

# PARAMETRICALLY FORCED SURFACE WAVES

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## 1. INTRODUCTION

Michael Faraday opens his *Diary* for July 1, 1831, with the observation that "Mercury on tin plate being vibrated in sunshine gave very beautiful effects of reflection" and goes on to report similar observations of surface waves on ink, water, alcohol, turpentine, milk, and white of egg covering a horizontal plate subjected to a vertical vibration (Faraday 1831a).<sup>1</sup> He also reports that when a vibrating vertical plate is dipped into a basin of water, "Elevations, waves or crispations immediately formed but of a peculiar character . . . beginning at the plate and projecting directly out from it . . . like the teeth of a very short coarse comb" and comments that this "experiment shews at once the cause of the ripple or crispation formed at surface of water in a drinking glass vibrating under the finger." On July 2, he examines the latter in some detail, and on July 5 he reports that similar waves can be excited by a superficially immersed, vibrating cork: "So soon as the cork touched the water a beautiful store of [radial] ridges formed all around it, running out 2, 3 or even 4 inches."

Faraday remarks that these waves had a frequency equal to one half that of the excitation, and we now would describe them as emerging from a symmetry-breaking bifurcation of a type that characterizes the motion of a simple pendulum subjected to a vertical oscillation of its pivot. If the acceleration of the pivot is  $\varepsilon g \cos 2\omega t$  ( $\varepsilon$  is a small parameter, and  $2\omega$  is

<sup>1</sup> In his subsequent report to the Royal Society, Faraday (1831b) comments that similar waves had been observed by Oersted, Wheatstone, Weber, "and probably others," and that he had "seen the water in a pail placed in a barrow, and that on the head of an upright cask in a brewer's van passing over stones, exhibit [similar waves]."

the angular frequency) and the angular displacement  $\theta$  is assumed to be small, the motion of the pendulum is governed by

$$\ddot{\theta} + \omega_0^2(1 + \varepsilon \cos 2\omega t)\theta = 0, \quad (1.1)$$

where  $\omega_0 = (g/L)^{1/2}$  is the frequency of small oscillations for  $\varepsilon = 0$ . The null solution  $\theta = 0$  is obvious (although not trivial), but if  $0 < \varepsilon \ll 1$ , (1.1) also admits the subharmonic solution

$$\theta = A \cos \frac{1}{2} \omega t \quad \text{for} \quad \omega^2 = \omega_0^2(1 \pm \frac{1}{2} \varepsilon). \quad (1.2a,b)$$

The question of which solution [ $\theta = 0$  or (1.2)] is realized is resolved by a stability analysis, which reveals that the null solution loses stability to the subharmonic solution (which equilibrates at a finite amplitude) in the frequency interval

$$\omega_0^2(1 - \frac{1}{2} \varepsilon) < \omega^2 < \omega_0^2(1 + \frac{1}{2} \varepsilon). \quad (1.3)$$

The introduction of the linear damping term  $2\delta\dot{\theta}$  (where  $\delta$  is the ratio of actual to critical damping) on the left-hand side of (1.1) leads to the replacement of  $\varepsilon$  by  $(\varepsilon^2 - \delta^2)^{1/2}$  in (1.3), from which it follows that the subharmonic response is possible only if  $\varepsilon > \delta$ .

Oscillations of the type just described are characterized by the adjective *parametric*, with reference to the imposed variation of one of the parameters (the effective gravitational acceleration in the present context), and arise in many contexts, such as with electronic oscillators (owing to the variation of a capacitor), seismometers, hula hoops, and yo-yos. We do not know who first observed such oscillations, but their scientific study in fluid mechanics manifestly dates from Faraday's experiments. The earliest example, other than Faraday's, of which we are aware is Melde's (1860) experiment, in which the tension in a string is varied periodically through attachment to one of the vibrating prongs of a massive tuning fork [see Section 68b in Rayleigh's *Theory of Sound* (1877)]. We designate those waves associated with an oscillation of the effective gravitational acceleration as *Faraday waves* and those waves with crests normal to a moving boundary (wavemaker) as *cross waves* (Section 6). The latter are sometimes described as *edge waves*, but this description usually is reserved for waves on a sloping beach (Section 7), originally discovered by Stokes (1846), which may be subharmonically excited by either an incoming wave or a disturbance moving parallel to the shore.

The excitation of Faraday waves has been analyzed by Rayleigh (1883a,b), Benjamin & Ursell (1954), Dodge et al. (1965), Ockendon & Ockendon (1973), Henstock & Sani (1974), Miles (1984a), Meron & Procaccia (1986), and Gu et al. (1988). Benjamin & Ursell (1954) reduce the description of small disturbances to Mathieu's equation and invoke the

known results for the stability of the solutions of that equation to confirm the conclusion of Faraday and Rayleigh [who had supported Faraday in opposition to the claim of Matthiessen (1868) that the resonant response was synchronous rather than subharmonic] that the free-surface response in a vertically oscillating container is subharmonic. [Mathieu's equation implies subharmonics of frequency  $\omega$ ,  $3\omega$ ,  $5\omega$ , . . . , in response to a  $2\omega$  excitation, but only the first subharmonic is consistent with the assumption of weak nonlinearity.] Ockendon & Ockendon (1973) extend the analysis of Benjamin & Ursell to small but finite amplitudes but do not explicitly calculate the interaction parameter ( $t_1$  in their notation) that measures nonlinear inertial effects (third-order terms in the equations of motion); they also determine the bifurcation structure of the evolution equations, including the qualitative effects of linear damping. Miles (1984a) calculates the interaction parameter [ $A$  in (3.3b) below] and incorporates linear damping. Gu et al. (1988) calculate the interaction parameter for a rectangular cylinder. The finite-amplitude analyses of Dodge et al. (1965) and Henstock & Sani (1974) are in error (see Miles 1984a), as is that of Meron & Procaccia (1986) (see Section 5 below).

We emphasize that the modifiers "nonlinear" and "finite-amplitude" in the preceding references and throughout the subsequent discussion refer to *weakly nonlinear* motions, for which terms of fourth and higher order are neglected in the equations of motion. Strong nonlinearity, which may imply such phenomena as wave breaking, lies almost entirely outside the realm of available analytical techniques.

Faraday waves have been of special interest in recent years as a (possibly) tractable problem in fluid mechanics to which the techniques of modern bifurcation and chaos theory are applicable after a normal-mode expansion. We therefore devote most of this review to those waves and consider cross-waves and edge waves only briefly.

Most of the following development, which deals with a homogeneous fluid, admits a straightforward extension to waves on the interface between two fluid layers, for which the effective value of  $g$  is proportional to the difference in densities of the two layers.

We do not attempt a systematic coverage of the Russian literature. See Nevolin (1985) for a review of the linear problem, with emphasis on acoustic and electromagnetic excitation. The paper by Ezerskii et al. (1986) appears especially intriguing, but we found their development frustratingly sparse.

## 2. VERTICALLY OSCILLATING BASIN

The following development is taken from Miles (1984a), emended and extended to incorporate capillarity. We assume a perfect fluid in a vertical

cylinder of cross section  $S$  and depth  $d$  that is subjected to the vertical displacement  $z_0 = a_0 \cos 2\omega t$  ( $0 < \omega^2 a_0 \ll g$ ). Damping is ultimately incorporated as a small perturbation. The implicit assumption that the free surface intersects the lateral boundary at a constant contact angle of  $90^\circ$  (see Section 2.3) poses serious problems if the lateral dimensions are not large compared with the capillary length (2.8 mm for clean water at  $20^\circ\text{C}$ ).

## 2.1 The Lagrangian

We pose the free-surface displacement (relative to the plane of the level surface, which is moving with the basin) in the form

$$\eta(\mathbf{x}, t) = \eta_n(t)\psi_n(\mathbf{x}) \quad (\mathbf{x} \text{ in } S), \quad (2.1)$$

where, here and subsequently (except as noted), repeated indices are summed over the participating modes, the  $\eta_n$  are generalized coordinates, the  $\psi_n$  are the orthonormal eigenfunctions determined by

$$(\nabla^2 + k_n^2)\psi = 0 \quad (\mathbf{x} \text{ in } S), \quad \mathbf{n} \cdot \nabla\psi = 0 \quad \text{on } \partial S, \quad \langle \psi_m \psi_n \rangle = \delta_{mn}, \quad (2.2a, b, c)$$

$k_n$  are the eigenvalues,  $\langle \rangle$  signifies an average over the cross section  $S$  of the cylindrical basin, and  $\delta_{mn}$  is the Kronecker delta. The corresponding Lagrangian, as calculated in Miles (1984a, Sect. 2) after incorporating capillary energy, is

$$L \equiv (\rho S)^{-1}(T - V) \quad (2.3a)$$

$$= \frac{1}{2}(\delta_{mn}a_n + a_{lmn}\eta_l + \frac{1}{2}a_{jlmn}\eta_j\eta_l + \dots)\dot{\eta}_m\dot{\eta}_n \\ - \frac{1}{2}(g + \ddot{z}_0)\eta_n\eta_n - \frac{1}{2}\hat{T}(\delta_{mn}k_n^2 - \frac{1}{4}\ell_{jlmn}\eta_j\eta_l + \dots)\eta_m\eta_n, \quad (2.3b)$$

where  $\rho$  is the fluid density,  $T$  and  $V$  are the kinetic and potential energies,  $g$  and  $\ddot{z}_0$  are the gravitational and imposed accelerations,  $\rho\hat{T}$  is the surface tension, and

$$a_n = (k_n \tanh k_n d)^{-1}, \quad (2.4)$$

$$a_{lmn} = [1 + \frac{1}{2}(k_l^2 - k_m^2 - k_n^2)a_m a_n] \langle \psi_l \psi_m \psi_n \rangle, \quad (2.5)$$

$$a_{jlmn} = \frac{1}{2}(k_i^2 + k_m^2 - k_j^2)(k_i^2 + k_n^2 - k_l^2)a_i a_m a_n \langle \psi_i \psi_j \psi_m \rangle \langle \psi_i \psi_l \psi_n \rangle \\ - (a_m + a_n) \langle \psi_j \psi_l \nabla \psi_m \cdot \nabla \psi_n \rangle, \quad (2.6)$$

$$\ell_{jlmn} = \langle (\nabla \psi_j \cdot \nabla \psi_l) (\nabla \psi_m \cdot \nabla \psi_n) \rangle. \quad (2.7)$$

We also find it convenient to introduce

$$\hat{a}_{jlmn} = (1 + k_i^2 l_{*}^2)^{-1} a_i \ell_{jlmn} l_{*}^2, \quad \ell_{lmn} = a_{lmn} - 4a_{mnl} \quad (2.8a, b)$$

and

$$\omega_n^2 = (gk_n + \hat{T}k_n^3) \tanh k_n d = (g/\alpha_n)(1 + k_n^2 l_*^2), \quad (2.9)$$

where  $l_* \equiv (\hat{T}/g)^{1/2}$  is the capillary length. The summation convention (that repeated indices are summed over the participating modes) applies in (2.1) and (2.3b) but not in (2.2) or (2.4)–(2.9).

## 2.2 Slow Modulation

The asymptotic (following the decay of transients) response to the excitation  $z_0 = a_0 \cos 2\omega t$  may be described by expanding  $\eta_n$  in a Fourier series with fundamental frequency  $\omega$  and slowly varying coefficients. The latter, say  $A_{mn}(\tau)$  and  $B_{mn}(\tau)$  (where  $\tau \equiv \varepsilon\omega t$  is a slow time and  $\varepsilon \equiv a_0/\alpha_1 \ll 1$  is a scaling parameter), may be determined by averaging  $L$  over a  $2\pi$  interval of  $\omega$  and then invoking Hamilton's principle for the average Lagrangian  $\langle L \rangle$  as a function of  $A_{mn}$ ,  $B_{mn}$ ,  $\dot{A}_{mn}$  and  $\dot{B}_{mn}$ , where, here and subsequently, the dot implies differentiation with respect to  $\tau$ . The participating modes may be resolved into a set of primary modes (typically one or two), for which a first approximation (typically not uniformly valid with respect to  $\omega - \omega_n$ ) may be obtained through a linear approximation, and a set of secondary modes that are excited through nonlinear interactions with the primary modes. If the nondimensional, dominant amplitudes are abbreviated by  $p_n$  and  $q_n$  and the average Lagrangian  $\langle L \rangle$  is truncated at either fourth order [as in (2.3)] or third order, depending on whether the dominant amplitudes are  $O(\varepsilon^{1/2})$  or  $O(\varepsilon)$ , respectively, and the amplitudes of the secondary modes eliminated in the former case or neglected in the latter, the  $p_n$  and  $q_n$  are found to be governed by a set of evolution equations of the Hamiltonian form

$$\dot{p}_n = -\partial H / \partial q_n \quad \text{and} \quad \dot{q}_n = \partial H / \partial p_n, \quad (2.10a,b)$$

in which  $H$  is either a quartic or a cubic in  $p_n$  and  $q_n$ . This average-Lagrangian formulation offers the advantage, vis-à-vis a perturbation expansion of the equations of motion, of preserving fundamental symmetries of the evolution equations (see the discussion in the third paragraph of Section 5).

Weak, linear damping may be incorporated (weak, nonlinear damping also could be incorporated) by introducing the dissipation function

$$D = \frac{1}{2} \alpha_n (p_n^2 + q_n^2), \quad (2.11)$$

where  $\alpha_n \equiv \delta_n/\varepsilon$  and  $\delta_n$  is the ratio of actual to critical damping for free oscillations in the  $n$ th mode [ $\delta_n$  is best determined experimentally, although theoretical estimates are available (Miles 1967)], and by replacing (2.10) by

$$\dot{p}_n = -\frac{\partial D}{\partial p_n} - \frac{\partial H}{\partial q_n}, \quad \dot{q}_n = -\frac{\partial D}{\partial q_n} + \frac{\partial H}{\partial p_n} \quad (n \text{ not summed}). \quad (2.12a,b)$$

### 2.3 The Meniscus

It is implicit in the formulation of the gravity-wave problem that, in the absence of capillary effects, the free surface intersects a vertical boundary normally. In fact, contact-angle hysteresis at the edge of the free surface may be significant for the damping of most surface waves of laboratory scale (see Benjamin & Ursell 1954, Case & Parkinson 1957, Keulegan 1959, Miles 1967, Mei & Liu 1973, Hocking 1987a,b). Moreover, the edge constraint associated with the meniscus dynamics also may affect the natural frequency of surface waves if the wavelength is sufficiently small ( $\lesssim 1$  cm).

Benjamin & Scott (1979) have suggested that the edge constraint for a rim-full container should be  $\eta = 0$ , have calculated the corresponding lowest natural frequency for two-dimensional deep-water waves in a rim-full channel, and have confirmed their prediction for channels of 4–20 mm breadth. Graham-Eagle (1983) has carried out the corresponding calculation for a circular cylinder. Benjamin & Scott also conjecture that the edge constraint  $\eta = 0$  may hold for a container that is not rim-full, but the evidence in support of this conjecture appears to be slight.

Hocking (1987a,b) has suggested that Benjamin & Scott's "pinned-end" edge condition be replaced by the "wetting" condition

$$\partial\eta/\partial t = \lambda(\partial\eta/\partial n), \quad (2.13)$$

where  $n$  is the inwardly directed normal to the vertical boundary and  $\lambda$  is a phenomenological parameter. This condition evidently includes the conventional and pinned-end conditions as limiting cases for  $\lambda = \infty$  and  $\lambda = 0$ , respectively. Hocking solves the two-dimensional problem for this condition, uses his results to calculate damping at the wall of a cylinder of general cross section, and obtains plausible agreement (for assumed values of  $\lambda$ ) with the measurements of Benjamin & Ursell (1954), Case & Parkinson (1957), and Keulegan (1959).

The effects of a nonclassical ( $\partial\eta/\partial n \neq 0$ ) edge constraint may be incorporated in a boundary-layer calculation (such as that of Hocking) of dissipation, but the nonorthogonality (in the sense that the integral of  $\eta_m\eta_n$  over the free surface for a pair of modes does not vanish for  $m \neq n$ ) of the natural modes renders the corresponding extension of the nonlinear dynamical analysis intractable. It therefore is important to use appropriate fluid-container combinations [e.g.  $n$ -butyl alcohol, rather than water, in plexiglass containers, as in Simonelli & Gollub's (1989) work] to minimize meniscus effects in centimeter-scale experiments.

### 3. SINGLE MODE

We now suppose that the motion is dominated by a single mode,  $n = 1$ . Letting

$$\eta_1 = l[p(\tau) \cos \omega t + q(\tau) \sin \omega t], \quad l = 2\varepsilon^{1/2} k_1^{-1} \tanh k_1 d, \quad (3.1a,b)$$

we obtain (Miles 1984a, Sect. 3)

$$H = \frac{1}{2}(p^2 - q^2) + \frac{1}{2}\beta(p^2 + q^2) + \frac{1}{4}A(p^2 + q^2)^2, \quad (3.2)$$

where

$$\beta = \frac{\omega^2 - \omega_1^2}{2\varepsilon\omega^3}, \quad A = \frac{1}{4}\{a_{1111}(2a_{1111} + 3\hat{a}_{1111}) + 2\kappa_n^{-1}a_{n111}^2 - \Omega_n^{-1}a_{n111}^2\}T^4, \quad (3.3a,b)$$

$$\Omega_n = \left( \frac{4\omega^2}{\omega_n^2} - 1 \right) \kappa_n, \quad \kappa_n = \frac{1 + k_n^2 l_{*}^2}{1 + k_1^2 l_{*}^2}, \quad T \equiv \tanh k_1 d, \quad (3.4a,b,c)$$

and  $n$  is summed over the secondary modes in (3.3b). (Note that  $\omega$  may be replaced by  $\omega_1$  in the denominator of the tuning parameter  $\beta$  within the present approximation.) Invoking (2.12), we obtain

$$\dot{p} + \alpha p = -[\beta - 1 + A(p^2 + q^2)]q, \quad \dot{q} + \alpha q = [\beta + 1 + A(p^2 + q^2)]p \quad (3.5a,b)$$

The parameter  $A$ , which determines the effect of finite amplitude on the resonant frequency according to

$$\omega^2 = \omega_1^2[1 - \varepsilon A(p^2 + q^2)], \quad (3.6)$$

is plotted vs.  $d/a$  in Figure 1 for the dominant axisymmetric and anti-symmetric gravity waves in a circular cylinder of radius  $a$ . It vanishes at a critical depth  $d = d_c$  (Tadjbakhsh & Keller 1960), and  $d \geq d_c$  ( $A \geq 0$ ) corresponds to a *soft/hard* spring in the terminology of mechanics. See Virnig et al. (1988) for experimental confirmation of this analogy in a rectangular tank.

#### 3.1 Stability and Bifurcation

A phase-plane analysis of (3.5) (Miles 1984a, Sect. 5) reveals that the null solution  $p = q = 0$  (plane free surface) is stable if and only if  $|\beta| > (1 - \alpha^2)^{1/2} \equiv \gamma$ , and that the finite-amplitude oscillation

$$\eta = \pm l|A|^{-1/2}(\gamma - \beta \operatorname{sgn} A)^{1/2} \sin[\omega t + \frac{1}{2} \cos^{-1}(\gamma \operatorname{sgn} A)] \quad (3.7)$$

is stable if  $\beta \operatorname{sgn} A < \gamma$ .

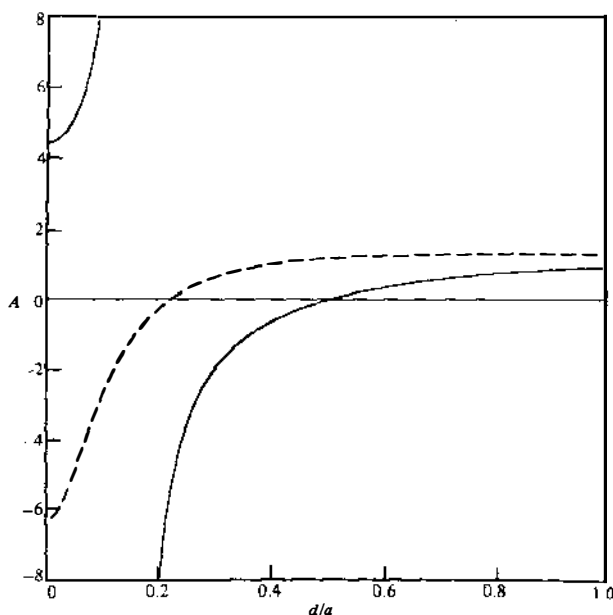


Figure 1 The interaction coefficient  $A$  for the dominant (gravity-wave) axisymmetric (---) and antisymmetric (—) modes in a circular cylinder of radius  $a$  and depth  $d$ . An internal resonance with a 2:1 frequency ratio occurs between these two modes at  $d/a = 0.1523$ . Similar resonances occur at  $d/a = 0.1981$  for the (0,1) and (0,3) modes and at  $d/a = 0.3470$  for the (0,1) and (0,4) modes, but these resonances are too narrow to be resolved on the scale of the present drawing (from Miles 1984a; courtesy of Cambridge University Press).

Both the null solution and (3.7) are stable if  $\beta \operatorname{sgn} A < -\gamma$ , and which is achieved depends on the initial condition; however, the basin of attraction of the null solution becomes increasingly dominant over that of (3.7) as  $\beta \operatorname{sgn} A$  decreases from  $-\gamma$ . There are no attractors other than the fixed points associated with these two motions; in particular, neither limit cycles (periodic oscillations of  $p$  and  $q$ , which imply slow modulation of the envelope of  $\eta_1$ ) nor chaotic motions are possible within the present approximation.

### 3.2 Comparison With Experiment

Benjamin & Ursell (1954) report fairly close agreement between the inviscid, theoretical prediction  $(\omega/\omega_1)^2 = 1 \pm 2ka_0$  ( $ka \gg 1$ ) of the stability boundary and their experiments in a  $2\frac{1}{8}$ "-diameter  $\times$  10"-deep Perspex (plexiglass) cylinder (see their Figure 1, wherein  $q = ka_0 = \epsilon$ ), with differences that are qualitatively consistent with small dissipation. Their quan-



titative estimate of the minimum value of  $a_0$  on the hypothesis that the dissipation is derived entirely from laminar (Stokes) boundary layers is an order of magnitude too low, and they conjecture that the actual dissipation is dominated by capillary hysteresis (see below). This conjecture is consistent with the hydrophobic combination of water (albeit "distilled") and Perspex (albeit "carefully cleaned").

Dodge et al. (1965) also report fairly close agreement between the inviscid, theoretical stability boundary and their experiments in a  $5.7'' \times 13''$  acrylic cylinder; however, they note that there is hysteresis between the transitions to and from the null state. They report good agreement between prediction and observation for the shape of the subharmonic wave. Virnig et al. (1988) report measurements of wave amplitudes for single-mode Faraday waves that are in reasonable agreement with the calculations of Gu et al. (1988).

Gollub & Meyer (1983) report experiments for single (for sufficiently small amplitudes), axisymmetric modes in a 2.41-cm-radius  $\times$  1-cm-deep circular cylinder. They measured four critical values of  $a_0$  ( $0 < A_c < A_0 < A_m < A_d$ ) as discrete functions of the modal wave number. Here  $A_c$  is the threshold of subharmonic motion;  $A_0$  is the threshold of precession, for which the center of the pattern is slightly ( $\sim 1$  mm) displaced from the axis and precesses at a low frequency ( $\ll \omega$ ) and the envelope of the carrier exhibits a periodic, almost monochromatic modulation;  $A_m$  is the threshold of a slow, azimuthal modulation of the axisymmetric mode accompanied by a broadening of the spectral lines of the envelope; and  $A_d$  is the threshold of chaotic motion, for which the power spectrum of the envelope is broad. For  $\omega/2\pi = 31$  Hz, it follows that  $A_0 = 1.64A_c$ ,  $A_m = 1.94A_c$ , and  $A_d = 2.37A_c$ .

The preceding theory predicts  $A_c = \delta/k$  (for  $kd \gg 1$ , as in Gollub & Meyer's experiments) but fails to predict the higher thresholds. For  $\omega/2\pi = 31$  Hz and  $k = 8.14 \text{ cm}^{-1}$ , it follows that  $A_c = 25 \text{ }\mu\text{m}$ , so that  $\varepsilon = kA_c = 0.02$  is indeed small, as assumed in the theory. A lower bound to the theoretical estimate of  $A_c = \delta/k$ , based on the assumption of Stokes boundary layers on the lateral surface of the cylinder (Miles 1967), is  $A_c = 18 \text{ }\mu\text{m}$ ; the assumption of an inextensible film on the free surface doubles this estimate to  $A_c = 36 \text{ }\mu\text{m}$ . These estimates suggest that dissipation at the free surface, including the meniscus, was relatively small in Gollub & Meyer's experiments.

The failure of the present theory to predict either periodic or chaotic modulation of the envelope suggests internal resonance between the primary mode and a secondary mode of the same natural frequency. (Internal resonance with a 2:1 frequency ratio would imply significant second-

harmonic content for the carrier, but the carrier in Gollub & Meyer's experiments appears to have been almost monochromatic.) Those modes with two nodal diameters are candidates by virtue of the proximity of the roots of the Bessel-function derivatives  $J'_0(ka)$  and  $J'_2(ka)$  ( $k_{0,s}a = 3.83, 7.02, 10.17, 13.32, 16.47, 19.62, 22.76, \dots$ ;  $k_{2,s}a = 3.05, 6.70, 9.97, 13.17, 16.35, 19.51, 22.67, \dots$ ), but we have not carried out a detailed investigation.

#### 4. INTERNAL RESONANCE: $\omega_2 \simeq 2\omega_1 \simeq 2\omega$

We now assume that  $\omega_2 - 2\omega_1 = O(\varepsilon\omega)$  for a particular pair of modes,  $n = 1$  and 2. The amplitudes of these primary modes then are  $O(\varepsilon)$ , while the secondary modes are excited only at  $O(\varepsilon^2)$ . Letting

$$\eta_n = l_n[p_n(\tau)\cos n\omega t + q_n(\tau)\sin n\omega t] \quad (n = 1, 2), \quad (4.1)$$

where

$$l_2 = (a_1/2a_2)^{1/2}l_1 \equiv |a_{112} - \frac{1}{4}a_{211}|^{-1}a_0, \quad (4.2)$$

assuming that  $4a_{211} < a_{112}$ , and proceeding as outlined in Section 2.2 (see Miles 1984a, Sect. 6), we obtain

$$H = \frac{1}{2}(p_1^2 - q_1^2) + \frac{1}{2}\beta_n(p_n^2 + q_n^2) - \frac{1}{2}(p_1^2 - q_1^2)p_2 - p_1q_1q_2, \quad (4.3)$$

wherein  $n$  is summed over 1, 2, and

$$\beta_n \equiv (2\varepsilon n\omega^2)^{-1}(n^2\omega^2 - \nu\omega_n^2) \quad (n = 1, 2). \quad (4.4)$$

The signs of  $p_2$  and  $q_2$  in (4.3) must be changed if  $4a_{112} > a_{211}$ . The Hamiltonian (4.3) and the corresponding evolution equations,

$$\begin{aligned} \dot{p}_1 + \alpha_1 p_1 + \beta_1 q_1 + p_2 q_1 - p_1 q_2 &= q_1, \\ \dot{q}_1 + \alpha_1 q_1 - \beta_1 p_1 + p_1 p_2 + q_1 q_2 &= p_1, \end{aligned} \quad (4.5a,b)$$

$$\dot{p}_2 + \alpha_2 p_2 + \beta_2 q_2 - p_1 q_1 = 0, \quad \dot{q}_2 + \alpha_2 q_2 - \beta_2 p_2 + \frac{1}{2}(p_1^2 - q_1^2) = 0, \quad (4.5c,d)$$

are isomorphic to those for a parametrically excited, internally resonant double pendulum (Miles 1985a, wherein the signs of  $\beta_1$  and  $\beta_2$  must be changed to agree with the present definitions).

##### 4.1 Stability and Bifurcation

It follows from the results for the pendulum that  $\beta_1^2 > 1 - \alpha_1^2$  is both necessary and sufficient for stability of the plane free surface. The stability conditions for finite-amplitude, monochromatic (at frequency  $\omega$ ) oscillations are algebraically complicated, and Miles (1984a, 1985a) obtains

explicit results only for exact internal resonance ( $\omega_2 = 2\omega_1$ ), in which case Hopf bifurcations, and hence limit cycles and chaotic motions, are impossible.

Gu & Sethna (1987) have analyzed (equivalents of) (4.5) in some detail and find that Hopf bifurcations *are* possible for  $0 < |\beta| < 2.67$  and  $0 < \alpha_1 < 0.243$ , where

$$\beta \equiv 2\beta_1 - \beta_2 = (\varepsilon\omega^2)^{-1}[(\frac{1}{2}\omega_2)^2 - \omega_1^2]. \quad (4.6)$$

They give numerical results for  $\alpha_1 = 2^{-1/2}\alpha_2 = 0.1$  and  $\beta = -1$ ; find Hopf bifurcations at  $\beta = -0.292$  and  $-0.471$ , which are, respectively, supercritical and subcritical; and show that chaotic motions are possible. [Nayfeh (1987) also finds a Hopf bifurcation by detuning the internal resonance for a particular case in a circular cylinder.] A more abstract analysis by Holmes (1986), based on somewhat different assumptions than those of Gu & Sethna, shows that chaotic motions are (in some parametric domain) an intrinsic property of parametrically excited, weakly nonlinear surface waves with 2:1 internal resonance.<sup>2</sup> [Two-dimensional waves in a shallow, rectangular tank (Miles 1985b), for which the resonances form a harmonic progression, may be an exception.]

## 4.2 Subharmonic Internal Resonance

An alternative to the preceding internal resonance is one between two modes with natural frequencies that approximate  $\frac{1}{2}\omega$  and  $\omega$  (vs.  $\omega$  and  $2\omega$ ), where  $2\omega$  is the forcing frequency. If the excitation  $a_0 \cos 2\omega t$  is replaced by  $a_0 \cos 4\omega t$  and  $\varepsilon = a_0/a_1$  is replaced by  $\varepsilon = a_0/a_2$ , the primary modes may be represented by (4.1), with the end results that  $\frac{1}{2}(p_1^2 - q_1^2)$  is replaced by  $p_2^2 - q_2^2$  in (4.3), and that chaotic motions are possible for exact resonance. See Becker & Milcs (1986) for the analysis of the corresponding double pendulum.

## 5. INTERNAL RESONANCE: $\omega_2 \simeq \omega_1$

We now suppose that  $\omega_2 - \omega_1 = O(\varepsilon\omega)$  for a particular pair of modes,  $n = 1$  and 2, and posit that

$$\eta_n = l[p_n(\tau) \cos \omega t + q_n(\tau) \sin \omega t] \quad (n = 1, 2), \quad (5.1)$$

<sup>2</sup>P. R. Sethna (private communication) comments that the work of Gollub & Meyer (1983), Holmes (1986), Gu & Sethna (1987), Gu et al. (1988), and Simonelli & Gollub (1989) “strongly suggest[s] global homoclinic and heteroclinic bifurcations” and that “a warning to the reader [of the present review] of the potential for complex phenomena at extremely long time scales occurring in those systems may be useful.”

where  $l$  is defined by (3.1b), and  $\tau \equiv \varepsilon \omega t$  and  $\varepsilon \equiv a_0/a_1$  are defined as in Section 2. Proceeding as in Section 2 above (cf. Miles 1984a, App. C), we obtain

$$H = \frac{1}{2}(p_n p_n - q_n q_n) + \frac{1}{2}\beta_n(p_n^2 + q_n^2) + \frac{1}{4}A_{mn}(p_m^2 + q_m^2)(p_n^2 + q_n^2) \\ + \frac{1}{2}B(p_1 q_2 - p_2 q_1)^2 + \frac{1}{2}C_n(p_n^2 + q_n^2)(p_1 p_2 + q_1 q_2), \quad (5.2)$$

where

$$\beta_n = (2\varepsilon\omega^2)^{-1}(\omega^2 - \omega_n^2) \quad (n = 1, 2), \quad (5.3)$$

$$A_{11} = \frac{1}{4}\{a_1(2a_{1111} + 3\hat{a}_{1111}) + 2\kappa_n^{-1}a_{n11}^2 - \Omega_n^{-1}\ell_{n11}^2\}T^4, \quad (5.4a)$$

$$A_{22} = \frac{1}{4}\{a_1(2a_{2222} + 3\hat{a}_{2222}) + 2\kappa_n^{-1}a_{n22}^2 - \Omega_n^{-1}\ell_{n22}^2\}T^4, \quad (5.4b)$$

$$A_{12} = A_{21} = [\frac{1}{4}a_1(a_{1122} + a_{2211} + 2a_{1221} + 2a_{1212} + 3\hat{a}_{1122} + 6\hat{a}_{1212}) \\ + \kappa_n^{-1}(\frac{1}{2}a_{n11}a_{n22} + a_{n12}^2) - \frac{1}{4}\Omega_n^{-1}\{\ell_{n11}\ell_{n22} + \frac{1}{2}(\ell_{n12} + \ell_{n21})^2\}]T^4, \quad (5.4c)$$

$$B = [\frac{1}{2}a_1(a_{1122} + a_{2211} - 2a_{1221} - 2a_{1212} - \frac{1}{2}\hat{a}_{1122} - \hat{a}_{1212}) \\ - \kappa_n^{-1}a_{n12}^2 + \frac{1}{2}\Omega_n^{-1}\ell_{n11}\ell_{n22}]T^4, \quad (5.5)$$

$$C_1 = [\frac{1}{2}a_1(a_{1112} + a_{1211} + 3\hat{a}_{1112}) + \kappa_n^{-1}a_{n12}a_{n11} \\ - \frac{1}{4}\Omega_n^{-1}\ell_{n11}(\ell_{n12} + \ell_{n21})]T^4, \quad (5.6a)$$

$$C_2 = [\frac{1}{2}a_1(a_{2221} + a_{2122} + 3\hat{a}_{2221}) + \kappa_n^{-1}a_{n12}a_{n22} \\ - \frac{1}{4}\Omega_n^{-1}\ell_{n22}(\ell_{n12} + \ell_{n21})]T^4, \quad (5.6b)$$

wherein the various parameters are defined by (2.4)–(2.9) and (3.4). The parameters  $C_{1,2}$  vanish for most configurations of practical interest (e.g. circular, elliptic, or rectangular cylinders), and we assume that  $C_{1,2} = 0$  throughout the subsequent development.

The canonical transformation

$$r_n = 2^{-1/2}(p_n + iq_n), \quad r_n^* = 2^{-1/2}(p_n - iq_n), \quad (5.7a,b)$$

carries (5.2) and (2.12) over to

$$H = \frac{1}{2}(r_n r_n^* + r_n^* r_n^*) + \beta_n r_n r_n^* + A_{mn} r_m r_m^* r_n r_n^* - \frac{1}{2}B(r_1 r_2^* - r_1^* r_2)^2, \quad (5.8)$$

$$\dot{r}_1 + \alpha_1 r_1 = i\{r_1^* + [\beta_1 + 2A_{11}|r_1|^2 + (2A_{12} + B)|r_2|^2]r_1 - Br_1^* r_2^2\}, \quad (5.9a)$$

$$\dot{r}_2 + \alpha_2 r_2 = i\{r_2^* + [\beta_2 + (2A_{12} + B)|r_1|^2 + 2A_{22}|r_2|^2]r_2 - Br_1^2 r_2^*\}, \quad (5.9b)$$

together with the complex conjugate of (5.9a,b). The corresponding energy equation is

$$\left(\frac{d}{d\tau} + 2\alpha_n\right)r_n r_n^* = i(r_n^* r_n^* - r_n r_n), \quad (5.10)$$

in which  $r_n r_n^* = \frac{1}{2}(p_n p_n + q_n q_n)$  represents the mean (over the period  $2\pi/\omega$ ) energy,  $2\alpha_n r_n r_n^*$  the mean dissipation rate, and  $i(r_n^* r_n^* - r_n r_n) = 2p_n q_n$  the mean rate at which work is done by the external driving force.

The present problem has been attacked by Meron & Procaccia (1986), Feng & Sethna (1989), Simonelli & Gollub (1989), and Umeki & Kambe (1989). Meron & Procaccia, starting from the equations of motion for the fluid, carry out a perturbation analysis that is equivalent to that of Miles (1984a) in neglecting terms of fourth or higher order in the free-surface boundary conditions; however, they also neglect “nonlinear terms containing the time derivative  $\partial_t$  [because] these terms are not expected to contribute significantly to the long-time behavior of the amplitude equations.” In fact, this last approximation is inconsistent with the remainder of their truncation, and this (and possibly other inconsistencies) leads to evolution equations in which (after trivial changes of notation) the coefficients of  $r_1 |r_2|^2$  and  $r_1^* r_2^2$  in their counterpart of (5.9a) differ from those of  $r_2 |r_1|^2$  and  $r_2^* r_1^2$ , respectively, in their counterpart of (5.9b).<sup>3</sup> This asymmetry in their evolution equations introduces a spurious term on the right-hand side of the energy equation (5.10), which implies a nonuniform validity of their approximation as  $t \uparrow \infty$ .

Simonelli & Gollub (1989) derive evolution equations equivalent to those of Meron & Procaccia (1986) from symmetry considerations, together with the conjecture that the coefficients on the right-hand side thereof are imaginary, which, they assert, “implies the system of equations to be Hamiltonian in the absence of damping”; however, they evidently overlook the fact that Hamiltonian symmetry also implies reciprocity of the coefficients of  $r_1 |r_2|^2$  and  $r_2 |r_1|^2$  and  $r_1^* r_2^2$  and  $r_2^* r_1^2$  in (5.9a,b).

Feng & Sethna’s (1989) evolution equations for gravity waves in an almost square tank are isomorphic to (5.9), with the additional symmetry  $A_{22} = A_{11}$  by virtue of their restriction to a pair of primary modes that transform into one another under a  $90^\circ$  rotation.

Umeki & Kambe (1989) have calculated (the equivalents of)  $A_{mn}$  and  $B$  in the present formulation for a circular cylinder and obtained numerical results for Ciliberto & Gollub’s (1985) configuration (see below).

<sup>3</sup> Dr. Meron (private communication) does not agree and says “I don’t see, however, any intrinsic symmetry which enforces the equality [of the coefficients of  $r_1 |r_2|^2$  and  $r_2 |r_1|^2$  and of  $r_1^* r_2^2$  and  $r_2^* r_1^2$  in (5.9a,b)], as exists, for example, in the case of a square cell: the coefficients are in general not identical.”

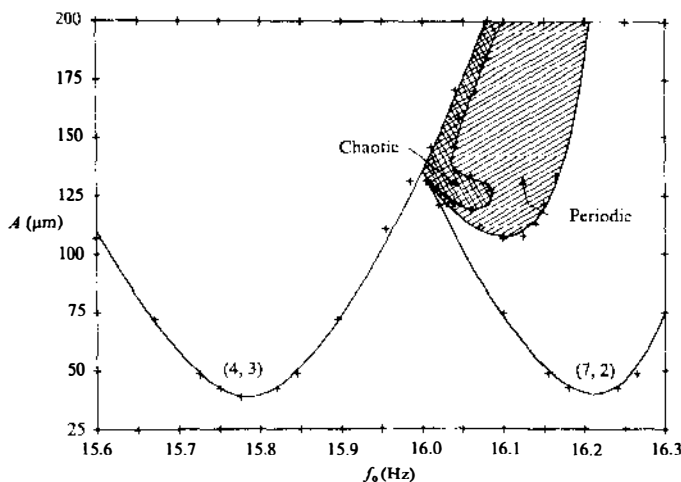
## 5.1 Stability and Bifurcation

The fixed points ( $\dot{r}_1 = \dot{r}_2 = 0$ ) of (5.9), at which (5.1) describes a monochromatic oscillation, may be classified as follows: (a)  $r_1 = r_2 = 0$ , corresponding to a plane free surface; (b<sub>1</sub>)  $|r_1| > 0$  and  $r_2 = 0$  or (b<sub>2</sub>)  $r_1 = 0$  and  $|r_2| > 0$ , corresponding to excitation of a single mode; and (c)  $|r_1| > 0$  and  $|r_2| > 0$ , corresponding to the joint excitation of two modes without mutual energy exchange. The stability boundaries between (a) and either (b<sub>1</sub>) or (b<sub>2</sub>) [ $\beta_n^2 = 1 - \alpha_n^2$  ( $n = 1, 2$ )] in an  $\varepsilon$  (or  $a_0$ )- $\omega$  plane correspond to pitchfork bifurcations at which the plane free surface loses stability to a monochromatic (at frequency  $\omega$ ) oscillation in a single mode, as in Section 3. The intersection of these two boundaries at (by definition) point is a co-dimension-two bifurcation, in the neighborhood of which either (b<sub>1</sub>) or (b<sub>2</sub>) could lose stability to (c) or to a limit cycle in which there is a slow (with time scale  $1/\varepsilon\omega$ ), periodic exchange of energy between the two modes. This limit cycle may lose stability to chaotic motion through a cascade of period-doubling bifurcations. A more surprising possibility (see below) appears to be a direct transition from either (b<sub>1</sub>) or (b<sub>2</sub>) to chaotic motion without (within the limits of experimental observation) intervening limit cycle(s) or period doublings.

The algebraic determination of the fixed points is straightforward but tedious. Feng & Sethna (1989) give the details for the special case  $A_{11} = A_{22}$  and  $\alpha_1 = \alpha_2 \ll 1$ . They give graphical results with  $\beta_1 + \beta_2$  and  $\beta_1 - \beta_2$ , which measure frequency and nonsquareness, respectively, as control parameters.

## 5.2 Comparison With Experiment

Ciliberto & Gollub (1985) report experiments with the (4,3) and (7,2) modes [the ( $l, m$ ) mode has  $l$  nodal diameters and  $m - 1 + \delta_0$  nodal circles] in a plexiglass circular cylinder of  $6.35 \pm 0.01$  cm radius filled to a depth of 1 cm with “distilled and de-ionized water that has been passed through both a carbon absorber to remove organic contaminants and a filter for particulates.” They do not specify the surface tension, but their data suggest that  $kl_* = 0.57$ , so that capillarity would have been nonnegligible. Their measured stability boundaries in an  $A = a_0$ ,  $f_0 = \omega/\pi$  plane are shown in Figure 2. It is notable that a direct transition between periodic oscillation of a single (4,3) mode and chaotic motion may occur above and to the right of the common intersection of the four stability boundaries in Figure 2. This is in contrast to the more typical scenario, in which the transition to chaos is through either a period-doubling cascade or a sequence of two or more additional oscillations of incommensurate frequency. It is, of course, possible that Ciliberto & Gollub’s sampling of



*Figure 2* Measured stability boundaries for the behavior of waves on a 1-cm layer of water in a circular cylinder of radius 6.35 cm as a function of driving amplitude  $A$  and frequency  $f_0$ . Stable surface patterns of the (4,3) and (7,2) modes occur in the regions marked (4,3) and (7,2). Slow-periodic and chaotic oscillations involving competition between these modes occur in the shaded regions (from Ciliberto & Gollub 1985; courtesy of Cambridge University Press).

their parameter space was not sufficiently fine to resolve these intermediate stages.

The calculation of the coefficients  $A_{mn}$  and  $B$  in (5.8) and (5.9) for Ciliberto & Gollub's (1985) modal pair has been carried out by Umeki & Kambe (1989). [The degenerate case of two modes differing only by a  $90^\circ$  rotation is trivial (in the present context) for parametric forcing in a circular cylinder.] Their stability boundaries have a richer bifurcation structure than, but are qualitatively consistent with, those of Ciliberto & Gollub. Meron & Procaccia (1986) assume numerical values of the relevant coefficients to obtain a phenomenological model. Their assumed values satisfy the necessary reciprocity conditions (see above), but they are rather special in assuming (the equivalent of)  $B = 0$  in (5.9) and differ significantly from those of Umeki & Kambe; however, their numerical predictions of the nonlinear stability boundaries near the critical point (the intersection of the linear stability boundaries) for the two modes are in qualitative agreement with those of Figure 2.

Feng & Sethna (1989) and Simonelli & Gollub (1989) have measured the stability boundaries for square and almost square cylinders. Feng & Sethna's experiments confirm some, and do not qualitatively contradict any, of their theoretical predictions. Simonelli & Gollub's experiments in

square containers are in qualitative accord with Feng & Sethna's theory for  $|\beta_1 - \beta_2| \ll 1$ . However, their experiments reveal a much richer structure than those of either Ciliberto & Gollub (1985) or Feng & Sethna. In particular, they find that the transition between the various states may be hysteretic and associate this with a subcritical bifurcation. For an almost square container, their experiments appear to be compatible with Feng & Sethna's theory, although direct comparison is difficult owing to different choices of control parameters. In particular, they find that the parameter space includes a region of time-dependent behavior but no region of a stable superposition of the two modes. Feng & Sethna do find the stable coexistence of the pure modes. In addition, they report slow rotational (about a vertical axis) motions not observed by Simonelli & Gollub. Experiments on standing waves in square cylinders also have been reported by Douady & Fauve (1988).

## 6. CROSS WAVES

Cross waves of the type excited by Faraday's (1831a) vibrating vertical plate dipped into a basin of water ("waves or crispations . . . beginning at the plate and projecting directly out from it . . . like the teeth of a very short coarse comb") are typically produced in the laboratory by a symmetric wavemaker in a rectangular channel when the driving frequency approximates twice one of the resonant frequencies of the transverse, standing-wave modes of the channel. In the simplest configuration, the basic wave motion (amplitude  $A_0$ ) induced by the wavemaker is independent of the transverse (across the channel) coordinate  $y$ , the cross waves (amplitude  $A_1$ ) are independent of the longitudinal (along the channel) coordinate  $x$ , and there are  $O(A_0 A_1^2)$  nonlinear interactions at both the free surface and the wavemaker. The nonlinear interaction at the free surface corresponds to that for the vertically oscillating container but is more complicated in that it cannot be represented simply by a variation of  $g$ ; the nonlinear interaction at the wavemaker has no counterpart in the vertically oscillating container.

The hypothesis that the cross wave is independent of the longitudinal coordinate  $x$  holds for a tank of length  $l$  if  $kl = O(1)$ , where  $k$  is the transverse wave number of the cross wave. The relative magnitudes of the basic and cross waves then are  $O(\varepsilon)$  and  $O(\varepsilon^{1/2})$ , respectively, where  $\varepsilon \equiv ka$  and  $a$  is the amplitude of the wavemaker motion. (All order estimates refer to the limit  $\varepsilon \downarrow 0$ .) If, on the other hand,  $kl = O(1/\varepsilon)$  and the reflection from the far end of the tank is neglected (presumably by virtue of an absorbing beach), both waves are  $O(\varepsilon)$ , and the cross wave exhibits a spatial variation with the length scale  $1/\varepsilon k$ .



## 6.1 Standing Cross Waves

Following Miles (1988),<sup>4</sup> we consider the excitation of gravity waves (capillarity is incorporated subsequently) of free-surface displacement  $\eta$  in a rectangular tank of width  $b$ , depth  $d$ , and length  $l$  in response to the wavemaker motion

$$x = af(z) \sin 2\omega t \quad (0 < y < b, -d < z < \eta) \quad (6.1)$$

on the assumptions that

$$ka \equiv \varepsilon \ll 1, \quad kd \gg 1, \quad kl = O(1), \quad (6.2)$$

where  $k = \pi/b$  for the dominant cross wave or  $n\pi/b$  ( $n = 2, 3, \dots$ ) for higher modes. We pose the solution for the velocity potential  $\phi$  and the displacement  $\eta$  in the form

$$(k\omega/g)\phi = \varepsilon\phi_0 + \varepsilon^{1/2}\phi_1 + \varepsilon\phi_{11} + O(\varepsilon^{3/2}) \quad (6.3a)$$

and

$$k\eta = \varepsilon\eta_0 + \varepsilon^{1/2}\eta_1 + \varepsilon\eta_{11} + O(\varepsilon^{3/2}), \quad (6.3b)$$

where the dimensionless variables  $(\phi_0, \eta_0)$  represent the linear, plane-wave ( $y$ -independent) solution of the problem,  $(\phi_1, \eta_1)$  represent the first-order cross-wave solution, and  $(\phi_{11}, \eta_{11})$  represent the second-order interaction of  $(\phi_1, \eta_1)$  with itself.

The plane-wave solution comprises a discrete propagating mode, which decays exponentially with depth, and a continuous spectrum of evanescent modes, which are oscillatory in  $z$  but decay exponentially away from the wavemaker. This solution may be determined from Havelock's (1929) solution of the wavemaker problem for a semi-infinite tank by neglecting the interaction of the evanescent modes with, and superimposing a reflected plane wave from, the end at  $x = l$  and is given by

$$\begin{aligned} \phi_0 = 4\kappa \cos 2\omega t \int_{-d}^0 f(\zeta) d\zeta & \left[ \frac{\cos 4\kappa(l-x)}{\sin 4\kappa l} e^{4\kappa(z+\zeta)} \right. \\ & \left. - \int_0^x e^{-\kappa x} \frac{(k \cos kz + 4\kappa \sin kz)(k \cos k\zeta + 4\kappa \sin k\zeta)}{\pi k(k^2 + 16\kappa^2)} dk \right], \quad (6.4) \end{aligned}$$

where  $\kappa \equiv \omega^2/g$ . The corresponding free-surface displacement is given by  $\omega\eta_0 = -\partial\phi_0/\partial t$  at  $z = 0$ .

<sup>4</sup> This problem was first treated by Garrett (1970), who linearized the boundary condition at the wavemaker and neglected the self-interaction of the cross wave in the free-surface conditions.

The second-order cross-wave solution (comprising the first-order solution and its self-interaction) may be inferred from Rayleigh's (1915) second-order solution for two-dimensional ( $y, z$  in the present problem) standing waves, which yields ( $A$  is the dimensionless counterpart of the amplitude  $A_1$ )

$$\phi_1 = \sqrt{2} A_\theta \cos ky e^{kz}, \quad \eta_1 = \sqrt{2} A \cos ky, \quad (6.5a,b)$$

$$\phi_{11} = -AA_\theta, \quad \eta_{11} = A^2 \cos 2ky, \quad (6.6a,b)$$

where

$$A = p(\tau) \cos \theta + q(\tau) \sin \theta, \quad \theta = \omega t, \quad \tau = \varepsilon \omega t. \quad (6.7a,b,c)$$

## 6.2 Hamiltonian Equations

Substituting (6.3) into Luke's (1967) Lagrangian

$$L = - \iiint [\phi_t + \frac{1}{2}(\nabla \phi)^2 + gz] dV \quad (6.8)$$

[see Miloh (1984) and Miles (1988) for the extension of Luke's variational principle to accommodate moving boundaries], averaging over the fast time  $\theta$ , and invoking Hamilton's principle, we find that  $p$  and  $q$  are canonical variables for the Hamiltonian

$$H = \beta_* pq + \frac{1}{2} \beta (p' + q^2) + \frac{1}{16} (p^2 + q^2)^2, \quad (6.9)$$

where

$$\beta = (\kappa - k)/2\kappa k = (\omega^2 - \omega_1^2)/(2g\omega^2), \quad \omega_1^2 = gk, \quad (6.10a,b)$$

and

$$\beta_* = l^{-1} \left[ \int_{-d}^0 f''(z) dz + (4\kappa l)^{-1} \left[ \int_{-d}^0 f''(z) e^{2\kappa z} dz - 2f(0) \right] \right]. \quad (6.11)$$

The two cases of principal interest are a plunger and a flap hinged about  $z = -d$ , for which  $f = 1$  and  $1 + (z/d)$ , respectively, and

$$2\kappa l \beta_* = 2\kappa d - 1 \quad \text{and} \quad (4\kappa d)^{-1} (2\kappa d - 1)^2, \quad (6.12a,b)$$

both of which vanish for  $\omega^2 = g/2d$  (in which case cross waves are not excited). The dominant effects of capillarity may be incorporated by multiplying  $gk$  by  $1 + k^2 l_*^2$  in (6.10b), where  $l_*$  is the capillary length [cf. (2.9)].

The canonical transformation (which is equivalent to a scale change and a  $\frac{1}{4}\pi$  phase shift of the complex amplitude  $p + iq$ )

$$\begin{bmatrix} p \\ q \end{bmatrix} = (2\beta_*)^{1/2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}, \quad \tau = \beta_*^{-1} T, \quad (6.13a,b)$$

yields

$$\partial P / \partial T = -\partial \hat{H} / \partial Q, \quad \partial Q / \partial T = \partial \hat{H} / \partial P, \quad (6.14a,b)$$

where

$$\hat{H} = \frac{1}{2}(P^2 - Q^2) + \frac{1}{2}(\beta/\beta_*)(P^2 + Q^2) + \frac{1}{4}(P^2 + Q^2)^2, \quad (6.15)$$

which is isomorphic to (3.2). It therefore follows from Section 3.1 that (a) the plane-wave solution ( $p = q = 0$ ) is stable if and only if  $|\beta| > (\beta_*^2 - \alpha^2)$ , where  $\alpha \equiv \delta/\varepsilon$  and  $\delta$  is the damping ratio for a free oscillation of the dominant transverse mode (i.e., the mode described by  $\phi_1$ ), and (b) the cross-wave solution is stable with a mean-square amplitude determined by

$$\langle A^2 \rangle = 2[(\beta_*^2 - \alpha^2)^{1/2} - \beta] \quad (6.16)$$

if  $\beta < (\beta_*^2 - \alpha^2)^{1/2}$ .

### 6.3 Spatially Decaying Cross Waves

We have already noticed that progressive plane waves in the neighborhood of one of the cutoff frequencies of a long channel may induce cross waves that exhibit a slow spatial modulation in the direction of propagation. The corresponding analytical problem has been considered by Mahony (1972), Jones (1984), Lichter & Chen (1987), Miles & Becker (1988), and Lichter & Bernoff (1988) and leads to a nonlinear Schrödinger equation for the slowly varying (in both space and time) envelope of the cross wave. Experiments have been carried out by Barnard & Pritchard (1972), Lichter & Shemer (1986), and Shemer & Lichter (1987). The numerical solutions of the nonlinear Schrödinger equation (Lichter & Chen 1987, Miles & Becker 1988) are in qualitative agreement with experiment.

The somewhat simpler problem of spatially modulated cross waves excited through vertical oscillation of a channel has been studied by Larraza & Putterman (1984) and Miles (1984b), who obtain results in quantitative agreement with the experiments of Wu et al. (1984).

### 6.4 Radial Cross Waves

The radial cross waves produced by Faraday have been studied by Schuler (1933) and Tatsuno et al. (1969). The latter measured the transition (from outwardly propagating axisymmetric waves to radially decaying cross waves) amplitude  $a_c$  of the vertical oscillation of spheres of four different diameters ( $D = 3, 4, 5, 6$  cm) versus the azimuthal wave number  $N$ . They found that the data collapse onto a single curve (within the experimental

error), along which  $a_c/D$  falls from 0.3+ to 0.005 as  $N$  increases from 2 to 32. Tatsuno et al. also found that the axisymmetric waves have the frequency, whereas the cross waves have half the frequency, of the spherical wavemaker.

No theoretical description is currently available, and the problem is significantly more complicated than that for the channel, in part because the radial cross waves are two dimensional (radial and azimuthal), in contrast to the one-dimensional cross waves in the channel, and in part because they are subject to radiation damping (J. M. Becker, private communication).

## 7. EDGE WAVES

The classical edge wave is that of Stokes (1846) on a uniformly sloping beach ( $z = -y \tan \beta$ ), for which the free-surface displacement and dispersion relation are given by

$$\eta(x, y, t) = a \exp(-ky \cos \beta) \cos(kx - \omega t), \quad \omega^2 = gk \sin \beta, \quad (7.1a, b)$$

where  $x$  and  $y$  are longshore and offshore coordinates,  $k|a| \ll 1$ , and  $\beta$  is the beach angle. This Stokes edge wave is the dominant member of a discrete set for which  $\omega^2 = gk \sin(2n+1)\beta$  and  $n = 0, 1, \dots$  up to the largest integer for which  $(2n+1)\beta < \frac{1}{2}\pi$  (Ursell 1952).

Guza & Davis (1974) show that a normally incident wave of frequency  $2\omega$  transfers energy, through a resonant interaction, to a standing edge wave of the form (7.1). Further analysis and experiment by Guza & Bowen (1975) reveal that the obliquity of the incoming wave induces an asymmetric, slow modulation, in both  $y$  and  $t$ , of the oppositely directed, traveling edge waves into which the standing edge wave may be resolved. Allowance for finite amplitude (Guza & Bowen 1976a) yields a steady-state equilibration, with radiation damping, of these slowly varying edge waves. Guza & Davis (1974) include boundary-layer damping of the edge waves, which determines a minimum amplitude of the incoming wave for energy transfer to the edge waves, in their calculation, but they omit the dispersive effect of the boundary layer. Yeh (1986) finds that including the latter effect improves the agreement between theory and his experiments (which were carried out in a much larger tank than those of Guza and his collaborators).

The results cited in the preceding paragraph suggest that discrete edge waves, which are analogous to the Faraday waves discussed in Sections 2–5 and, especially, the cross waves discussed in Section 6.2, may provide a realistic model for such problems as the development of cusps on beaches of limited lateral extent, for which reason they remain of considerable oceanographic interest. [See Guza (1985) for a brief review (written in

1981) of edge waves in the context of oceanography, with special reference to laboratory modeling.] However, Akylas (1983) finds that the standing edge wave induced by a normally incident wave is unstable with respect to longshore modulation of scale  $O(1/k\varepsilon)$ , where  $\varepsilon$  is the slope of the incident wave, and that the resulting envelope develops into a sequence of solitons. And Foda & Mei (1988) find that longshore modulation of  $O(1/k\varepsilon^3)$  of the incident wave also may transfer energy to standing edge waves. Similar results presumably hold for oblique incidence. It appears, then, that a full treatment of the interaction between incident swell and edge waves on a sloping beach is even more complicated than that of cross waves in a long tank (Section 6.3) and beyond the scope of the present review.

All of the results cited in the preceding two paragraphs are for a slope that is uniform and small, but it can be shown (Miles 1989a,b) that the model is valid for any beach for which  $\sigma \ll 1$  and  $kl \gg 1$ , where  $\sigma = \tan \beta$  is the slope at the shoreline,  $k$  is the wave number of the edge wave [as in (7.1)], and  $l$  is the characteristic length for the variation of the slope.

A more serious limitation on much of the existing work on edge waves may be the neglect of viscous damping of the incoming and reflected waves. This neglect implies perfect reflection, whereas typical laboratory experiments [see Mahony & Pritchard (1980) for discussion] yield reflection coefficients that are smaller than 1/3 for wave periods smaller than 1 s. On the other hand, Guza & Bowen (1976b) report almost perfect reflection for periods of around 3 s, which suggests that dissipation may be relatively unimportant for sufficiently long waves.

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