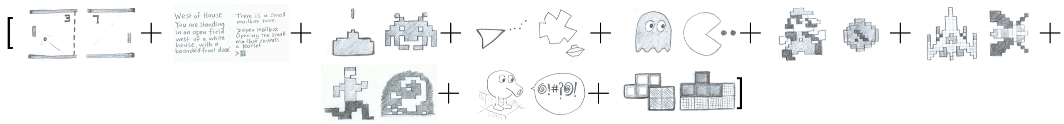

122 Matrixology (Linear Algebra)—Practice exam #4
University of Vermont, Fall Semester,

Name: ______________________________

➤ Total points: 36 (3 points per question); Time allowed: 165 minutes. Smile.
➤ Current brains only: No pensieves, calculators, or similar gadgets allowed.
➤ For full points, *please show all working clearly*.

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1. Draw the 'big picture' of how $\mathbf{A}\vec{x} = \vec{b}$ works when $\mathbf{A}$ is an $m \times n$ matrix. Indicate on your diagram the following:

   (a) Which space is R^m and which is R^n.

   (b) Row space, column space, nullspace, and left nullspace.

   (c) The dimensions of the above subspaces in terms of r, m, and n.

   (d) How $\mathbf{A}$ maps vectors.

   (e) Where the vectors $\vec{x} = \vec{x}_r + \vec{x}_n$ and $\vec{b} = \vec{p} + \vec{e}$ live.

   (f) The appropriate orthogonality of subspaces.

2. For the four general cases of $\mathbf{A}\vec{x} = \vec{b}$ below:

   (a) give an example reduced row echelon form matrix $\mathbf{R_A}$;

   (b) sketch the appropriate cartoon abstract 'big pictures';

   (c) indicate the number of possible solutions (0, 1, or ∞);

   (d) and note whether or not nullspace and left nullspace are equal to $\{\vec{0}\}$ (Y/N).

case	example $\mathbf{R_A}$	big picture	# solutions	$N(\mathbf{A}) = \{\vec{0}\}$?	$N(\mathbf{A}^{\mathrm{T}}) = \{\vec{0}\}$?
$m = r$ $n = r$					
$m = r$, $n > r$					
$m > r$, $n = r$					
$m > r$, $n > r$					

3. Given a matrix $\mathbf{A}$ and its transpose $\mathbf{A}^{\mathrm{T}}$ have the following reduced row echelon forms, respectively,

$$\mathbf{R}_{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{R}_{\mathbf{A}^{\mathrm{T}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

answer the following questions.

(a) $m =$ _____, $n =$ _____, $r =$ _____,

dim $C(\mathbf{A}^{\mathrm{T}}) =$ _____, dim $C(\mathbf{A}) =$ _____,

dim $N(\mathbf{A}) =$ _____, dim $N(\mathbf{A}^{\mathrm{T}}) =$ _____.

(b) Find bases for $\mathbf{A}$'s row space and column space.

(c) Find a basis for $\mathbf{A}$'s nullspace

4. $\mathbf{LU}$ decomposition:

Find $\mathbf{U}$ for the following matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 4 & 1 & 4 \\ -4 & 11 & 0 \end{bmatrix}.$$

Write down each row operation, the multipliers l_{21}, l_{31}, and l_{32}, and the corresponding elimination matrices $\mathbf{E}_{21}$, $\mathbf{E}_{31}$, and $\mathbf{E}_{32}$.

5. This question carries on with the the preceding question's $\mathbf{A}$.

(a) What are the pivots of $\mathbf{A}$?

(b) Write down a general formula for $|\mathbf{A}|$ in terms of its pivots (remembering that in general, row swaps may be needed to reduce $\mathbf{A}$ to $\mathbf{U}$), and compute the determinant of the $\mathbf{A}$ we have here.

(c) Write down the inverses of the elimination matrices and compute $\mathbf{L} = \mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{32}^{-1}$.

6. Least squares approximation:

   (a) Given

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix},$$

   solve the normal equation $\mathbf{A}^{\mathrm{T}}\mathbf{A}\vec{x}^* = \mathbf{A}^{\mathrm{T}}\vec{b}$.

   (b) Find $\vec{p}$ and $\vec{e}$, the components of $\vec{b}$ that live in column space and left nullspace respectively.

7. The Gram-Schmidt process:

   Consider the subspace $\mathbf{S}$ of R^3 that is spanned by the following two linearly independent vectors:

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

   Find an orthonormal basis vectors ($\hat{q}_1$ and $\hat{q}_2$) for $\mathbf{S}$ using the (exciting) Gram-Schmidt process.

(b) Consequently, for the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$ find the factorization $\mathbf{A} = \mathbf{QR}$ (i.e., find $\mathbf{Q}$ and $\mathbf{R}$).

8. (a) Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$$

(b) Write down $\mathbf{A}$'s diagonalized counterpart $\mathbf{\Lambda}$ and the transformation matrices $\mathbf{S}$ and $\mathbf{S}^{-1}$.

(c) Hence determine $\mathbf{A}^n$ where n is arbitrary.

9. Computing determinants: Given

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix},$$

(a) Write down the minor matrices $\mathbf{M}_{12}$, $\mathbf{M}_{22}$, and $\mathbf{M}_{32}$, compute the cofactors C_{12}, C_{22}, and C_{32}, and hence find $\det(\mathbf{A})$.

(b) Also find $|\mathbf{A}|$ by reducing $\mathbf{A}$ to an upper triangular matrix with 1's on the leading diagonal.

10. Positive Definite Matrices

Let $f(x, x_2, x_3) = 2x^2 + x_2^2 + 6x_3^2 + 2x_1x_2 - 4x_1x_3 + 4x_2x_3$.

(a) Rewrite $f(x_1, x_2, x_3)$ as $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $\mathbf{A}$ is a symmetric matrix.

(b) Determine the signs of eigenvalues by finding the pivots.

(c) Write down the definition of positive definiteness. Is this matrix positive definite?

11. Singular Value Decomposition

(a) Consider the matrix:

$$\mathbf{A} = \frac{1}{5}\begin{bmatrix} 9 & 12 \\ 8 & -6 \end{bmatrix}.$$

Determine the singular value decomposition of $\mathbf{A}$, i.e., find the three matrices $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}^{\mathrm{T}}$ such that $\mathbf{A} = \mathbf{U\Sigma V}^{\mathrm{T}}$.

(Reminder: $\mathbf{A}\hat{v}_i = \sigma_i \hat{u}_i$ and $\mathbf{A}^{\top}\mathbf{A}\hat{v}_i = \sigma_i^2 \hat{v}_i$.)

(b) The Big Picture: Illustrate how $\mathbf{A}$ maps between the happy basis vectors that are the $\hat{v}_i$'s and $\hat{u}_i$'s. (Please draw the particular Big Picture not the abstract Big picture.)

Complete your picture by adding a unit circle in row space and the ellipse that $\mathbf{A}$ creates in column space by transforming this circle.

12. (a) True or False (2 pts):

i. The nullspace of a nontrivial 1×3 matrix $\mathbf{A}$ is a 2-D plane in $\mathbb{R}^3$: ____.

ii. The product $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is symmetric for any $n \times n$ matrix $\mathbf{A}$: ____.

iii. An $n \times n$ matrix cannot be diagonalized if one or more eigenvalues of $\mathbf{A}$ are 0: ____.

iv. The matrix $\mathbf{M} = [\vec{v}_1|\vec{v}_2]$ transforms a vector's representation from the basis $\{\vec{v}_1, \vec{v}_2\}$ to the natural basis: ____.

v. The determinant of a matrix $\mathbf{A}$ is equal to the sum of $\mathbf{A}$'s eigenvalues: ____.

vi. The matrices $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ have different eigenvalues: ____.

(b) Find the determinant of the following matrix (1 pt):

$$\mathbf{A}_n = \begin{bmatrix} \cos(1) & \cos(2) & \cdots & \cos(n) \\ \cos(n+1) & \cos(n+2) & \cdots & \cos(2n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(n(n-1)+1) & \cos(n(n-1)+2) & \cdots & \cos(n^2) \end{bmatrix}$$

The Triumphant Bonus Page: