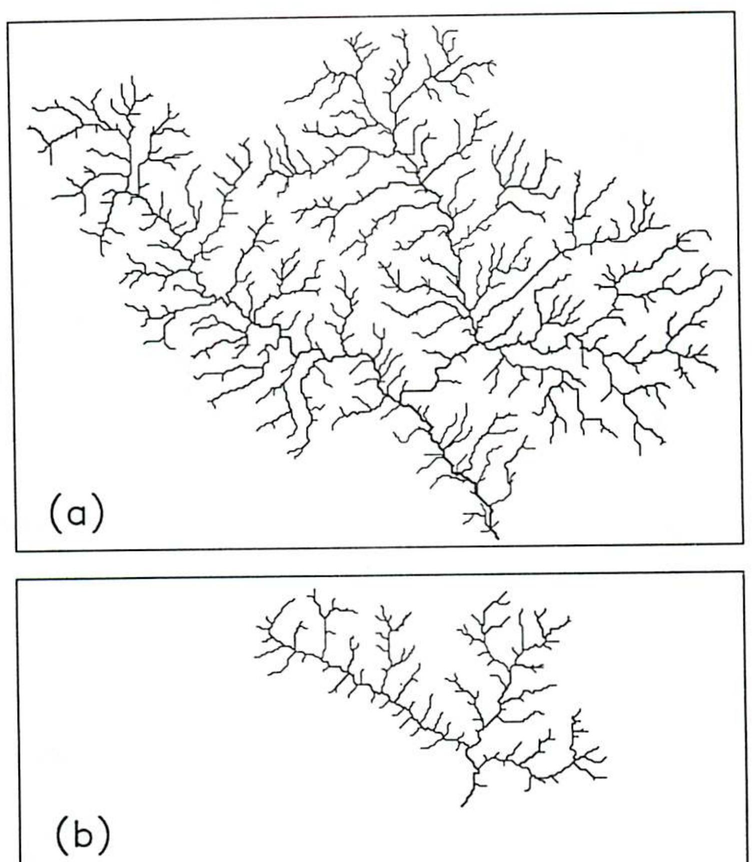

Fig. 4. Drainage network for (a) Magra river basin and (b) Entella river basin

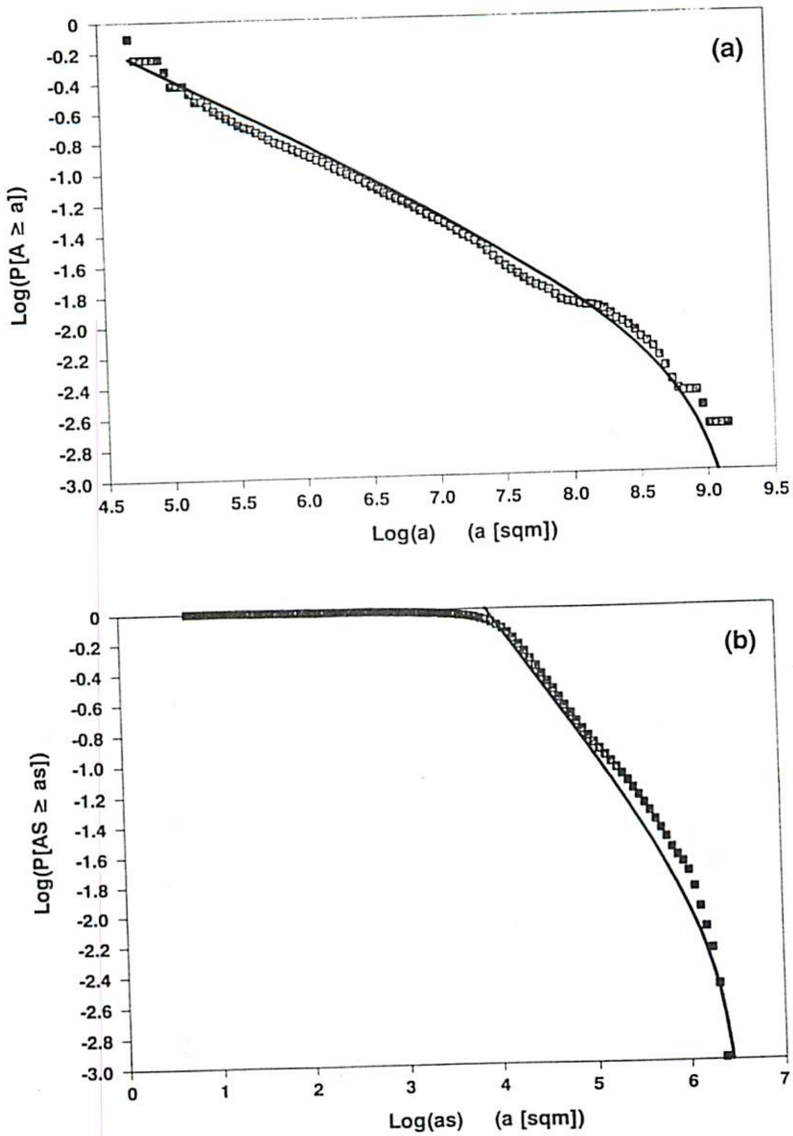

Fig. 5. Magra river basin: cumulative probability distribution and cumulative frequency distribution for (a) contributing area and (b) energy

ANALYSIS OF NATURAL RIVER NETWORKS

The cumulative frequency distributions for contributing area and energy can be obtained from the analysis of natural river networks throughout the use of DEMs. Two different basins in Italy are analysed here as obtained from the interpretation of a DEM in a square grid with 225 m to a side. The drainage structure of the two basins is described in Figure 4, whereas Table I reports the relevant characteristics of the river networks as obtained from the analysis of the network structure of the basins with no filtering procedure. This is equivalent to assuming that the channel maintenance area is equal to the pixel area (225×225 m). The observed cumulative frequency distributions for contributing area and energy are given in Figures 5 and 6. In the same figures the corresponding theoretical cumulative probability distributions are drawn, as derived from Equations (17) and (23), with reference to the characteristics of the two basins reported in Table I. The

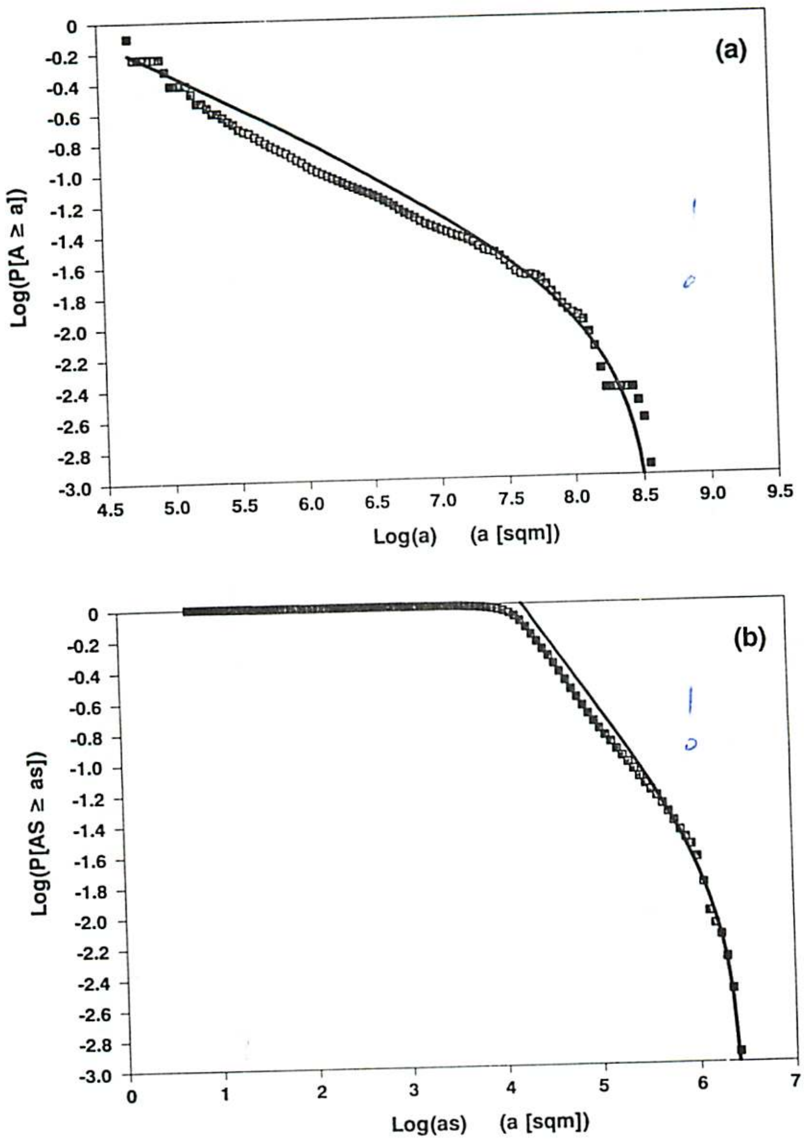

Fig. 6. Entella river basin: cumulative probability distribution and cumulative frequency distribution for (a) contributing area and (b) energy

results show a good agreement between the predicted and the observed distributions for contributing area and energy.

CONCLUSIONS

The invariance and scaling properties of the cumulative probability distributions of stream length, stream order, contributing area (i.e. discharge mass) and energy dissipation per unit length of channel can be described by the present approach. The structure proposed here for the cumulative probability distributions for areas and energy is able to fit the observed distributions and their deviations from straight lines at large values of contributing area. The value of the exponent for the distribution of contributing area can be linked to the fractal dimension of single rivers and river networks. The value of the exponent for the distribution of energy can be linked to the fractal dimension of single rivers, river networks and to the ratio $\log R_a / \log R_s$.

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APPENDIX

Distribution of contributing area

From the definition of contributing area ratio, R_a, we obtain

$$\frac{A_{\omega_*}}{A_\Omega} = R_a^{-(\Omega-\omega_*)} \tag{A1}$$

and, consequently,

$$-(\Omega-\omega_*) = \frac{\log(A_{\omega_*}/A_\Omega)}{\log R_a} \tag{A2}$$

It follows that

$$\begin{aligned}\left(\frac{R_l}{R_b}\right)^{-(\Omega-\omega_*)} &= \left(\frac{R_l}{R_b}\right)^{\frac{\log(A_{\omega_*}/A_\Omega)}{\log R_a}} = \exp\left[\frac{\log(A_{\omega_*}/A_\Omega)}{\log R_a}\log\left(\frac{R_l}{R_b}\right)\right] = \exp\left[\log\left(\frac{A_{\omega_*}}{A_\Omega}\right)\frac{\log R_l - \log R_b}{\log R_a}\right] \\ &= \exp\left[\log\left(\frac{A_{\omega_*}}{A_\Omega}\right)\frac{\log R_l}{\log R_a}\left(1-\frac{\log R_b}{\log R_l}\right)\right] = \exp\left[-\frac{d}{2}(D-1)\log\left(\frac{A_{\omega_*}}{A_\Omega}\right)\right] = \left(\frac{A_{\omega_*}}{A_\Omega}\right)^{-\frac{d}{2}(D-1)}\end{aligned} \tag{A3}$$

and, finally, we obtain

$$\left(\frac{R_l}{R_b}\right)^{-(\Omega-\omega_*)} = \left(\frac{A_{\omega_*}}{A_\Omega}\right)^{-\frac{d}{2}(D-1)} \tag{A4}$$

Distribution of energy

From Equation (A1) and from the definition of the slope ratio, R_s, we obtain

$$\frac{A_{\omega_*}S_{\omega_*}}{A_\Omega S_\Omega} = (R_a R_s)^{-(\Omega-\omega_*)} \tag{A5}$$

and, consequently,

$$-(\Omega-\omega_*) = \frac{\log(A_{\omega_*}S_{\omega_*}/A_\Omega S_\Omega)}{\log(R_a R_s)} \tag{A6}$$

Following the same procedure used to derive Equation (A3), with Θ defined in Equation (22), we obtain

$$\left(\frac{R_l}{R_b}\right)^{-(\Omega-\omega_*)} = \left(\frac{A_{\omega_*}S_{\omega_*}}{A_\Omega S_\Omega}\right)^{-\frac{d}{2}(D-1)\frac{\Theta}{\Theta-1}} \tag{A7}$$