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SCALING, UNIVERSALITY, AND GEOMORPHOLOGY

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■ **Abstract** Theories of scaling apply wherever similarity exists across many scales. This similarity may be found in geometry and in dynamical processes. Universality arises when the qualitative character of a system is sufficient to quantitatively predict its essential features, such as the exponents that characterize scaling laws. Within geomorphology, two areas where the concepts of scaling and universality have found application are the geometry of river networks and the statistical structure of topography. We begin this review with a pedagogical presentation of scaling and universality. We then describe recent progress made in applying these ideas to networks and topography. This overview leads to a synthesis that attempts a classification of surface and network properties based on generic mechanisms and geometric constraints. We also briefly review how scaling and universality have been applied to related problems in sedimentology—specifically, the origin of stromatolites and the relation of the statistical properties of submarine-canyon topography to the size distribution of turbidite deposits. Throughout the review, our intention is to elucidate not only the problems that can be solved using these concepts, but also those that cannot.

INTRODUCTION

Geomorphology is the science devoted to the pattern and form of landscapes (Dietrich & Montgomery 1998, Scheidegger 1991). Studies range from physical theories for landscape evolution to inference of earth history from surface structures. In any such study, it is imperative to distinguish what makes landscapes alike from what makes them different. Here we review theoretical aspects of this challenge and their relevance to observational measurements.

Extensive studies over the last half-century have led geomorphologists to discover a wealth of empirical ‘‘laws’’ that seem to indicate that some properties of landscapes are invariant. These laws are especially prevalent in fluvial geomorphology—the branch of the subject that deals with how water, and rivers in par-

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ticular, shapes landscapes. Indeed, the literature of river networks is replete with observations that have been elevated to laws, attached to names such as Horton, Hack, and Tokunaga (Horton 1945, Hack 1957, Tokunaga 1966, Abrahams 1984, Rodríguez-Iturbe & Rinaldo 1997). Looking beyond river networks, one sees that topography naturally encodes more information. Here, too, there have been claims that statistical measures such as power spectra and correlation functions have a generic form that may also be invariant (e.g. see Turcotte 1997).

Many of these empirical measures are expressed as scaling laws. Scaling laws are typically of power law form and indicate an invariance under appropriate changes of scale. Some may also be related to fractal dimensions (Mandelbrot 1983); they have surely gained much of their popularity from their association with fractal geometry.

Our purpose here, however, is to stress not the geometric interpretation of these scaling laws, but rather their significance in terms of physical and geological processes. This is often a difficult task. For example, Hack's law states that the length l of the main stream of a river basin scales like a power h of the basin's area a (i.e. $l \propto a^h$). From the pioneering studies of Hack (1957) to the present day (Gray 1961, Rigon et al 1996, Maritan et al 1996b), one often finds that $h \simeq 0.6$. Various questions arise: Why is this a power law relation? Why is $h \simeq 0.6$? How much variation is there? Could there be one universal exponent that all networks strive for, perhaps an elegant fraction such as $h = 3/5$ or $h = 7/12$? Is it possible that $l \propto a^{0.6}$ will end up in a metaphorical zoo, left to be named, classified, and admired, but not understood?

The notion of universality allows one to go beyond zoology. The idea, in a nutshell, is that many complicated phenomena, sometimes from vastly different fields, exhibit the same scaling laws. When one pares away the details, common generic processes can often be identified. One has then determined a universality class. The utility of the idea lies in what is learned from the identification of the generic mechanisms.

Universality is of unquestionable power and beauty in fundamental fields where one seeks the "essential" mechanisms common to diverse phenomena. However, in many areas of science one is interested not only in generic mechanisms but also in the details that make one system different from another. We argue that geomorphology, and indeed many of the earth sciences, are in the latter category. Universality's importance nevertheless remains undiminished: it allows us to simultaneously distinguish what different systems (e.g. two river basins) have in common in addition to what makes them different.

This review is intended to be pedagogical and reasonably self-contained rather than exhaustive and encyclopedic. We begin by discussing the concepts of scaling and universality in the context of their applications to networks and topography. We then concentrate on studies of networks and topography proper. Because observational data rarely allows access to time-dependent statistics, our focus is on stationary (steady-state) properties. Here, theories of random networks and surfaces receive much emphasis. In the case of networks, we attempt a synthesis of widely scattered theoretical results and propose how they may explain observations of scale dependence in Hack's law. We suggest that slow "crossovers" from one scale-dependent regime to another may account for observed variations in the Hack exponent.

Our discussion of topography is framed by the consideration of universality classes demarcated by stochastic partial differential equations for surface evolution. Here the theories are again quite general, perhaps so much so that their applicability to geomorphology could be rightly questioned. We attempt a balanced presentation that points out the possible limitations in addition to the obvious advantages of generic theories. The latter receive implicit emphasis through a brief review of how these surface evolution models have also been applied to related problems in sedimentology—namely, the origin of ancient sedimentary structures known as stromatolites and the relation of submarine topography to the size distribution of turbidite deposits.

We conclude the review with a brief discussion of open problems. We point out that a key missing link is the relationship between theories for networks and theories for surfaces. Even more crucial is the need for dynamical theories of networks and surfaces whose time-dependent aspects may be tested by accessible data.

In closing this introduction, we refer the reader to the recent review of river networks by Rinaldo et al (1998) and the extensive overview provided in the recent book by Rodríguez-Iturbe & Rinaldo (1997). Not wishing to cover the same ground in our discussion of networks, we endeavor to provide complementary and novel thoughts, and refer to these other treatments and the references therein at appropriate places in our review.

SCALING

Consider the following problem in fluvial geomorphology: If one doubles the length of a stream, how does the area drained by that stream change? Or, inversely, how does basin shape change when we compare basins of different drainage areas? The concept of scaling addresses such questions.

Basin Allometry

Figure 1 shows two river basins along with a sub-basin of each. A basin can be defined at any point on a landscape. Embedded within any basin is a multitude of sub-basins. In considering our preceding question, we must first define some dimensions. A reasonable way to do this is to enclose each basin by a rectangle with dimensions $L_\parallel$ and $L_\perp$ as illustrated in Figure 1*a*. $L_\parallel$ is the longitudinal extent of the basin, and $L_\perp$ is the basin's characteristic width. By this construction, the area a of a basin is related to these lengths by

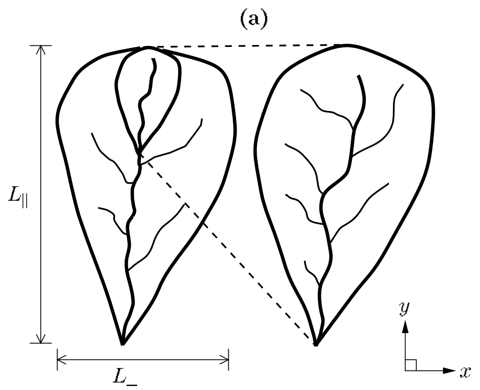

Figure 1 A pair of river basins, each with a sub-basin scaled up for comparison with the original. The basins in (*a*) are self-similar. The basins in (*b*) are not.

$$a \propto L_\parallel L_\perp. \tag{1}$$

Measurements taken from real river basins suggest that $L_\perp$ scales like a power H of $L_\parallel$ (Ijjasz-Vasquez et al 1994, Maritan et al 1996b). In symbols,

$$L_\perp \propto L_\parallel^H. \tag{2}$$

Substituting Equation 2 into Equation 1, we obtain

$$a \propto L_\parallel^{1+H}. \tag{3}$$

Equations 2 and 3 are scaling laws. Respectively, they describe how one length scales with respect to another, and how the total area scales with respect to one of the lengths.

Figure 1*a* corresponds to $H = 1$, known as geometric similarity or self-similarity. As the latter appellation implies, regardless of size, basins look the same. More prosaically, lengths scale like widths.

The case $H \neq 1$ is called allometric scaling. Originally introduced in biology by Huxley & Teissier (1936) "to denote growth of a part at a different rate from that of a body as a whole," its meaning for basin size is illustrated by Figure 1*b*. Here, $0 \leq H < 1$, which means that if we examine basins of increasing area, basin shape becomes more elongate. In other words, because $(1 - H)/(1 + H) > 0$ and

$$L_\perp/L_\parallel \propto L_\parallel^{-(1-H)} \propto a^{-(1-H)/(1+H)}, \tag{4}$$

the aspect ratio $L_\perp/L_\parallel$ decreases as basin size increases.

Random Walks

Our next example is a random walk (Feller 1968, Montroll & Shlesinger 1984). We describe it straightforwardly here, noting that random walks and their geomorphological applications will reappear throughout the review.

The basic random walk may be defined in terms of a person who has had too much to drink stumbling home along a sidewalk. The disoriented walker moves a unit distance along the sidewalk in a fixed time step. After each time step, our inebriated friend spontaneously and with an even chance turns about face or maintains the same course, and then wanders another unit distance only to repeat the same erratic decision process. The walker's position x_n after the nth step, relative to the front door of his or her local establishment, is given by

$$x_n = x_{n-1} + s_{n-1} = \sum_{k=0}^{n-1} s_k, \tag{5}$$

where each $s_k = \pm 1$ with equal probability and $x_0 = 0$.

There are many scaling laws associated with random walks. Probably the most important of these describes the root-mean-square distance that the average walker has traveled after n steps. Since x_n is the sum of independent increments, its variance $\langle x_n^2 \rangle$ is given by the sum of the individual variances,

$$\langle x_n^2 \rangle = \sum_{i=1}^{n} \langle s_i^2 - \langle s_i \rangle^2 \rangle, \tag{6}$$

where $\langle \cdot \rangle$ indicates an average over an ensemble of walkers. Since $\langle s_i \rangle = 0$ and $s_i^2 = 1$, we have $\langle x_n^2 \rangle = n$. Defining $r_n = \langle x_n^2 \rangle^{1/2}$, we obtain

$$r_n = n^{1/2}. \tag{7}$$

Generalizing the scaling (Equation 7) to continuous time t and space x, we have $r(t) \propto t^{1/2}$. Now note that

$$r(t) = b^{-1/2} r(bt). \tag{8}$$

In other words, if one rescales time and space such that $t \to bt$ and $x \to b^{1/2}x$, the statistics of the random walk are unchanged. Figure 2 illustrates the meaning of these rescalings. The two random walks shown—the lower being a portion of the upper rescaled—are said to be statistically equivalent.

More generally, functions $f(x)$ that satisfy equations of the form $f(x) = b^{-\alpha} f(bx)$ are called self-affine (Mandelbrot 1983). This relation need not be exact and indeed usually only holds in a statistical sense. An example already given is the scaling of basin widths found in Equation 2. Also, when f measures the elevation of a surface at position x, α is called the roughness exponent (Barabasi & Stanley 1995).

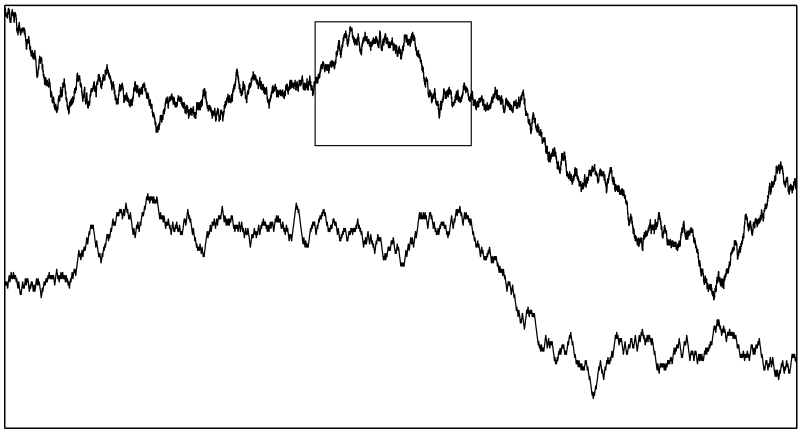

Figure 2 Two example random walks where the lower walk is the inset section of the upper walk "blown up." Random walks are statistically equivalent under the rescaling of Equation 8. Here, $b = 1/5$ so the rescaling is obtained by stretching the horizontal axis by a factor of 5 and the vertical axis by $5^{1/2}$.

Probability Distributions

What is the size distribution of river basins? In other words, if one chooses a random position on a landscape, what is the probability that an area a drains into that point? As we will see, scaling laws appear once again, this time in the form of probability distributions.

Imagine that the boundaries on each side of a basin are directed random walks. In this context, a directed random walk is one in which the random motion is always in the x-direction of Figure 1, while the y-direction plays the role of time. Taking the left boundary to be $\phi_l(y)$ and the right boundary to be $\phi_r(y)$, a basin is formed when these two walks intersect (i.e. a pair of sots collide). Because ϕ_l and ϕ_r are independent, the difference $\phi(y) = \phi_r(y) - \phi_l(y)$ is yet another random walk. We see then that the distribution of basin sizes may be related to the probability that the random walk $\phi(y)$ returns to its initial position after n steps for the first time. This is the classic problem of the first return time of a random walk. As the number of steps becomes large, the asymptotic form of the solution is (Feller 1968)

$$P(n) = \frac{1}{2\sqrt{\pi}} n^{-3/2}. \tag{9}$$

In terms of basin parameters, we may take $n \propto l \propto L_\parallel$, where l is the length of the main stream. Note that the assumption $l \propto L_\parallel$ is valid only for directed random walks; this is discussed further in the following section on networks. We therefore have the distribution of main stream lengths,

$$P_l(l) \propto l^{-3/2}. \tag{10}$$

Because the typical width of such a basin of length l scales like $l^{1/2}$ (see Equation 7), the typical area $a \propto l^{3/2}$. Thus the probability of basin areas is

$$P_a(a) = P_l[l(a)]\,\frac{dl}{da} \propto a^{-4/3}. \tag{11}$$

As expressed by Equation 2, basin widths scale in general like $L_\parallel^H$, where $0 \leq H < 1$ rather than the fixed $H = 1/2$ of random walks. In keeping with this observation, the distributions for area and main stream also generalize. Thus we write

$$P_l(l) \propto l^{-\gamma} \text{ and } P_a(a) \propto a^{-\tau}. \tag{12}$$

Furthermore, the exponents τ and γ are not independent and their connection lies in the aforementioned Hack's law, one of the most well-known scaling laws of river networks. Hack's law expresses the variation of average main stream length $\bar{l}$ with area a,

$$\bar{l} \propto a^h, \tag{13}$$

where h is known as Hack's exponent. The averaged value $\bar{l}$ is required here since there are noticeable statistical fluctuations in Hack's law (Maritan et al 1996b, Rigon et al 1996, Dodds & Rothman 2000). For the simple random model we have $\bar{l} \propto a^{2/3}$ and therefore Hack's exponent $h = 2/3$ (Takayasu et al 1988, Huber 1991). Now, as per Equation 11, we can write

$$P_a(a) = P_l[\bar{l}(a)]\,\frac{d\bar{l}}{da} \tag{14}$$

$$\propto a^{h(1-\gamma)-1}. \tag{15}$$

Using Equation 12, we obtain our first scaling relation,

$$\tau = h(1 - \gamma) - 1. \tag{16}$$

In general, scaling relations express exponents as algebraic combinations of other exponents. Such relations abound in theories involving scaling laws, and as such provide important tests for both theory and experiment.

Scaling Functions

In any physical system, scaling is restricted to a certain range. For example, basins as small as a water molecule are clearly out of sanity's bounds. At the other extreme, drainage areas are capped by the size of the overall basin, which is dictated by geology. In the customary terminology, we say that scaling breaks down at such upper cutoffs because of finite-size effects.

As it turns out, this feature of scaling is important both in theory and in practice. Measurements of exponents are made more rigorous and more can be achieved with limited system size. In the case of river networks, Maritan et al (1996b) demonstrated how finite-size scaling can be used to derive a number of scaling relations. We outline the basic principle below.

Consider the probability distribution of basin areas $P_a(a) \propto a^{-\tau}$. We can more generally write it as

$$P_a(a) = a^{-\tau} f(a/a_*), \tag{17}$$

where f is referred to as a scaling function and a_* is the typical largest basin area. The behavior of the present scaling function is

$$f(x) \propto \begin{cases} c \ \text{for}\ x \ll 1 \\ 0 \ \text{for}\ x \gg 1 \end{cases}. \tag{18}$$

Therefore, for $a \ll a_*$ we have the power law scaling of $P_a(a)$, whereas for $a \gg a_*$, the probability vanishes.

We enjoy the full worth of this construction when we are able to examine systems of varying overall size. We can recast the form of $P_a(a)$ with lengths by noting from Equation 3 that $a_* \propto L_\parallel^D$ where we have set $1 + H = D$. This gives

$$P_a(a|L_\parallel) = L_\parallel^{-D\tau} \tilde{f}(a/L_\parallel^D), \tag{19}$$

where $\tilde{f}$ is a new scaling function which, because of Equation 12, has the limiting forms

$$\tilde{f}(x) \propto \begin{cases} x^{-\tau} \ \text{for}\ x \ll 1 \\ 0 \ \text{for}\ x \gg 1 \end{cases}. \tag{20}$$

We now have two exponents involved. By examining basins of different overall size L we obtain a family of distributions upon which we perform a scaling collapse. Rewriting Equation 19, we have

$$L_\parallel^{\tau D} P_a(a|L_\parallel) = \tilde{f}(a/L_\parallel^D), \tag{21}$$

so that plots of $L_\parallel^{D\tau} P_a$ against $a/L_\parallel^D$ should lie along one curve, namely the graph of the scaling function $\tilde{f}$. Thus, by tuning the two exponents τ and D to obtain the best data collapse, we are able to arrive at strong estimates for both.

UNIVERSALITY

We would like to know precisely which aspects of a system are responsible for observed scaling laws. Sometimes, seemingly different mechanisms lead to the same behavior. If there is truly a connection between these mechanisms, then it must be at a level abstract from raw details. In scaling theory, such connections exist and are heralded by the title of universality. In this section, we consider several examples of universality. This will then lead us into problems in geomorphology proper.

More Random Walks: Crossover Phenomena

First, consider once again the drunkard's walk. Suppose that instead of describing the walk as one with discrete steps of unit length, the walker instead lurches a distance s_n at time n, with s_n now drawn from some probability distribution $P(s)$. Take, for example, $P(s) \propto \exp\{-s^2/2\sigma^2\}$, a Gaussian with variance σ^2. We then find from Equation 6 that $\langle x_n^2 \rangle = n\sigma^2$, and once again the characteristic excursion $r_n \propto n^{1/2}$. Thus the scaling is the same in both cases, even though the details of the motion differ. Loosely stated, any choice of $P(s)$ will yield the same result, as long as the probability of an extremely large step is extremely small. This is an especially simple but nonetheless powerful instance of universality, which in this case derives directly from the central limit theorem.

Real random walks may of course be more complicated. The archetypal case is Brownian motion. Here we consider the random path taken by a microscopically small object, say a tiny sphere of radius r and mass m, immersed in a liquid of viscosity μ. Within the fluid, random molecular motions induced by thermal agitations act to give the particle random kicks, thus creating a random walk. A classical model of the process, owing to Langevin, is expressed by the stochastic differential equation (Reif 1965, Gardiner 1985)

$$m \frac{dv}{dt} = -\alpha v + \eta(t). \tag{22}$$

Here v is the velocity of the particle, $\eta(t)$ is uncorrelated Gaussian noise, and $\alpha = 6\pi r\mu$ is the hydrodynamic drag that resists the motion of the sphere. Now note that the existence of the drag force creates a characteristic time scale $\tau = \alpha/m$. For times $t \ll \tau$, we expect viscous damping to be sufficiently unimportant so that the particle moves in free flight with the thermal velocity characteristic of molecular motion. On the other hand, for times $t \gg \tau$, the effect of any single kick should damp out.[1] Solving Equation 22 for the mean-square excursion $\langle x^2 \rangle$, one finds (Reif 1965, Gardiner 1985)

$$\langle x^2 \rangle \propto \begin{cases} t^2 \text{ for } t \ll \tau \\ t \text{ for } t \gg \tau \end{cases}. \tag{23}$$

The first of these relations describes the ballistic phase of Brownian motion, whereas the second describes the diffusive phase. The point here is that there is a crossover from one type of behavior to another, each characterized by a particular exponent. The existence of the ballistic phase at small times, like the diffusive

[1]Although the essence of the problem is captured here, the full story is in truth richer (e.g. Alder & Wainwright 1970).

phase at large times, is independent of the details of the motion. Brownian motion thus provides an elementary example of a dynamical process that can fall into one of two classes of motion, depending on which processes dominate at which times. In general, such crossover transitions range from being sharp to being long and drawn out. Later in this review, we argue that crossovers are an important feature of Hack's law.

Surface Evolution

Universality can manifest itself in ways more subtle and much deeper than the simple random walks discussed previously. To illustrate this, we consider an example of particular relevance to geomorphology: classes of surface evolution.

We consider a surface $h(\vec{x}, t)$, where h is the height at position $\vec{x}$ at time t. A wide variety of models of growing or eroding surfaces may be modeled by the stochastic equation (Kardar et al 1986, Barabasi & Stanley 1995, Halpin-Healy & Zhang 1995, Marsili et al 1996)

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} |\nabla h|^2 + \eta(\vec{x}, t). \tag{24}$$

The individual effect of each of the three terms on the right-hand side is portrayed in Figure 3.

The first term represents diffusion, i.e. the tendency of bumps to smooth out. The second term reflects the tendency of a surface to grow or erode in a direction normal to itself. Its quadratic form results from retaining the leading-order nonlinearity that accounts for the projection of this growth direction on the (vertical) axis on which h is measured; the coefficient λ is related to the velocity of growth, or alternatively, the erosion rate. Finally, $\eta(\vec{x}, t)$ is stochastic noise, uncorrelated in both space and time and with zero mean and finite variance. It represents time-

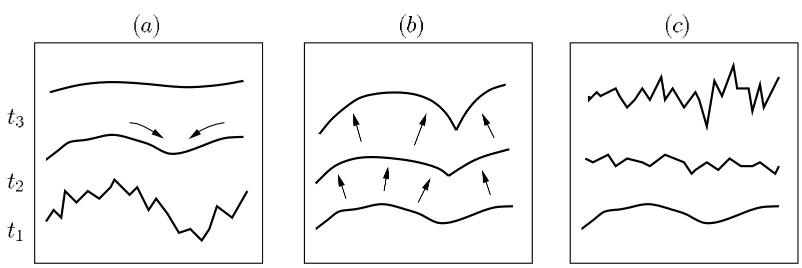

Figure 3 Three mechanisms of surface growth corresponding to the three terms of Equation 24. Diffusive smoothing is depicted in (*a*), growth normal to the surface in (*b*), and random growth in (*c*). Surfaces are shown for each mechanism at times t_1, t_2, and t_3 where $t_1 < t_2 < t_3$. In (*a*), the arrows represent the flux of deposited material, and in (*b*) they indicate the direction of growth (for $\lambda > 0$).

and space-dependent inhomogeneities in material properties (e.g. soil type) and forcing conditions (e.g. rainfall rate).

Equation 24 is written such that h represents the fluctuations of height in a frame of reference moving with velocity λ. These fluctuations contain the statistical signature of the physical growth process. In particular, three classes of surface evolution may be described by Equation 24. These classes may be characterized by the behavior of the height-height correlation function

$$C(r, t) = \langle |h(x + r, t) - h(x, t)|^2 \rangle_x^{1/2}. \tag{25}$$

The function C, here written under the assumption that $C(\vec{x}) = C(x)$, with $x = |\vec{x}|$, measures the roughness of a surface, or more precisely, the root-mean-square height fluctuation over a distance r. We are interested in the scaling behavior with respect to both space and time.

The first class of surface evolution models, called random deposition, is obtained by setting $\nu = \lambda = 0$, leaving only the noise term. Because the variance of the height of each point grows linearly and independently with time, so does the variance of the height difference of any two points, independent of their separation r. Thus

$$C(r, t) \propto t^{1/2}. \tag{26}$$

The second class is noisy diffusion, where now only $\lambda = 0$. It represents random fluctuations mediated by diffusive smoothing. As such, at a particular time there is a length scale below which diffusion balances noise and above which noise continues to increase the roughness. Indeed, one may show that Equation 24 is invariant with respect to the self-affine transformations $t \to b^z t$ and $x \to b^\alpha x$, and that (Family 1986)

$$C(r, t) = r^\alpha f(kt/r^z) \tag{27}$$

describes simultaneously the dependence of C on both r and t. Here, k is a constant with appropriate dimensions and $f(y)$ is a scaling function, as discussed in the previous section on scaling. For the particular case of noisy diffusion, f may be calculated analytically. However, the preceding physical arguments suffice to point out that for large values of y, i.e. $kt \gg r^z$, $f(y)$ approaches a constant. On the other hand, for $kt \ll r^z$, the r-dependence in the two factors on the right side of Equation 27 must cancel, yielding $C(r,t) \propto t^{\alpha/z}$. The actual values of α and z depend on the dimensionality of the growing or eroding surface. For one-dimensional surfaces, such as the transects of Figure 3, $\alpha = 1/2$ and $z = 2$ (Edwards & Wilkinson 1982, Family 1986, Barabasi & Stanley 1995). For the more relevant situation of a two-dimensional surface, one finds that again $z = 2$ but that $C(r,t) \propto (\log r)^{1/2}$. Because this r-dependence is very small, it is often taken as equivalent to having $\alpha = 0$.

Real geological surfaces are often thought to exhibit fractal properties (e.g. Turcotte et al 1998), which in the present context means $\alpha > 0$. Clearly, noisy diffusion would then be an inadequate model. Including the nonlinear term of Equation 24, however, changes affairs considerably. The dynamical scaling equation (Equation 27) is still satisfied, but now, in the case of one-dimensional interfaces, one may obtain the exact solution $\alpha = 1/2$ and $z = 3/2$, for any $\lambda \neq 0$ (Kardar et al 1986). For the two-dimensional case, theory predicts that $\alpha + z = 2$, but there is no complete analytic solution. Thus far, numerical studies have estimated that $0.2 \lesssim \alpha \lesssim 0.4$. Indeed, a large literature has evolved to understand the behavior of the KPZ equation (introduced by Kardar, Parisi, and Zhang) in higher dimensions (Krug & Spohn 1992, Barabasi & Stanley 1995, Halpin-Healy & Zhang 1995).

The point of this brief review of surface roughening models is that discrete universality classes of interface dynamics may be defined. Each class is defined by the dominant physical mechanisms and the scaling exponents that they produce. It is important to note additionally that Equation 24 is not as specialized as it may naïvely seem. For example, instead of surface diffusion we may find surface tension. Or a microscopic process of local rearrangement could exist, and its large scale simply behaves diffusively. No matter what their origin, all these processes are described by $\nabla^2 h$. Equally if not more important, the growth normal to the interface, represented by the nonlinear term, is relevant to a variety of processes in which the microscopic evolution of a surface includes processes that depend on the amount of the exposed surface area and not just its inclination. As we have already stated, the nonlinear term is the leading-order nonlinearity in an expansion that takes account of this growth factor. It turns out that one may establish the irrelevance of higher-order terms. ‘‘Irrelevance’’ in this context means that the values of the scaling exponents α and z do not change. This is an integral part of the notion of universality, because it means that asymptotically—e.g. at long times and large length scales—different models, different equations, and hence different processes can behave the same way.

A Little History

Scaling and universality are deep ideas with an illustrious past. Therefore, a brief historical perspective is in order.

In essence, scaling may be viewed as an extension of classical dimensional analysis (Barenblatt 1996). Our interest, however, is strongly influenced by studies of phase transitions and critical phenomena that began in the 1960s. Analogous to the present situation with river networks, equilibrium critical phenomena at that time presented a plethora of empirical scaling exponents for which there was no fundamental ‘‘first principles’’ understanding. Kadanoff and others then showed how an analysis of a simple model of phase transitions—the famous Ising model of statistical mechanics—could yield the solution to these problems (Kadanoff et al 1966). Their innovation was to view the problem at different length scales and search for solutions that satisfied scale invariance.

These ideas were richly extended by Wilson's development of the calculational tool known as the renormalization group (Wilson & Kogut 1974). This provided a formal way to eliminate short-wavelength components from problems, while at the same time finding a ''fixed point'' from which the appropriate scaling laws could be derived. The renormalization group method then showed explicitly how different microscopic models could yield the same macroscopic dynamics, i.e. fall within the same universality class.

These ideas turned out to have tremendous significance well beyond equilibrium critical phenomena. [See, for example, the brief modern review by Kadanoff (1990) and the pedagogical book by Goldenfeld (1992).] Of particular relevance to geomorphology are the applications in dynamical systems theory. An outstanding example is the famous period-doubling transition to chaos, which occurs in systems ranging from the forced pendulum to Rayleigh-Bénard convection (Strogatz 1994). By performing a mathematical analysis similar to that of the renormalization group, Feigenbaum (1980) was able to quantitatively predict the way in which a system undergoes period-doubling bifurcations. The theory applies not only to a host of models, but also to widely disparate experimental systems.

Underlying all this work is an effort to look for classes of problems that have common solutions. This is the essence of universality: If a problem satisfies qualitative criteria, then its quantitative behavior—scaling laws and scaling relations—may be predicted. In the remainder of this review, we explore these concepts as they apply to geomorphology.

RIVER NETWORKS

River networks were introduced in the discussions of scaling and universality as a principal source of scaling phenomena in geomorphology. The list of scaling laws for river networks is indeed a lengthy one. As we will describe, however, the use of certain assumptions demonstrates that the total content of these laws comes down to the values of only a few scaling exponents; all other exponents are connected via scaling relations. In particular, we view Hack's law as central to the description of river networks.

After these scaling laws are marshalled together, which is an important step in its own right, one is left with two rather deep questions: What is the source of scaling in river networks? And does the scaling exhibited by river networks belong to a single universality class, a discrete set, or even a continuum? It is fair to say that the answers are not yet known. An explanation of the former question would presumably lead to an elucidation of the latter. To this end, we present and examine model networks for which analytic results exist. These networks epitomize basic universality classes of river networks. This will in turn lead us to a critical analysis of Hack's law.

Scaling Laws and Scaling Relations

We now fill in some gaps of further definitions of scaling laws pertinent to river networks. We have already seen Hack's law (Equation 13), probability distributions for area and length (Equation 12), and the scaling of basin widths and areas with respect to longitudinal length (Equations 2 and 3). In what follows, we introduce a set of assumptions that allow for the derivation of these laws and the relevant scaling relations. The main outcome is that we will at the end be able to express the universality class of a network in two numbers.

Horton's Laws Horton (1945) was the first to develop a quantitative treatment of river network structure. The basic idea is to assign indices of significance to streams, affording a means of comparing stream lengths, drainage areas, and so on. A later improvement by Strahler (1957) led to the following method of stream ordering, depicted in Figure 4. Source streams are defined as first-order stream segments. Deletion of these from the network produces a new set of source streams, which are then the second-order stream segments. The process is iterated until we have labeled all stream segments. In this framework, a sub-basin is of the same order as its main stream.

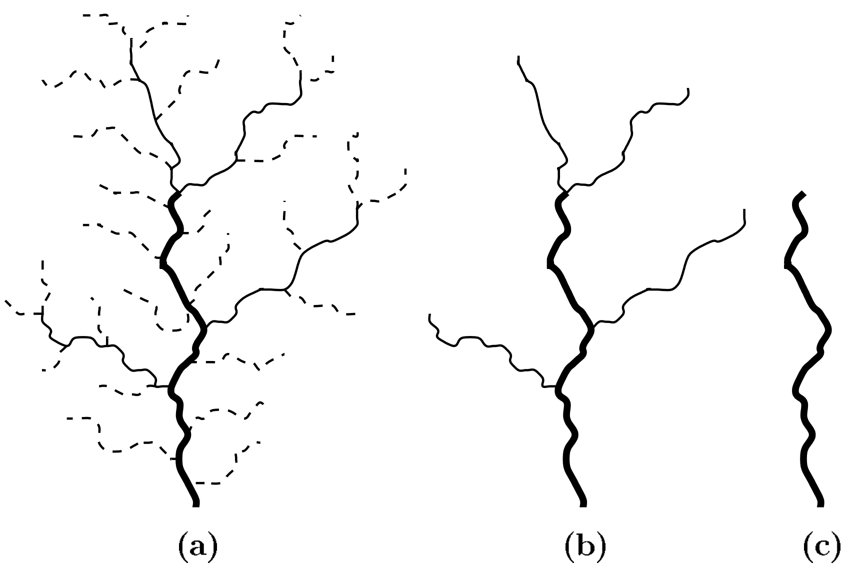

Figure 4 Horton-Strahler stream ordering. (*b*) is created by removing all source streams from the network in (*a*), and these same streams are denoted as first-order stream segments. The new source streams in the pruned network of (*b*) are labeled as second-order stream segments and are themselves removed to give (*c*), a third-order stream segment.

Given such an ordering, natural quantities to measure are n_ω, the number of stream segments for a given order ω; $\bar{\ell}_\omega$, the average stream segment length; $\bar{a}_\omega$, the average basin area; and the variation in these numbers from order to order.

The following ratios are generally observed to be independent of order ω:

$$\frac{n_\omega}{n_{\omega+1}} = R_n, \; \frac{\bar{\ell}_{\omega+1}}{\bar{\ell}_\omega} = R_\ell, \; \frac{\bar{l}_{\omega+1}}{\bar{l}_\omega} = R_l, \text{ and } \frac{\bar{a}_{\omega+1}}{\bar{a}_\omega} = R_a. \tag{28}$$

The ratios R_ℓ and R_l are simply connected. The stream ordering has broken down main stream lengths into stream segments, so we have $\bar{l}_\omega = \sum_{k=1}^{\omega} \bar{\ell}_k$. A trivial calculation shows that if $R_\ell = \bar{\ell}_{\omega+1}/\bar{\ell}_\omega$ then $R_\ell = R_l$ $(= \bar{l}_{\omega+1}/\bar{l}_\omega)$ (Dodds & Rothman 1999).

Tokunaga's Law The Horton ratios (Equation 28), although indicative of the network structure, do not give the full picture. We cannot construct a network using the Horton ratios alone because we do not know which streams connect with which. The network perhaps suggested by the Horton ratios is one where all streams of order ω flow into streams of order $\omega + 1$, a true hierarchy. But this is misleading because streams of a certain order entrain streams of all lower orders.

A more direct description was later developed by Tokunaga (Tokunaga 1966, 1978, 1984; Peckham 1995; Newman et al 1997; Dodds & Rothman 1999). The same stream ordering is applied as before, and we now consider $\{T_{\omega,\omega'}\}$, the so-called Tokunaga ratios. These represent the average number of streams of order ω' that are side tributaries to streams of order ω. Real networks have a self-similar form, so we first have $T_{\omega,\omega'} = T_{\omega-\omega'} = T_\nu$. Tokunaga's law goes further than this by stating that the Tokunaga ratios may be derived from just two network-dependent parameters, T_1 and R_T:

$$T_\nu = T_1(R_T)^{\nu-1}. \tag{29}$$

As it turns out, we can further argue that Horton's laws plus uniform drainage density are equivalent to Tokunaga's law. Both afford the same type of network (Dodds & Rothman 1999). First, this assumption leads to the result that $R_a \equiv R_n$ so that only two Horton ratios are independent. Moreover, an invertible transformation between the remaining pairs of parameters may be deduced to be (Tokunaga 1978)

$$R_n = 1/2(2 + R_T + T_1) + 1/2[(2 + R_T + T_1)^2 - 8R_T]^{1/2}, \tag{30}$$

$$R_\ell = R_T. \tag{31}$$

Scaling Relations We now assume that (*a*) a network obeys Tokunaga's law, (*b*) drainage density is uniform, and (*c*) single streams are self-affine. The latter two are defined as follows. Drainage density is a measure of how finely a landscape is dissected by channels. Another parameter that effectively expresses this is the typical length of a hillslope separating drainage divides from channels. The last assumption incorporates self-affinity of single streams. The main stream of a basin is reported to scale as (Tarboton et al 1988, La Barbera & Rosso 1989, Tarboton et al 1990, Maritan et al 1996b)

$$l \propto L_{\parallel}^{d}. \tag{32}$$

From this microscopic picture of network connection, we can build up to the scaling laws of the macroscopic level (Dodds & Rothman 1999). Exponents are found in terms of (T_1, R_T, d), or equivalently, (R_n, R_l, d). More precisely, only the ratio $\log R_l/\log R_n$ is needed, so all scaling exponents may be found in terms of

$$(\log R_l/\log R_n, d) \equiv (h, d), \tag{33}$$

where h is the Hack exponent. Thus, we have a degeneracy—the two parameters R_l and R_n are bound together to give only one value. We therefore find more information in these microscopic, structural descriptions than in the macroscopic power laws. Table 1 lists all exponents and their algebraic connection to these fundamental network parameters.

TABLE 1 A list of scaling laws for river networks*

Law	Name or description	Scaling relation
$T_\nu = T_1(R_T)^{\nu-1}$	Tokunaga's law	(T_1, R_T)
$n_{\omega+1}/n_\omega = R_n$	Horton's law of stream numbers	see Equation 31
$\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$	Horton's law of stream segments lengths	$R_\ell = R_T$
$\bar{l}_{\omega+1}/\bar{l}_\omega = R_l$	Horton's law of main stream lengths	$R_l = R_T$
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	Horton's law of stream areas	$R_a = R_n$
$l \propto L^d$	self-affinity of single channels	d
$l \propto a^h$	Hack's law	$h = \log R_l/\log R_n$
$a \propto L^D$	scaling of basin areas	$D = d/h$
$L_\perp \propto L^H$	scaling of basin widths	$H = d/h - 1$
$P(a) \propto a^{-\tau}$	probability of basin areas	$\tau = 2 - h$
$P(l) \propto l^{-\gamma}$	probability of stream lengths	$\gamma = 1/h$
$\Lambda \propto a^\beta$	Langbein's law	$\beta = 1 + h$
$\lambda \propto L\phi$	variation of Langbein's law	$\phi = d$
$\tilde{\Lambda} \propto a^{\tilde{\beta}}$	as above	$\tilde{\beta} = 1 + h$
$\tilde{\lambda} \propto L^{\tilde{\varphi}}$	as above	$\tilde{\phi} = d$

*The first five laws require Horton-Strahler stream ordering, whereas the rest are independent of this construction. All laws and quantities are defined in the text. The assumptions required to deduce all of these scalings are comprised of the two italicized relations, Tokunaga's law and the self-affinity of single channels, and the assumption of uniform drainage density.

In our discussion of scaling, we introduced probability distributions of drainage area and main stream length. There we derived the scaling relation $\tau = h(1 - \gamma) - 1$ (Equation 16). With the preceding assumptions, we may further show that (Dodds & Rothman 1999)

$$\tau = 2 - h \text{ and } \gamma = 1/h. \tag{34}$$

Thus we have only one independent exponent and two scaling relations. We note that other constructions may lead to the same result (Maritan et al 1996b, Meakin et al 1991).

One final collection of scaling laws revolve around $\tilde{\Lambda}$, the total distance along streams from all stream junctions in the network to the outlet of a basin. Empirical observations suggest that $\tilde{\Lambda} \propto \alpha^{\tilde{\beta}}$, and this is often referred to as Langbein's law (Langbein 1947). We can show that $\tilde{\beta} = 1 + h$ follows from our basic assumptions.

Known Universality Classes of River Networks

We next describe basic network models that exemplify various universality classes of river networks. These classes form the basis of our ensuing discussion of Hack's law. We consider networks for non-convergent flow, random networks of directed and undirected nature, self-similar networks, and ''optimal channel networks.'' We also discuss binary trees to illustrate the requirement that networks be connected with surfaces. We take universality classes to be defined by the pair (h,d), and these exponents are sufficient to give the exponents of all macroscopic scaling laws. The network classes described as follows are provided in Table 2, along with results for real river networks.

TABLE 2 Theoretical networks with analytically known universality classes*

Network	***h***	***d***
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
Optimal channel networks (I)	1/2	1
Optimal channel networks (II)	2/3	1
Optimal channel networks (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

*The universality class of river networks is defined by the pair of exponents (h,d) where h is Hack's exponent (Equation 13) and d is the scaling exponent that represents stream sinuosity (Equation 32). Each network is detailed in the text. All other scaling exponents may be obtained via the scaling relations listed in Table 1. The range of these exponents for real river networks is shown for comparison.

Note that in some cases, different models belong to the same universality class. Indeed, this is the very spirit of universality. The details of the models we outline in this section are important only to the models themselves.

Non-convergent Flow [(h,d) = (1, 1)] Figure 5*a* shows the trivial case of non-convergent flow where $(h,d) = (1, 1)$. By non-convergent, we mean the flow is either parallel or divergent. Basins are effectively linear objects, and thus we find that drainage area is proportional to length. This universality class corresponds, for example, to flow over convex hillslopes, structures that are typically dominated by diffusive processes rather than erosive ones. Importantly, channels in long valleys will also belong to this class. After such a channel forms, drainage area and length will increase regularly until a junction with a comparable channel is reached. The flow here is non-convergent at the level of the network itself.

Directed Random Networks [(h,d) = (2/3, 1)] We next present what we deem to be the simplest possible network entailing convergent flow that is physically reasonable. This is the directed random network first introduced by Scheidegger (1967). Scheidegger originally considered the ensemble of networks formed on a triangular lattice when flow from each site is randomly chosen to be in one of two directions. This may be reformulated on a regular square lattice with the choices given in Figure 5*b*. Because of universality, the same scaling arises independent of the underlying lattice. Now, these networks are essentially the same as those we discussed in the introductory section on scaling. In both cases, basin boundaries and main streams are directed random walks. We have thus already derived the results $h = 2/3$, $\tau = 4/3$, and $\gamma = 3/2$. Other exponents follow from the scaling relations. Furthermore, $d = 1$ because the networks are

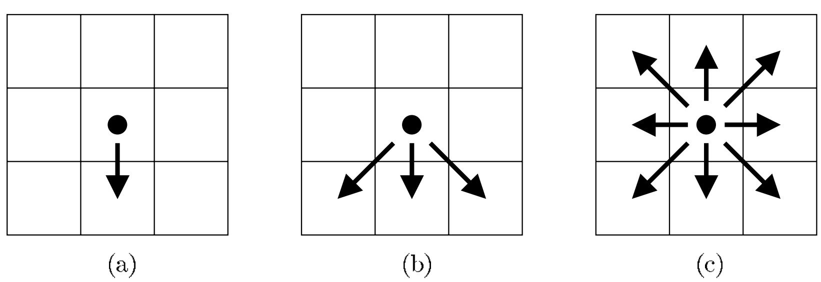

Figure 5 Possible directions of flow for three networks whose statistics belong to differing universality classes. Diagram (*a*) provides the trivial class where flow is essentially non-convergent; diagram (*b*) corresponds to directed random networks; and diagram (*c*) corresponds to undirected random networks. Note that although diagram (*a*) literally pictures perfect parallel flow, it figuratively symbolizes any set of flow lines that do not converge.

directed. Our first universality class is therefore defined by the pair of exponents $(h,d) = (2/3, 1)$.

Undirected Random Networks [(h,d) = (5/8, 5/4)] If we relax the condition of directedness, then we move to a set of networks belonging to a different universality class. These networks were first explored by Leopold & Langbein (1962). They were later theoretically studied under the moniker of random spanning trees by Manna et al (1992), who found that the universality class is described by $(h,d) = (5/8, 5/4)$. The possible flow directions are shown for both directed and undirected random networks in Figures 5*b* and 5*c*.

Branching Trees [(h,d) = (?, ?)] The networks mentioned previously are built on two-dimensional lattices. We take an aside here to discuss a case where no clear link to a two-dimensional substrate exists. Consider a binary branching tree and all of its possible sub-networks (Shreve 1966, 1967). This seems a logical model since most river networks are composed of confluences of two streams at forks—very rarely does one see even trifurcations, let alone the conjoining of four streams.

However, binary trees are not as general as one might think. That river networks are trees is evident, but they are special trees in that they fill all space (see Figure 6). If links between forks are assumed to be roughly constant throughout a network, then drainage density increases exponentially. Conversely, if drainage density is held constant, then links grow exponentially in length as we move away

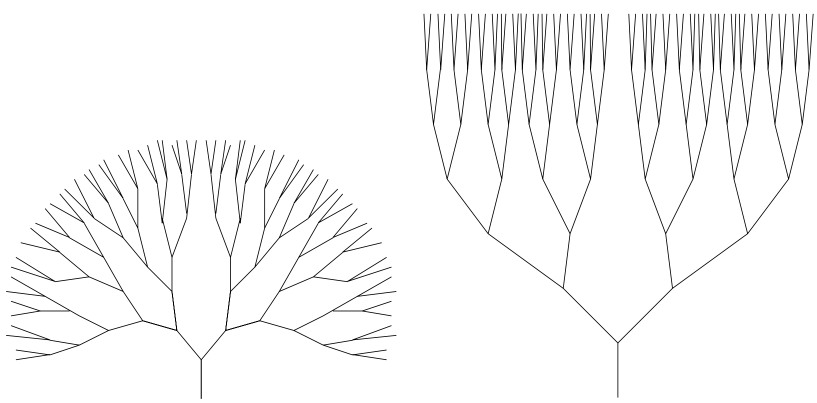

Figure 6 Two examples of a binary tree rendered onto a plane. The exponential growth of the number of branches means that the networks fill up space too quickly. The two usual assumptions of binary trees—that individual links are similar in length and that drainage density is uniform—cannot both be maintained.

from the outlet into the network. Thus, we cannot consider the binary tree model to be a representation of real river networks.[2]

Nevertheless, we briefly persist with this unrealizable model because it is of historic importance and does allow for some interesting analysis. If we do make the unphysical assumption that we may ascribe unit lengths and areas to each link, then the Horton ratios can be calculated as $R_n = 4$ and $R_l = 2$ (Shreve 1966, 1967; Tokunaga 1978). This gives the Hack exponent $h = \log R_l/\log R_n = 1/2$. Other avenues have arrived at this same result, which came to be known as Moon's conjecture (Moon 1980, Waymire 1989). Because main stream lengths are proportional to basin length, we have the universality class $(h,d) = (1/2, 1)$.

One final comment regarding binary trees concerns the work of Kirchner (1993), who found that river network scaling laws are ‘‘statistically inevitable.’’ The problem with this seemingly general and hence rather damning result is that the basis of the study was the examination of binary tree sub-networks. Thus, no conclusions regarding real river networks may be drawn from Kirchner's work. As we hope has been clearly demonstrated, any reasonable method for producing general ensembles of networks must have the networks associated with surfaces. Moreover, work by Costa-Cabral & Burges (1997) has confirmed variability of network laws for one particular model that works along these lines.

Self-Similar Basins [(h,d) = (1/2, 1)] An example of a network that belongs to the universality class $(h,d) = (1/2, 1)$ and is embedded in a surface is the so-called Peano basin (Rodríguez-Iturbe & Rinaldo 1997). Its definition is an iterative one demonstrated by the first three ‘‘basins’’ in Figure 7. A modified version without trifurcations is illustrated on the right, so that all junctions are the usual forks of river networks. Technically, this also allows for the proper application of Tokunaga's description, which quite reasonably presumes that all junctions are forks.

It is a simple exercise to show that for the Peano basin, $n_\omega = 3\cdot4^{\Omega-\omega-1}$ and $\bar{l}_\omega = 2^{\omega-1}\bar{l}_1$ for $\omega = 1, \ldots, \Omega - 1$ where Ω is the overall basin order. Thus we have $R_n = 4$ and $R_l = 2$, and hence $\log R_l/\log R_n = 1/2$. An inspection of Tokunaga's ratios shows that $T_1 = 1$ and $R_T = 2$, in agreement with the transformation of Equations 30 and 31. Since main stream length rapidly approaches $L_\parallel$, we have essentially $l = L_\parallel$, so that $d = 1$. The important point to note here is that, by construction, the basins are of unit aspect ratio. Because $d = 1$, we see that Hack's exponent is by necessity 1/2.

Thus, the Peano basin belongs to what we call the self-similar universality class, defined by $(h,d) = (1/2, 1)$. As with other simple models, the Peano basin is not something we would expect to find in nature. Nevertheless, the general class of self-similar basins is a very reasonable one. Indeed, that basins of all

[2]Binary trees are examples of Bethe lattices, which have been well studied in percolation theory (Stauffer & Aharony 1992). Solutions to percolation problems show that they resemble infinite-dimensional space, not two-dimensional space.

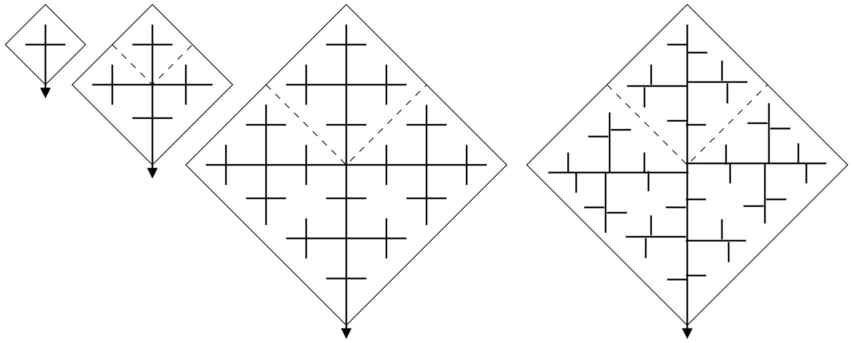

Figure 7 The Peano basin, a member of the $(h, d) = (1/2, 1)$ universality class. The first three basins show the basic construction, with each larger basin built out of four of those from the previous level. The rightmost basin shows a slight perturbation to remove the trifurcations. Each basin's outlet is at its bottom.

sizes should be geometrically similar is what would be expected by straightforward dimensional analysis.

Optimal Channel Networks [(h,d) = (2/3, 1), (1/2, 1), or (3/5, 1)] Another collection of networks with well-understood universality classes makes up the optimal channel networks (see Rodríguez-Iturbe & Rinaldo 1997, Rinaldo et al 1998, and references therein). These models, known as OCNs, are based on the conjecture that landscapes evolve to a stationary state characterized by the minimization of the energy dissipation rate $\dot{\varepsilon}$, where

$$\dot{\varepsilon} \sim \sum_i a_i s_i \sim \sum_i a_i^{1-\theta}. \tag{35}$$

Here a_i and s_i are the contributing area and the slope at the ith location on a map and are identified with a thermodynamic flux and force, respectively. The second approximation comes from the empirical observation that $\langle s \rangle_a \sim a^{-\theta}$, where the average is taken over locations with the same contributing area, and typically $\theta \simeq 0.5$ (Horton 1945, Flint 1974, Rodríguez-Iturbe & Rinaldo 1997).

The conjecture of optimality is controversial. Although it is appealing to seek a variational formulation of fluvial erosion (Sinclair & Ball 1996, Banavar et al 1997), it seems unlikely that the existence of a variational principle could be proved or disproved.[3] It remains nevertheless interesting to consider ramifications of such a conjecture.

[3]A comparison with fluid mechanics is instructive. Here one starts with the Navier-Stokes equations, so precise derivations are possible. For example, in the case of creeping (Stokes) flow with fixed boundaries, the flow field does indeed minimize energy dissipation rate (see Lamb 1945, art. 344). On the other hand, if the boundaries can move, cases may be found in which the flow field maximizes, rather than minimizes, dissipation (Hinch 1988).

Maritan et al (1996a) have shown that OCNs based on the formulation in Equation 35 fall into two distinct universality classes, denoted respectively here by I and II, depending on the value of θ. In the simplest case ($\theta = 0$), we find that the OCNs belong to the universality class of directed random networks. On the other hand, for $0 < \theta \leq 1/2$, the OCNs fall into the self-similar class. A third class (III) is made possible by extending the model to include a fixed, random erosivity at each site. This final class is deduced as $(h,d) = (3/5, 1)$.

Much of the literature on OCNs is devoted to numerical investigations. As it turns out, the universality classes given previously are not necessarily obtained, and differing exponents are reported. The reason is that the minimization process is fraught with local minima. Furthermore, the results depend on the details of the numerical method itself. As we discuss later, the actual scaling of real networks may be somewhat deceptively masked by long crossovers between distinct regimes of scaling. It is conceivable that a similar effect occurs with OCNs. Locally, physical processes such as the one suggested in the OCN formulation may conspire to produce certain scaling exponents, whereas at large scales, exponents in keeping with random networks may become apparent.

Summary Table 2 provides a summary of the foregoing networks and their corresponding universality classes. As the table shows, we have identified five distinct universality classes for river networks. Ranges for h and d for real river networks are also indicated. In scaling theory, the importance of exact results cannot be overlooked. The measurement of scaling exponents is a notoriously fickle exercise. For example, one might find that regression analysis gives a tight error bound over any given variable range but that the choice of the range greatly affects the estimate. Thus, we need persuasive reasoning to reject these known universality classes of networks and composite versions thereof. For the remainder of this section we explore the possibility of their existence in nature.

Real River Networks I: Hack's Law for Maximal Basins

Hack's original paper was concerned with basins with drainage areas less than 10^3 km^2 (Hack 1957). He found $h = 0.6$ but also noted fluctuations with h, which could be as large as 0.7 in some regions. Later, measurements by Gray (1961) found $h = 0.57$ in the midwestern area of the United States. Unfortunately, exponents are often reported without error bars, and since we are concerned with distinguishing values like 0.50, 0.57, and 0.67, estimations of error are imperative.

Although both Hack and Gray sampled from areas of differing geologies, they restricted their work to localized areas of the United States and included data taken from sub-basins contained within a single basin. In contrast to this, later work by Mueller (1972, 1973), Mosley & Parker (1973), and Montgomery & Dietrich (1992) compared individual basins from around the world. We suggest it is vital to discriminate between such intra-basin and inter-basin measurements.

We refer to the latter as the maximal basin version of Hack's law—the cross-comparison of continent draining basins. Denoting maximal by a tilde, we have

$$\bar{l} = \tilde{c}\tilde{a}^{\tilde{h}}. \tag{36}$$

Similarly, the mainstream length now scales as

$$\tilde{l} \propto \tilde{L}_{\|}^{\tilde{d}}. \tag{37}$$

Curiously, the reported results here point to a maximal basin Hack exponent of approximately 0.5. The oft-cited findings of Mueller (1973) further claim a crossover from 0.6 to 0.5 scaling in the maximal basin version of Hack's law. Upon inspection of the data used by Mueller, it is evident that considerable error in exponents must be acknowledged. Nevertheless, Figure 8 shows the maximal basin Hack's law for 37 networks from around the world. We find the exponent $\tilde{h} = 0.50 \pm 0.06$, which suggests that the world's largest river basins are self-similar. Note that without proper knowledge of $\tilde{d}$, we must be careful since self-similar basins belong specifically to the universality class $(\tilde{h}, \tilde{d}) = (1/2, 1)$.

A simple argument for why the world's largest basins would be self-similar is as follows. The shapes of the drainage areas of these networks are dictated by geologic processes. The Mississippi, for example, is bounded by the Rocky

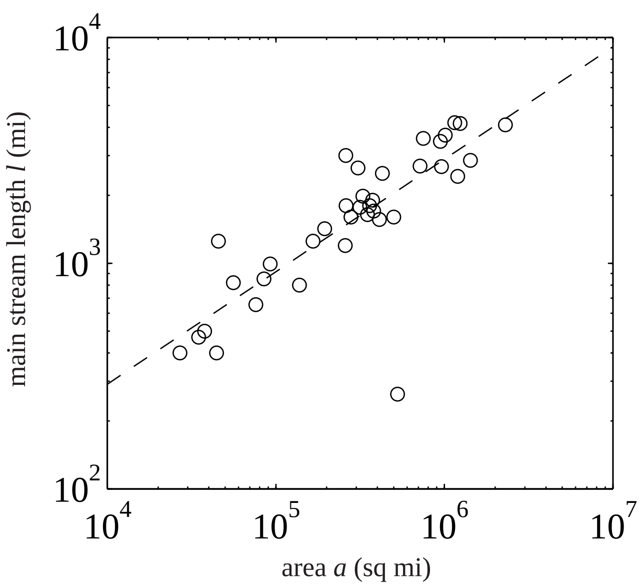

Figure 8 The maximal basin version of Hack's law $\tilde{l} = \tilde{c}\tilde{a}^{\tilde{h}}$ for 37 of the world's largest basins (data taken from Leopold 1994). Hack's exponent $\tilde{h}$ is estimated to be 0.50 and the prefactor to be 3.0. A confidence interval of one standard deviation gives the corresponding ranges as $0.44 < \tilde{h} < 0.56$ and $1.3 < \tilde{c} < 6.6$.

Mountains and the Appalachians, structures of predominantly tectonic origin. The aspect ratios of these basin shapes are thus rather variable. They depend partly on how the dominant river orients itself in the basin, though this is also consequent of the overall geology. It thus seems reasonable that most continent-scale basins would not deviate too far from having unit aspect ratios. Note that the tendency, if any, would be toward thinner basins. If $\tilde{L}_{\parallel} \lesssim \tilde{L}_{\perp}$, then only one basin would be expected. On the other hand, if $\tilde{L}_{\parallel} \ll \tilde{L}_{\perp}$, as in the case of a coastal mountain range, then multiple dominant basins will arise. The transition from single to many basins as the ratio $\tilde{L}_{\parallel}/\tilde{L}_{\perp}$ decreases is an interesting problem in itself. Furthermore, at this scale, main streams are expected to be proportional to overall basin length because they are predominantly directed. Hence, we suggest that geologically constrained maximal basins belong to the universality class $(\tilde{h}, \tilde{d}) = (1/2, 1)$.

Real River Networks II: Hack's Law for Single Basins

What do these arguments imply for the original formulation of Hack's law? It is by no means evident how the scalings of intra-basin and inter-basin versions are related. In fact, the scaling for Hack's law would seem to be a complicated one. We first conjecture the existence of up to four separate scaling regimes connected by three crossover regions, as depicted in Figure 9. These scaling regimes are associated with non-convergent hillslope flow, $(h = h_h = 1)$, an intermediate region of short-range order with unknown scaling $(h = h_?)$, random networks $(h = h_r)$, and geologic controls $(h = h_g)$.

Hillslopes We proceed in our description from smallest to largest basins. At the lower limit, we are in the realm of unchannelized, diffusive hillslopes. As we noted in our discussion of non-convergent flow, "basins" therefore have no width and are equivalent to lines of flow. We thus have $l \propto a$ and the universality class $(h, d) = (1, 1)$. Any measurement taken from a map that resolves hillslope structure will exhibit this linear phase of Hack's law. Note, however, that for a landscape where long, linear channels exist side by side, this hillslope transition will be subsumed into an even more pronounced linear regime of Hack's law.

This first crossover is potentially an important one. Recall that by the present definition, main stream length l is measured to the drainage divide and not to a channel head. This obviates problems associated with the qualitative definition of channel head position. Nevertheless, the position and movement of channel heads are important signatures of the nature of erosional processes (Dietrich & Dunne 1993). Hack's law therefore gives, in theory, a simple measure of channel head position from the position of the first crossover at the end of the $h = 1$ regime. The typical hillslope length l_1 is indicated in Figure 9. Because hillslope length gives drainage density, we equivalently have a measure of the latter and hence its fluctuations.

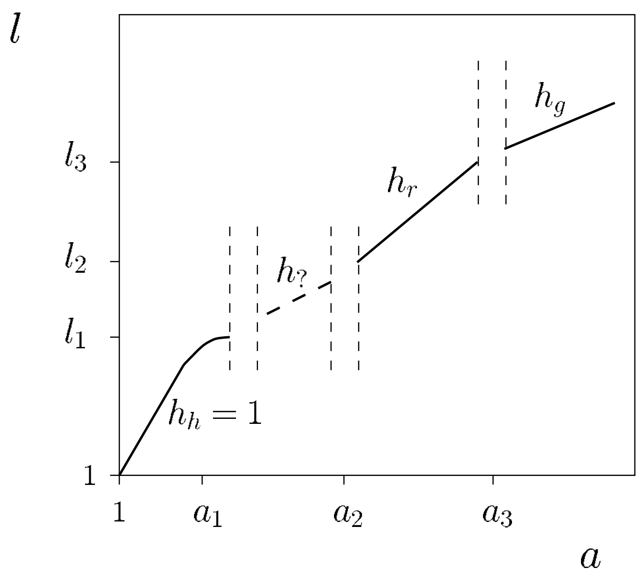

Figure 9 Conjecture for the full extent of Hack's law. Shown on a double logarithmic plot are four possible scaling regimes joined by three crossovers indicated by gaps. Smallest basin areas and stream lengths pertain to hillslopes (non-convergent flow) where $h_h = 1$. After channels begin at around (a_1, l_1), the rigidity in how the smallest streams fit together leads to a rapid drop in the exponent h. The existence of robust scaling in this intermediate region is an unresolved problem, hence $h = h_?$). As basin size increases toward (a_2, l_2), boundaries become more flexible and less correlated so that Hack's law moves via a potentially long crossover toward a random universality class where $h_\tau = 5/8$ or $2/3$. Finally, as the basins reach the size of the system beyond some a_3, geologic boundaries become important and there is a regime where the scaling may change yet again to a value $1/2 < h_g < 1$.

Crossover to Short-Range Order At the end of the hillslope regime, the first channels form when flow fully changes from non-convergent to convergent. Then appears a spread of basins with different areas that have similar main stream lengths. Hence, Hack's law flattens out from the slope of $h = 1$. These first-order streams are subject to strong ordering constraints. Neighboring source streams that feed into the same second-order stream are roughly parallel. Second-order streams are to a lesser degree similarly positioned; because their separation is greater, more sinuous formations are possible. This marks the beginning of what is potentially a long crossover in Hack's law. Initially, we may have $h \simeq 1/2$ because of the orderedness of basins. However, we may well have a non-trivial scaling owing to the physics of the situation; this is a question yet to be resolved.

Crossover to Randomness Whatever the case, we find that at larger scales, correlations in stream and basin shape decrease. This suggests an approach to one of the random universality classes. Thus, Hack's exponent would increase toward an h_r in keeping with the overall stream structure. The random regime is indicated in Figure 9, holding between a_2 and a_3 an inner and outer basin size. As we discussed, $h_r = 2/3$ for directed networks and $h_r = 5/8$ for non-directed. Note that d would increase from 1 to 5/4 for non-directed networks.

It is important to note that the existence of a definite crossover to a random universality class would give a length scale that marks the extent of network correlations. In principle, if one could further extract a measure of the rate at which networks correlate—i.e. the rate at which the cumulative effects of processes such as erosion, landslides, and diffusion migrate throughout a basin—then one would have an estimate of network age.

Crossover to Geologic Constraints Eventually, basins reach the size and shape of the geologically constrained maximal basin. Here, the scaling may well change again. Several possibilities arise. For a maximal basin that is long and thin, we could have a return to $h = 1$ if interior basins are relatively wide. A maximal basin with an aspect ratio closer to 1 could see h drop back to 1/2. For a basin lying within a wide drainage region where only $\tilde{L}_\parallel$ is set by tectonic controls, the scaling may remain unchanged. Thus, for Hack's law, the most general feature identifiable with geology is this last crossover itself, with a range of scalings as a possibility.

In addition to Hack's law, other scaling laws such as those for area and length probability distributions will pass through corresponding regimes of scaling. Scaling relations will thus hold within certain ranges of basin variables, which should in principle be reconcilable with observations of elements such as drainage density and regional geology.

What we have provided here is an unabashedly heuristic argument for the form of Hack's law. In practice, different regions will have the four scaling regimes present to varying degrees. Clearly, further empirical work on Hack's law is desirable to establish the validity of these claims.

TOPOGRAPHY

We have thus presented a flavor of the rich phenomenology of river networks. But in truth, the surfaces from which these networks derive are a more fundamental indicator of geomorphological evolution. The reasoning is simple: from any elevation field $h(\vec{x})$, a unique drainage network may be constructed; however, no unique elevation field may be associated with a given network.

We now arrive at the question of what to quantify about topography. A crucial point is that once we have a collection of topographic measures, we need to understand how they relate to the scaling laws of river networks. Networks represent connectivity, and their physical and hydrological significance is obvious. But connectivity results from correlations, so explicit consideration of how patches of topography are correlated to other patches of topography should be useful too.

We thus return to the basic measure of topographic connectedness, the height-height correlation function (Equation 26). We now need its dependence on direction, so we write

$$C(\vec{r}) = \langle |h(\vec{x} + \vec{r}) - h(\vec{x})|^2 \rangle_{\vec{x}}^{1/2}. \tag{38}$$

Next, we consider some generic forms of $C(\vec{r})$ and some theoretical models that one may associate with them.

Self-affine Topography

At sufficiently large scales, it is often reasonable to assume that $C(\vec{r})$ is isotropic—i.e. $C(\vec{r}) = C(r)$, where $r = |\vec{r}|$. Over the past two decades, numerous investigations have provided evidence that $C(r) \sim r^{\alpha}$ over some range of length scales, with the roughness exponent α between 0 and 1 (Newman & Turcotte 1990, Turcotte 1997, Mark & Aronson 1984, Matsushita & Ouchi 1989, Ouchi & Matsushita 1992, Chase 1992, Lifton & Chase 1992, Barenblatt et al 1985, Gilbert 1989, Norton & Sorenson 1989). It follows from our discussion of scaling that this implies statistical similarity of the topography, giving $h(x) \simeq b^{-\alpha}h(bx)$. Such topography is called self-affine, and the exponent α may be related to a fractal dimension (Mandelbrot 1983, Barabasi & Stanley 1995, Turcotte 1997).

However, in marked contrast to the situation with network scaling laws, it is extremely rare that α may be unambiguously defined over several orders of magnitude. Figure 10 shows a typical example where no simple straight-line segment may be identified. At small values of r, there is a *tendency* toward a relatively steep slope, and at large values, a relatively small slope. One finds this ambiguity often in the literature. Indeed, in the references just cited, one often finds large values of the roughness exponent ($0.70 \lesssim \alpha \lesssim 0.85$) at small scales, and small values ($0.30 \lesssim \alpha \lesssim 0.55$) at large scales, with the crossover at approximately 1 km, as in Figure 10.

Whether this is a genuine crossover from one type of scaling to another—or even whether one should expect any power law scaling at all—is a subject of much debate, with no conclusion to date. The evidence for scaling is sufficiently good, however, to motivate its theoretical justification.

Next we review classes of stochastic partial differential equations that yield predictions for roughness exponents. Deterministic equations for erosion, though of considerable interest and utility, have not led to such predictions (Smith & Bretherton 1972, Willgoose et al 1991, Howard 1994, Izumi & Parker 1995, Banavar et al 1997). Given that deterministic formulations of fully developed turbulence yield scaling laws approximately consistent with measurements (Frisch 1995), it would be especially interesting to know if any deterministic erosion model could do the same.

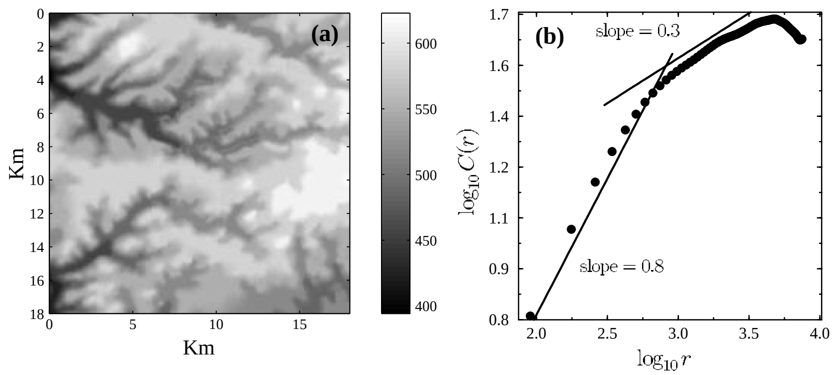

Figure 10 (*a*) Digital elevation map of an area of the Appalachian Plateau, in Northwest Pennsylvania. Elevations are given in meters. The spatial resolution is 90 m. (*b*) Averaged height-height correlation function $C(r)$ for the landscape in Figure 10*a*, where r is oriented in the vertical direction of (*a*). Logarithms are computed from quantities measured in units of meters. Straight lines are given to guide the eye, not to imply power law scaling. From Pastor-Satorras & Rothman (1998b).

Stochastic Equation Models

The simplest nonlinear surface evolution model is given by Equation 24. Introduced by Kardar, Parisi, and Zhang (KPZ) in 1986, its rich phenomenology and the theoretical challenges posed by this model led subsequently to an enormous literature [for reviews, see Krug & Spohn (1992), Halpin-Healy & Zhang (1995), Barabasi & Stanley (1995)]. As we already discussed, the KPZ equation embodies only simple notions of smoothing, stochasticity, and growth normal to the interface. Since the KPZ equation is associated with $0.2 \lesssim \alpha \lesssim 0.4$ for two-dimensional surfaces, and its derivation may be broadly associated with many geologic processes, Sornette & Zhang (1993) proposed it as the generic mechanism responsible for many observations of $0.2 \lesssim \alpha \lesssim 0.4$ made from eroded surfaces.

This raises a conundrum of interest central to this review. If indeed $C(r) \sim r^{\alpha}$ with a KPZ-consistent α, is any progress of geomorphological interest made by its identification with the KPZ universality class? The answer is not obvious; it depends strongly on one's scientific taste as well as on the problem one wishes to solve. Here we advocate a pragmatic point of view: If such a classification allows either the solution or better formulation of a problem, then progress is indeed made.

One straightforward way to proceed is to ask how we may explain the wide variety of circumstances in which α is not close to 0.4. Are there, for example, different universal mechanisms responsible for the aforementioned observations of $0.70 \lesssim \alpha \lesssim 0.85$?

Noting that these large values of α are observed at small length scales, Pastor-Satorras & Rothman (1998a,b) proposed that the differences may result from the inherent anisotropy of fluvial erosion. The central idea, illustrated by Figure 11, is that at small scales an unambiguous downhill (i.e. ‘‘dip’’) direction may be identified. The problem thus contains a preferred direction, and one expects that both the governing dynamics and statistics such as $C(\lambda)$ should reflect this broken symmetry. Below, we review some of the ramifications of this idea.

By assuming (*a*) this asymmetry, (*b*) the conservation of material, and (*c*) stochastic material heterogeneities, the following stochastic equation may be derived:

$$\frac{\partial h}{\partial t} = (\nu + \mu_0)\frac{\partial^2 h}{\partial x_\parallel^2} + \nu\frac{\partial^2 h}{\partial x_\perp^2} + \frac{\mu_2}{3}\frac{\partial^2 h^3}{\partial x_\perp^2} + \eta. \tag{39}$$

This stochastic partial differential equation represents anisotropic linear diffusion (where diffusion is emphasized in the $x_\parallel$-direction) supplemented by a cubic nonlinearity when $\mu_2 \neq 0$. The ‘‘bare’’ diffusivity is given by ν, while the enhancement in the $x_\parallel$ direction is given by μ_0. The nonlinear term is the leading-order nonlinearity of an expansion that takes account of the mean effect of the contributing area on the erosion rate. As in Equation 24, η once again represents an uncorrelated noise, but here we consider it as a function of space only. The addition of noise is crucial because it allows us to make some simple predictions concerning the anisotropy of correlations.

In the absence of nonlinearity (i.e. when $\mu_2 = 0$), a linear anisotropic noisy diffusion equation is obtained. Because the equation is linear, the statistics it yields may be predicted exactly. The most pertinent result is that the ratio of the correlations in the two principal directions scales like

$$\frac{C_\perp}{C_\parallel} \sim \left(\frac{\nu + \mu_0}{\nu}\right). \tag{40}$$

In other words, since the preferred direction gives $\mu_0 > 0$, the topography is quantitatively rougher, at all scales and by the same factor, in the perpendicular

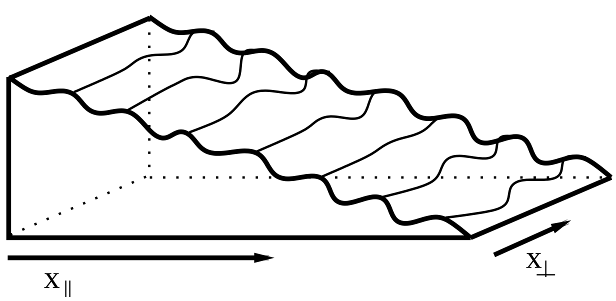

Figure 11 Schematic configuration of an anisotropic landscape.

direction than in the parallel direction. Rough empirical support of this prediction is shown in Figure 12.

The nonlinear case ($\mu_2 > 0$) provides much richer fare. The application of the dynamic renormalization group (e.g. Barabasi & Stanley 1995, Medina et al 1989) shows that the topography should be self-affine, with roughness exponents that depend on direction. That is,

$$C_{\parallel}(x_{\parallel}) \sim x_{\parallel}^{\alpha_{\parallel}} \text{ and } C_{\perp}(x_{\perp}) \sim x_{\perp}^{\alpha_{\perp}} \tag{41}$$

for correlations along fixed transects $x_{\perp}^{0} = $ const. and $x_{\parallel}^{0} = $ const., respectively. First-order estimates of the roughness exponents are

$$\alpha_{\perp} = \frac{5}{6} \simeq 0.83 \text{ and } \alpha_{\parallel} = \frac{5}{8} \simeq 0.63. \tag{42}$$

Evidence for this anisotropic scaling is shown in Figure 13. Figure 13*a* shows submarine topography of a portion of the continental slope off the coast of Oregon. Here the slope results from the relatively abrupt increase in the depth of the seafloor as the continental shelf gives way to the deeper continental rise. The main feature of the topography is a submarine canyon. In this region, submarine canyons are thought to have resulted from seepage-induced slope failure (Orange et al 1994), which occurs when excess pore pressure within the material overcomes the gravitational and friction forces on the surface of the material, causing the slope to become unstable. Slope instabilities then create submarine avalanches, which themselves can erode the slope as they slide downward.

Figure 13*b* shows the plots of $C(x_{\parallel})$ and $C(x_{\perp})$ computed from the topography in Figure 13*a*. One sees that the least-squares estimates of the roughness exponents, $\alpha_{\perp} \simeq 0.78$ and $\alpha_{\parallel} \simeq 0.67$, exhibit a good fit to the theoretical predictions (Equation 42).

We may thus tentatively conclude that Equation 39 provides the identification of two universality classes of anisotropic erosion. The linear case is characterized by the difference in prefactors given by Equation 40, whereas the nonlinear case is characterized by the values of the scaling exponents $\alpha_{\perp}$ and $\alpha_{\parallel}$ given by Equation 42. Empirical work by Chan & Rothman (in preparation) indicates that the former case has wide generality and may be used to identify the characteristic length scale that governs the anisotropy.

Applications to Sedimentology

The "world view" afforded by stochastic surface-evolution equations is most useful when little is known about the detailed dynamical processes that create a surface. As such, this view can help identify the origin of certain geomorphological patterns created by some sedimentary systems. Here we briefly review two such cases.

Our first example is that of turbidite deposition. Turbidites are the sedimentary deposits that result from underwater avalanches known as turbidity currents. Tur-

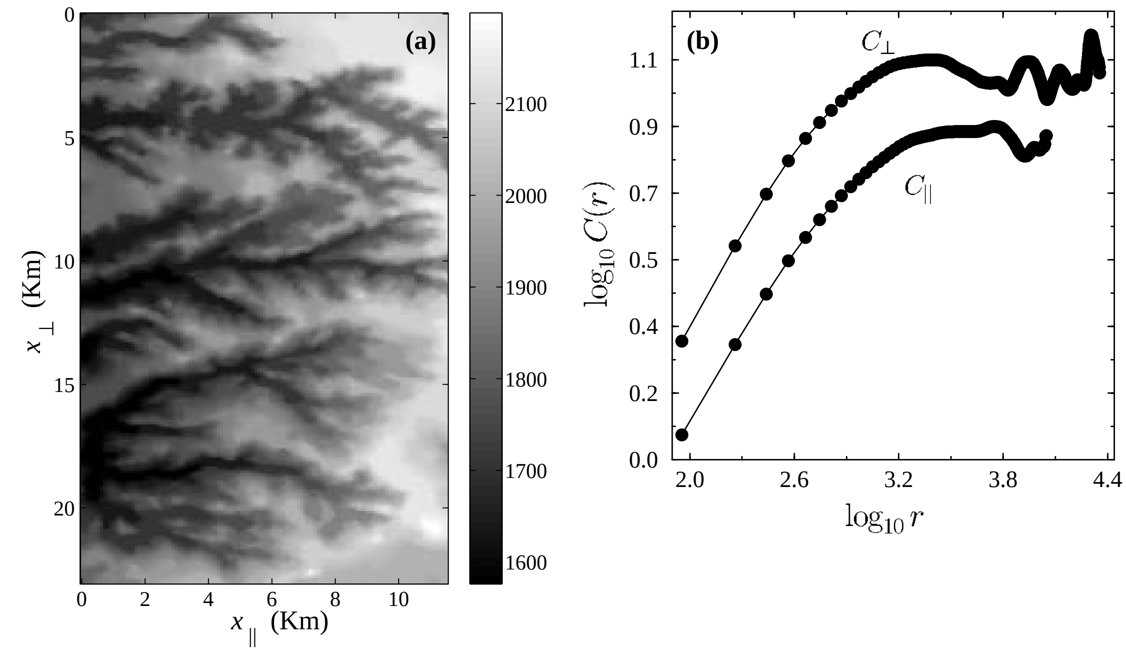

Figure 12 (a) Digital elevation map of an area near Marble Canyon in northeast Arizona. Elevations are given in meters, and the spatial resolution is 90 m. (b) Height-height correlation functions computed along the parallel ($C_\parallel$) and perpendicular ($C_\perp$) directions for the landscape shown in Figure 12a. Logarithms are computed from quantities measured in meters. Since the two correlation functions approximately differ only by a vertical shift along a logarithmic axis, the prediction of Equation 40 is roughly satisfied. From Pastor-Satorras & Rothman (1998a).

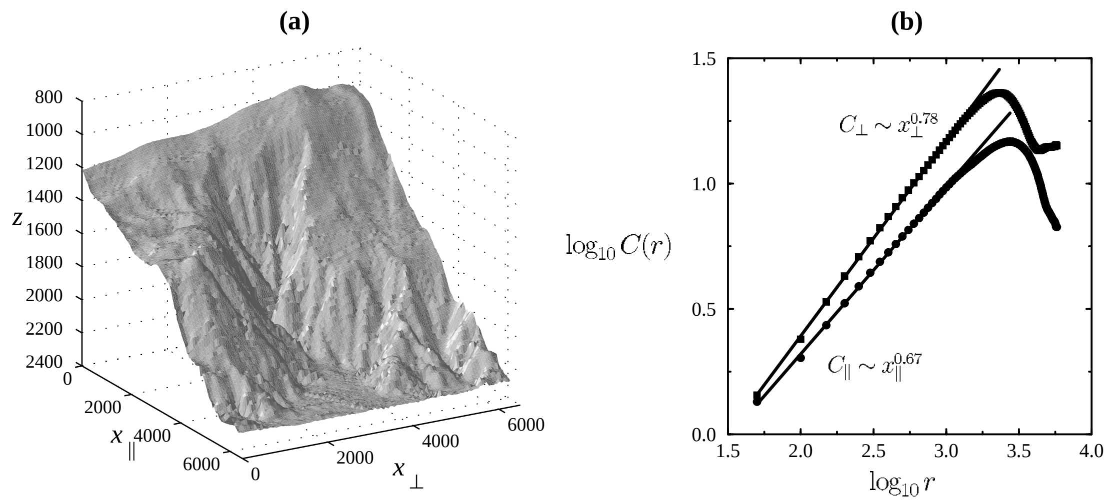

Figure 13 (*a*) Digital map of a submarine canyon off the coast of Oregon, located at coordinates 44°40′ N, 125°45′ W. The vertical axis represents the depth *z* below sea level. The spatial resolution is 50 m. All units are given in meters. (*b*) Height-height correlation functions computed along the parallel ($C_\parallel$) and perpendicular ($C_\perp$) directions for the topography shown in (*a*). Solid lines are least-squares fits to the scaling region. The logarithms are computed from quantities measured in meters. From Pastor-Satorras & Rothman (1998a,b).

bidity currents are often initiated as slope instabilities in submarine canyons, such as the one shown in Figure 13*a*. Empirical studies of the size distribution of turbidite deposits has shown that they may sometimes be characterized by power law distributions (Hiscott et al 1992, Rothman et al 1994, Rothman & Grotzinger 1995). Specifically, one finds that the turbidite event size s can scale like $P(s) \propto s^{-\tau_1}$, with τ_1 a characteristic exponent that depends in part on allometric relations governing the spreading of turbidity currents (Rothman et al 1994, Rothman & Grotzinger 1995).

Because the model given by Equation 39 has successfully characterized the topographic fluctuations of a real submarine canyon, it is natural to ask whether the exponents that characterize the surface roughness can be related to the avalanche size-distribution exponent τ_1. Viewing Equation 39 as a model of forced nonlinear diffusion, Pastor-Satorras & Rothman (1998a) constructed a scaling argument to obtain the scaling relation

$$\tau_1 = 2 - \frac{1}{1 + \alpha_\parallel/\alpha_\perp}. \tag{43}$$

We see that the size-distribution exponent τ_1 depends only on the anisotropy of the correlations via the ratio $\alpha_\parallel/\alpha_\perp$. This equation has not yet been tested, owing to the practical difficulties of obtaining measurements of related canyons and turbidites. It would be of considerable practical interest, however, if one could indeed use relations like Equation 43 to predict turbidite size distributions from the fluctuations of submarine topography.

Our second example is a particularly unusual sedimentary rock known as a stromatolite. Stromatolites are laminated, accretionary structures whose laminae exhibit complex patterns (Walter 1976, Grotzinger & Knoll 1999). An example is shown in Figure 14. The origin of these patterns is unknown, but they are commonly thought to be associated with sediment-binding or precipitation mechanisms induced by ancient microbial mats or biofilms. Because stromatolites as old as 3.5 Gyr have been found, they are considered as evidence for early life on Earth (Schopf 1983). As such, a fundamental understanding of how features of the patterns relate to theories for their origin would be of considerable interest.

Some patterns formed by stromatolites can be characterized as successive single-valued interfacial profiles $h(x)$, where h is the height of the interface and x its position. Grotzinger & Rothman (1996) analyzed such patterns over length scales ranging from centimeters to meters. Their analysis found that the average power spectrum of the height fluctuations followed a scaling law that is equivalent, in real space, to finding $C(r) \propto r^\alpha$, with $\alpha \simeq 0.5$. Reasoning that the stromatolite laminae could have formed by a combination of sedimentary fallout, chemical precipitation, and diffusive smoothing, they concluded that a purely physical model based on the KPZ equation was both plausible and approximately consistent with their data analysis.

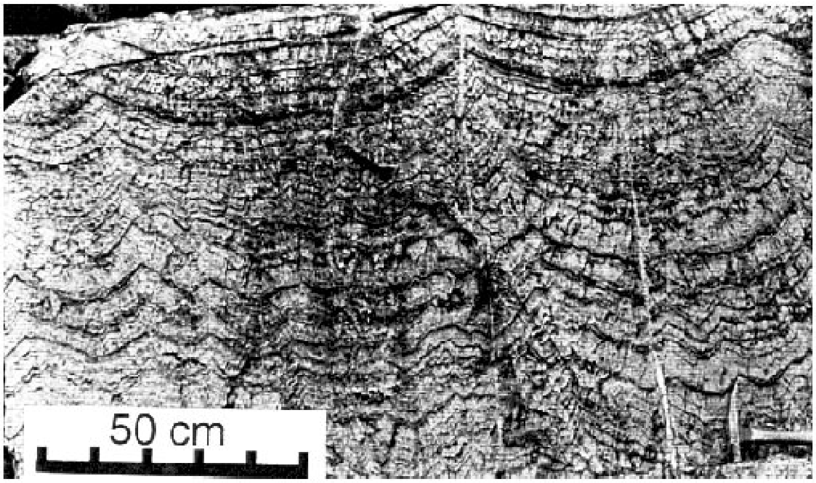

Figure 14 A vertical cross-sectional cut through a stromatolite. The dark jagged lines running roughly horizontally represent the interfaces between successive stromatolite laminae. From Grotzinger & Rothman (1996).

As we can see, an argument based on scaling and universality has shown that certain complex patterns in rocks may derive from physical principles rather than early forms of life. However, it is of crucial importance to note that the mechanisms of the KPZ equation are sufficiently general that they apply as well to the growth of bacteria colonies, for example, as they do to the accumulation of sediment. Thus, what was learned in this stromatolite study was in fact more subtle: It showed that a statistic (i.e. an exponent derived from a power spectrum) can be measured, and that the result could be equally consistent with growth mechanisms that are either biological or physical. Rather than ruling out a particular detailed mechanism, such arguments based on scaling and universality instead frame the debate. This is an important step toward resolving the unfinished business of addressing the precise mechanisms by which these sedimentary structures form (Grotzinger & Knoll 1999).

Topographic Networks

Our last ambition here is to explore the connection between the two main subjects of this review: surfaces and river networks. The partnership between a landscape and its network is a significant and complicated coevolution. In both cases, we have talked about categorization into universality classes. Although we have seen that this is not always possible, and that nature is not filled with perfect scaling laws, we are nevertheless led to consider the link between universality classes of surfaces and networks. Real river networks are defined by surfaces. Therefore, if a surface belongs to class X, then what may we say of its network? This is a question not yet answered, and here we outline some basic ideas.

We can theoretically break down topography into three major classes: surfaces with zero, finite, and infinite correlation length. In addition, and in keeping with our previous observations, we may also specify a degree of anisotropy. We introduce $C_a(\lambda)$, a new correlation function. This is the autocorrelation function of the surface and is related to Equation 38. It is defined as

$$C_a(\vec{r}) = \langle |h(\vec{x} + \vec{r})h(\vec{r})|^2\rangle_{\vec{x}}^{1/2}. \tag{44}$$

A general scaling form of the autocorrelation function may be written as

$$C_a(r) \sim e^{-r/\xi} r^{-\alpha}. \tag{45}$$

Here, ξ is a measure of the extent of correlations in the surface and is referred to as the correlation length. Heights on the surface separated by $r \gg \xi$ are essentially uncorrelated, whereas for $r \ll \xi$, a power law relation holds. Next, we consider how networks behave as a function of ξ and α.

Uncorrelated Surfaces [$\xi = 0$] A surface with no correlations gives rise to a random network. For example, on a two-dimensional lattice we assign to each site a height randomly chosen from the interval [0, 1]. With sufficient tilting of this random carpet, a random directed network will appear (Dodds & Rothman 1999). We would thus have $h(\xi = 0) = 2/3$ and $d(\xi = 0) = 1$. If, however, the carpet is lifted up at its center, we would obtain a random network that is directed radially. Thus, the underlying geologic structure is important in determining the universality class of the network, even in this case of zero correlations.

Correlated Surfaces [$0 < \xi < \infty$] To move away from random networks we must evidently find correlations present in surfaces, and this is apparently true of real landscapes. Consider a surface with a finite correlation length—that is, one where correlations exist but are limited in extent. Such a surface must at large scales exhibit the characteristics of a random one. This is of course true whether or not the correlations are isotropic or anisotropic. The absolute limiting case for the earth is that of correlations on the size of continents, and more typically on the scale set by tectonic action. The barriers to such massive connectedness are strong. Although fluvial activity tends to develop correlations down through a network, it is the transverse extent that is slow to evolve. Diffusive movement of material is well capped by the age of the earth and the slowness of contributing processes.

Crossovers in statistics would therefore be an integral feature here. Although surfaces would be effectively random at large scales, they would follow the statistics of self-affine surfaces for small scales. The magnitude of ξ would dictate the extent of this scaling. In finding transitions in surface correlations, we would expect to see the same in network scaling laws. A finite correlation length evident in $C_a(r)$ should, for example, appear also in Hack's law. Referring back to Figure 9, we see that ξ should be on the order of l_2.

Self-affine Surfaces [$\xi = \infty$] In theory, of course, we do have perfect self-affine surfaces. This appears to be an area that warrants further investigation, and indeed some progress has been made (Goodchild & Klinkenberg 1993). It is conceivable, for example, that scaling relations exist that combine surface and network exponents, i.e. $h = h(\alpha)$ and $d = d(\alpha)$. If so, we would then have an idealized pairing of landscapes and networks to use as a basis for understanding real structures.

Anisotropic Correlations Equation 45 may be straightforwardly generalized to allow for anisotropic correlations. Then we have two correlation lengths $\xi_\perp$ and $\xi_\parallel$ and two scaling exponents $\alpha_\perp$ and $\alpha_\parallel$. Limiting cases may be understood in the case of directed flow. For example, when $\xi_\perp \ll \xi_\parallel$—i.e. when transverse correlations are short relative to correlations in the overall flow direction—then we would expect the hillslope universality class $(h, d) = (1, 1)$. Scaling relations would also extend to become functions of the two scaling exponents, $h = h(\alpha_\perp, \alpha_\parallel)$ and $d = d(\alpha_\perp, \alpha_\parallel)$.

CONCLUSION

This review has advocated the use of simple models for determining how and why geomorphological systems exhibit certain scaling laws. The discussion has been focused almost entirely on theories for the steady-state or final structure of rivers and topography. We have emphasized how assumptions of randomness can be a useful point of departure for developing insight and more sophisticated theories.

Our review leaves many important issues untouched. Foremost among these is the need to formulate models of equivalent simplicity for the dynamic evolution of geomorphic systems. Here stochastic equations for surface growth provide some insight; however, there remains no clear idea of how their predictions of time dependence could be related to available measurements. The numerical simulation of hypothesized erosion equations (e.g. Willgoose et al 1991, Howard 1994) can be useful, but the dependence of results on parameter choices and model details makes its relevance to real systems difficult to ascertain.

In contrast, the applicability of scaling and universality depends simply on whether the qualitative criteria that define universality classes is indeed present in real systems. For example, we have shown that if a landscape is uncorrelated beyond a certain length scale, then precise predictions for network structure beyond that length scale may be made. Indeed, our review has pointed out the existence of several critical length scales that delimit certain scaling regimes. It may well be that the key to extending this approach to dynamics will lie in the identification of how these length scales change with time. The current explosion in the availability of high-resolution digital elevation maps promises that exciting progress toward the resolution of these issues will be made in the near future.

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