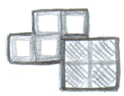
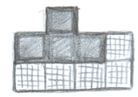


MATH 122: Matrixology (Linear Algebra)
Solutions to Level Tetris (1984), 10 of 10
University of Vermont, Fall 2016



1. (Q 4, 6.5) Show that the function $f(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients.

   Do this by rewriting $f(x_1, x_2)$ as $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and finding the pivots of $\mathbf{A}$ and noting their signs (and explaining why the signs of the pivots matter).

   Write f as a difference of squares and find a point (x_1, x_2) where f is negative.

   Note of caution: All of this signs matching for pivots and eigenvalues falls apart if we have to do row swaps in our reduction.

   Solution:

   First, we can rewrite our function as

   $$\begin{bmatrix} x_1 & x_2 \end{bmatrix} f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   We need to do one step of row reduction to reveal the pivots:

   $$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \underset{\text{R2'= R2 -2 R1}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

   The pivots are 1 and -1 so we must have one positive and one negative eigenvalue: f is therefore not positive definite.

   Completing the square:

   $$f(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2 = (x_1 + 2x_2)^2 - 4x_2^2 + 3x_2^2 = 1 \cdot (x_1 + 2x_2)^2 - 1 \cdot (x_2)^2.$$

   Note the appearance of the pivots 1 and -1 in front of the squares. As we saw in class, the $\mathbf{LU}$ factorization of symmetric matrices, $\mathbf{A} = LDL^{\mathrm{T}}$, is behind all of this.

   □

2. (Q 9, 6.5) Find the 3 by 3 matrix $\mathbf{A}$ and its pivots, rank, eigenvalues, and determinant:

   $$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

Is this matrix positive definite, semi-positive definite, or neither?

Solution:

Expanding $4(x_1 - x_2 + 2x_3)^2$ we have

$$4x_1^2 + 4x_2^2 + 16x_3^2 - 8x_1x_2 - 16x_2x_3 + 16x_3x_1$$

and this can be written as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

We can now find the pivots of $\mathbf{A}$ (much easier than finding the eigenvalues):

$$\begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix} \underset{\substack{\text{R2'} \\ = \text{R2 -} \\ \text{-1 R1}}}{\rightsquigarrow} \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 8 & -8 & 16 \end{bmatrix} \underset{\substack{\text{R3'} \\ = \text{R3 -} \\ \text{2 R1}}}{\rightsquigarrow} \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The pivots are 4, 0, 0 and our matrix is therefore semi-positive definite.

Some bonus sneaky grooviness: we can see straight away that $\mathbf{A}$ is a rank one matrix:

$$\mathbf{A} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \end{bmatrix} = 24 \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}.$$

We now have $\mathbf{A}$ in its spectral decomposition form:

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_n \hat{v}_i \hat{v}_i^{\mathrm{T}}.$$

So the eigenvalues are 24, 0, and 0, which means that $\mathbf{A}$ is semi-positive definite.

Another way to see this: we know from the pivots that two of the eigenvalues are 0. Since the trace of $\mathbf{A}$ is the sum of the eigenvalues, we have that the trace of $\mathbf{A}$ must be $\lambda_1 + 0 + 0 = \lambda_1$. Checking $\mathbf{A}$, we have $\lambda_1 = 24$.

The determinant of $\mathbf{A}$ is zilch since we have 0 eigenvalues.

□

3. (following set of questions based on Q 7, Section 6.7)

   Singular Value Decomposition = Happiness.

   Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) What are m, n, and r for this matrix?

(b) What are the dimensions of $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$?

(c) Calculate $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$.

Solution:

(a) $m = 2$, $n = 3$, and $r = 2$.

(b) $\mathbf{U}$ is 2x2, $\mathbf{\Sigma}$ is 2x3, and $\mathbf{V}$ is 3x3.

(c)

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

□

4. For the matrix $\mathbf{A}$ given above, find the eigenvalues and eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$, and thereby construct $\mathbf{V}$ and $\mathbf{\Sigma}$.

See this tweet for some post-it based help:
https://twitter.com/matrixologyvox/status/593540446845947904

Solution:

Okay, we have to solve $|\mathbf{A} - \lambda I| = 0$. Using the 'big formula' and going across the top row (to take advantage of the 0 in the (1,3) entry), we have:

$$0 = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)(1-\lambda) - (1)(1)] - (1)(1-\lambda) - (0)(1)$$
$$= -\lambda^3 4\lambda^2 - 3\lambda$$
$$= -\lambda(\lambda - 3)(\lambda - 1).$$

Our eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_1 = 0$. Ordering for largest to smallest is important here.

We notice a couple of things: (1) The eigenvalues are all ≥ 0. This is good as these are the squares of our singular values, the σ_i. (2) One eigenvalue is 0. This makes sense as the rank $r = 2$ which means that we have two non-zero singular values.

Our singular values are the square roots of the eigenvalues:

$$\sigma_1 = \sqrt{3} \text{ and } \sigma_2 = 1.$$

Note that there are only two singular values as $\mathbf{A}$ is 2x3.

Next task: find the eigenvectors.

(a) For $\lambda_1 = 3$, we solve $(\mathbf{A}^{\mathrm{T}}\mathbf{A} - 3I)\vec{v}_1 = \vec{0}$.

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \vec{v}_1 = \vec{0}.$$

You can do this be inspection, or by systematically finding the nullspace vector, or however you please. By inspection: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Normalizing, we have $\hat{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(b) For $\lambda_2 = 1$, we solve $(\mathbf{A}^{\mathrm{T}}\mathbf{A} - I)\vec{v}_2 = \vec{0}$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{v}_2 = \vec{0}.$$

By inspection: $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and the normalized eigenvector is $\hat{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(c) For $\lambda_3 = 0$, solve $(\mathbf{A}^{\mathrm{T}}\mathbf{A} - 0I)\vec{v}_3 = \vec{0}$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{v}_3 = \vec{0}.$$

By inspection: $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and the normalized eigenvector is $\hat{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

We can now write down $\mathbf{V} = [\hat{v}_1|\hat{v}_2|\hat{v}_3]$:

$$\mathbf{V} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

And the $\mathbf{\Sigma}$ matrix is

$$\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

□

5. For the same $\mathbf{A}$, now find the basis $\{\hat{u}_i\}$ using the essential connection $\mathbf{A}\hat{v}_i = \sigma_i \hat{u}_i$.

   Construct $\mathbf{U}$ from the basis you find.

   Again see this tweet for some post-it based help: https://twitter.com/matrixologyvox/status/593540446845947904

   Solution:

   We multiply the $\hat{v}_i$ for which $\sigma_i > 0$ by $\mathbf{A}$ to find the $\hat{u}_i$. We'll need to pull the σ_i out to find the $\hat{u}_i$. Recall that $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$. First off:

$$\mathbf{A}\hat{v}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$= \sqrt{3} \frac{1}{\sqrt{3}} \frac{1}{\sqrt{6}} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$= \sqrt{3} \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$= \sqrt{3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \sigma_1 \hat{u}_1.$$

Notice how when we pull out σ_1, we (almost magically) end up with a happy little unit vector.

Second vector:

$$\begin{aligned}
\mathbf{A}\hat{v}_2 &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= 1 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \sigma_2 \hat{u}_2.
\end{aligned}$$

Smashing. Note that $\hat{u}_1^{\mathrm{T}} \hat{u}_2 = 0$ and we have an orthonormal basis for R^2.

Finally,

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

□

6. Next find the $\{\hat{u}_i\}$ in a different way by finding the eigenvalues and eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$.

Solution:

Eigenvalues:

$$\begin{aligned}
0 = |\mathbf{A}\mathbf{A}^{\mathrm{T}} - \lambda I| &= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \\
&= (2-\lambda)^2 - 1 \\
&= (2-\lambda-1)(2-\lambda+1) \\
&= (1-\lambda)(3-\lambda),
\end{aligned}$$

where we have used the difference of perfect squares.

So $\lambda_1 = 3$ and $\lambda_2 = 1$ which again gives $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

Eigenvector time (la-la-la-la) for $\lambda_1 = 3$:

$$\begin{aligned}
\vec{0} &= (\mathbf{A}\mathbf{A}^{\mathrm{T}} - \lambda_1 I)\vec{u}_1 \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3I \\
&= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.
\end{aligned}$$

By inspection, we have $\hat{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Next,

$$\begin{aligned}\vec{0} &= (\mathbf{A}\mathbf{A}^{\mathrm{T}} - \lambda_2 I)\vec{u}_2 \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1I \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.\end{aligned}$$

This gives $\hat{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note that we could have chosen $\hat{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, the negative of the one we have above.

In fact, we always need to compute $\mathbf{A}\hat{v}_i$ to find out which direction $\hat{u}_i$ should take. Beyond this, we don't need to compute the $\hat{u}_i$ directly ever as once we have $\vec{v}_i$ we need only multiply by $\mathbf{A}$ (as per the previous question). We found the u's directly here to (1) see that both ways give the same thing and (2) punish ourselves just a little more.

□

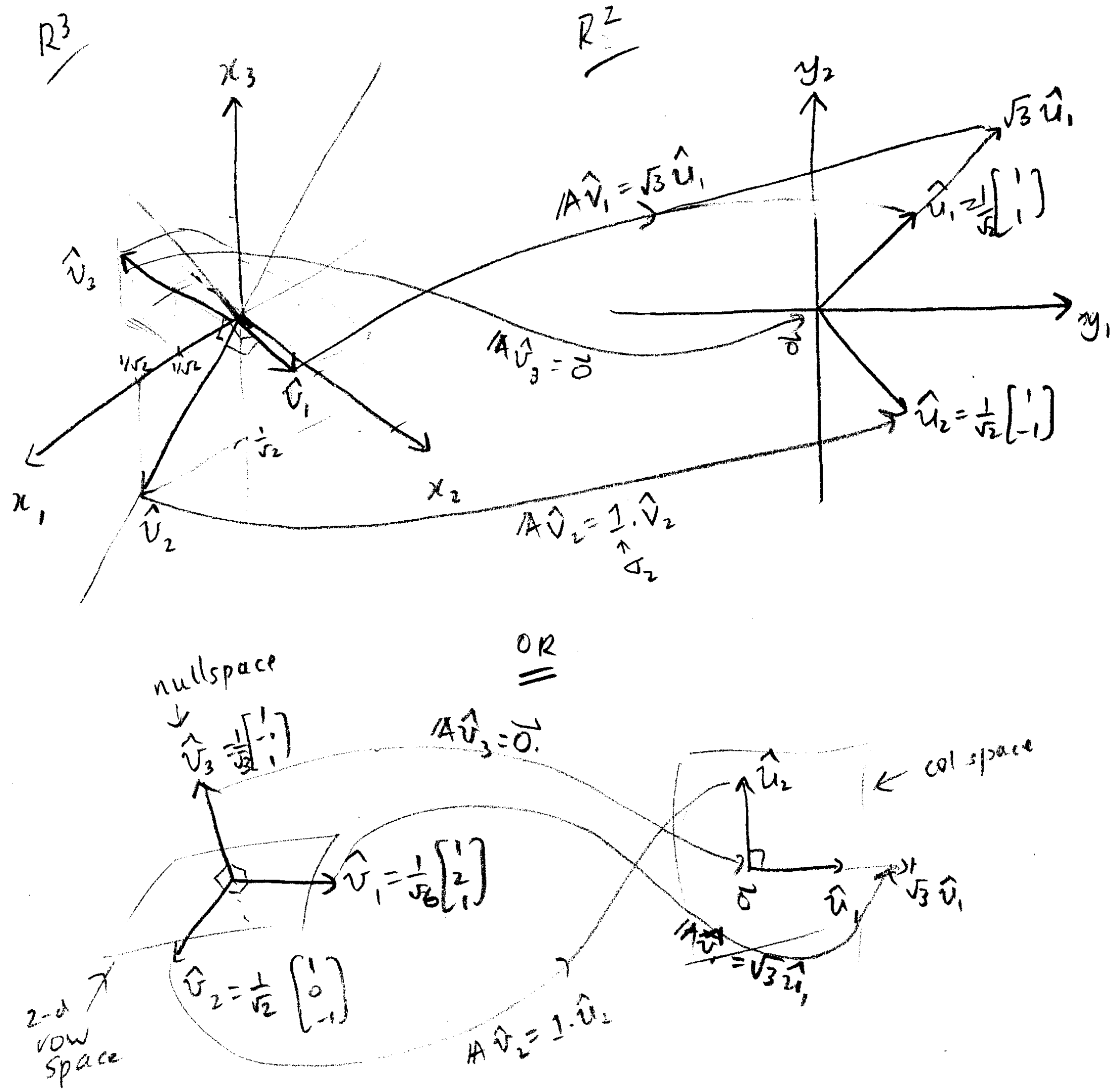
7. (a) Put everything together and show that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$.
   (b) Draw the 'big picture' for this $\mathbf{A}$ showing which $\hat{v}_i$'s are mapped to which $\hat{u}_i$'s.
   (c) Which basis vectors, if any, belong to the two nullspaces?

Solution:

(a)

$$\begin{aligned}\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= \begin{bmatrix} 1/2 + 1/2 & 1 + 0 & 1/2 - 1/2 \\ 1/2 - 1/2 & 1 + 0 & 1/2 + 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.\end{aligned}$$

(b)



(c) Left nullspace is just $\{\vec{0}\}$.

$\mathbf{A}$'s nullspace has dimension 1 and has the basis vector $\hat{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

□

8. Finally, for this same $\mathbf{A}$, perform the following calculation:

$$\sigma_1 \hat{u}_1 \hat{v}_1^{\mathrm{T}} + \sigma_2 \hat{u}_2 \hat{v}_2^{\mathrm{T}} + \ldots + \sigma_r \hat{u}_r \hat{v}_r^{\mathrm{T}}$$

where r is the rank of $\mathbf{A}$.

You should obtain $\mathbf{A}$...

Solution:

$$\sigma_1 \hat{u}_1 \hat{v}_1^{\mathrm{T}} + \sigma_2 \hat{u}_2 \hat{v}_2^{\mathrm{T}} = \sqrt{3}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 1\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \mathbf{A}.$$

□

9. Matlab question.

   Verify the signs you found for the pivots of $\mathbf{A}$ in question 1 by using Matlab to find $\mathbf{A}$'s eigenvalues.

   Solution:

   Using Matlab, we find $\lambda_1 = -0.2361$ and $\lambda_1 = 4.2361$:

```
>> eig([ 1 2; 2 3])
ans =
   -0.2361
    4.2361
```

   One positive and one negative, matching the signs of the pivots.

□

10. Matlab question.

    Use Matlab to compute the SVD for the matrix $\mathbf{A}$ you explored in questions 3–8.

    Solution:

```
>> [U,Sigma,V] = svd([ 1 1 0 ; 0 1 1])
U =
-0.7071   -0.7071
-0.7071    0.7071
Sigma =
```

```
1.7321         0         0
0         1.0000         0
V =
-0.4082   -0.7071    0.5774
-0.8165    0.0000   -0.5774
-0.4082    0.7071    0.5774
```

□

11. (The bonus one pointer)

    Where does the fearsome kiwi rank among among rattites and what's unusual about the kiwi egg?

    Solution:

    The kiwi is the smallest of all struthious birds.

    A kiwi egg can weight up to 1/4 of the mother's own weight, which is believed to be the highest ratio of all birds.

□