

Specimen

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LETTER TO THE EDITOR

On Blum's Four-dimensional Geometric Explanation for the 0·75 Exponent in Metabolic Allometry

It has frequently been empirically observed that the best-fit exponent to which interspecific variation in mass is raised, in relation to energy expenditure (heat loss) of resting animals, is in the region of 0·73–0·75 (Kleiber, 1961; Brody, 1945; Hemmingsen, 1960; Stahl, 1967; Boddington, 1978; Robinson *et al.*, 1983; Lasiewski & Dawson, 1967; Aschoff & Phol, 1970; Calder, 1974). There has been much recent debate (Heusner, 1982, 1983; Donhoffer, 1986; Reiss, 1986; Calder, 1986) about why such empirical exponents exceed a theoretical value of 0·66, which might be anticipated on the basis of dimensional similarity and the fact that heat loss is a surface phenomenon (Rubner, 1883), the so-called "surface law".

McMahon (1973, 1975) has indicated that animals compared across species are not geometrically similar. Larger animals for example have stouter limbs to support their mass in earth's gravity. On this basis a model of "elastic similarity" was developed and this model predicts an interspecific scaling exponent of 0·75 directly.

An alternative approach (Blum, 1977) also predicts the 0·75 exponent directly. The argument by Blum is simple. The surface of an n -dimensional object (A) is a function of the object's volume (V). Generally,

$$A \propto V^{(n-1)/n}, \quad (1)$$

for a 3-D object therefore,

$$A \propto V^{(3-1)/3} \propto V^{2/3}, \quad (2)$$

which is familiar as the surface law of Rubner (1883).

If there were four dimensions, however,

$$A \propto V^{(4-1)/4} \propto V^{3/4}. \quad (3)$$

In other words the area of a four-dimensional object scales to its volume raised to the exponent 0·75 and not 0·66. We can therefore explain the scaling of energy expenditure to mass as a four-dimensional animal replacing heat loss across its surface.

Several subsequent papers have addressed what the physical reality of the fourth dimension may be (Boddington, 1978), and what the other consequences of such four-dimensional geometric modelling may be (Hainsworth, 1981). Recent allometric reviews (Peters, 1983; Calder, 1984; Reiss, 1989) all include Blum's four-dimensional geometry as a possible explanation of the scaling of resting energy expenditure to mass.

The problem with Blum's theory is mostly one of conceptualization (see e.g. Calder, 1984). It is outwith our everyday experience to consider 4-D objects. It is the contention of the present paper that firstly the area of a fourth-dimensional object has no bearing on the arguments relating to heat loss across an animal's body surface, and secondly that the acceptability of Blum's theory has occurred simply because of semantic problems with the terms area and volume, and what these terms actually mean when we are considering objects with higher dimensionality compared with our everyday usage.

To illustrate this problem let us consider Blum's model of the relation between area and volume for a two-dimensional object. Given (1)

$$A \propto V^{2-1/2} \propto V^{1/2}. \quad (4)$$

The eqn (4) states that the area of a 2-D object is proportional to its volume raised to the power 0.5. Yet it is clear that by definition 2-D objects do not have a volume, consequently eqn (4) would also imply they cannot have any area either. It is instructive to consider why this problem occurs with Blum's model.

The number of dimensional attributes an object has is equal to the number of dimensions it has. For example, a one-dimensional object only has the attribute length. A two-dimensional object has a linear attribute and an area. A three-dimensional object has a linear attribute, an area and a volume, etc. The problem with Blum's approach is that he uses terms with everyday meanings (area and volume) to represent something different. In Blum's approach, volume does not mean "the third dimensional attribute of an object" but instead means "the highest dimensional attribute". Similarly the area means not "the second dimensional attribute" but "the next to the highest dimensional attribute".

Once we know this we can readily understand eqn (4). Although this equation actually says the area of a 2-D object is proportional to its volume raised to the power 0.5, what it means is the second highest dimensional attribute of a 2-D object is proportional to the highest dimensional attribute raised to the power 0.5. In other words the length of a 2-D object is proportional to the square root of area, or area is proportional to length squared.

Now that we have cleared up the semantic problems with Blum's use of the terms volume and area we can now consider exactly what eqn (3) means. This equation appears to say that area is proportional to volume raised to the power 0.75, but it actually means the second highest attribute of a four-dimensional object (= volume or mass) is proportional to the highest dimensional attribute (= hypervolume) raised to the power 0.75.

Since heat exchange occurs across the body surface, and body surface plays no part in eqn (3) then this relationship has absolutely no bearing on the relationship between energy expenditure, body mass and the surface law. The error obviously occurs because volume in the equation is not equable with mass (but hypervolume) and area is not equable to body surface (but volume). Indeed since the interrelationships of lower dimensional attributes remain intact, independent of the number of higher dimensions, it remains true that the area (in its everyday usage) of an object is proportional to volume (in its everyday usage) raised to the power 0.66, even if

a higher dimensional reality exists. This latter fact also accords with empirical observations (see Calder, 1984 for review). Blum's model cannot explain the scaling of energy expenditure to body mass.

I am grateful to R. Rucker for writing the book "*The Fourth Dimension and How to Get There*" which teaches the way to understand the relation between the third and fourth dimensions is to consider the relation between the second and third dimensions.

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