Optimal Supply Networks III: Redistribution

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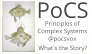
Principles of Complex Systems, Vols. 1, 2, 3D, 4 Fourever, V for Vendetta

Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Institute
University of Vermont | Santa Fe Institute



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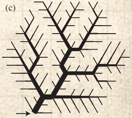

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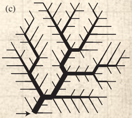
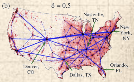


Outline

Many sources, many sinks

How do we distribute sources?



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How do we distribute sources?

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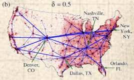

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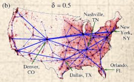

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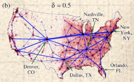

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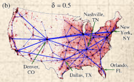
- Obvious: if density is uniform then sources are best distributed uniformly.



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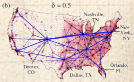

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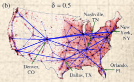
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?



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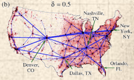
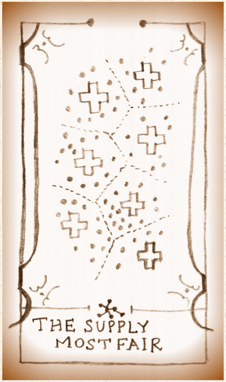
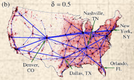
- Obvious: if density is uniform then sources are best distributed uniformly.
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um *et al.* (2009) [6], and work cited by them.



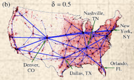

THE JOHN DORY
THE SUN
THE NETWORKS COMPLEX
"I am the Monarch of the Sea."
THE ARK OF THE RANDOM NETWORK
THE NETWORKS FREE OF SCALE
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
THE UNKNOWN MECHANISM
THE BRAID OR THE BRANCH
THE LAW OF OPTIMAL FORKS
THE CHURCH OF QUARTEROLOGY
D-1
D
THE TRUTHICIDE
THE FAMILY OF PANCAKE COWS
THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES


THE SUPPLY MOST FAIR


(b)
δ = 0.5
Nashville, TN
New York, NY
Denver, CO
Dallas, TX
Orlando, FL

Optimal source allocation: Size-density law

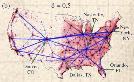
Solidifying the basic problem



Optimal source allocation: Size-density law

Solidifying the basic problem

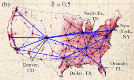
- Given a region with some population distribution ρ, most likely uneven.



Optimal source allocation: Size-density law

Solidifying the basic problem

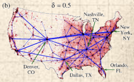
- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.



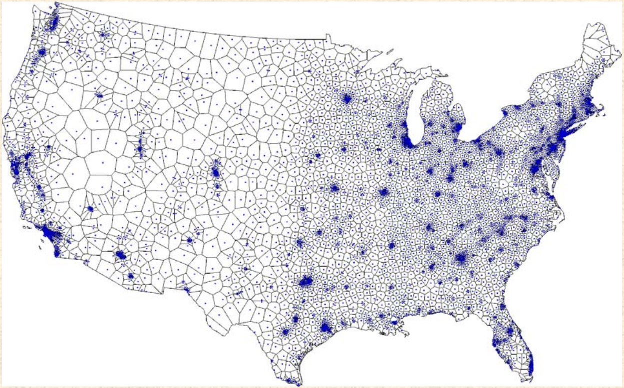
Optimal source allocation: Size-density law

Solidifying the basic problem

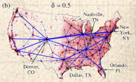
- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?



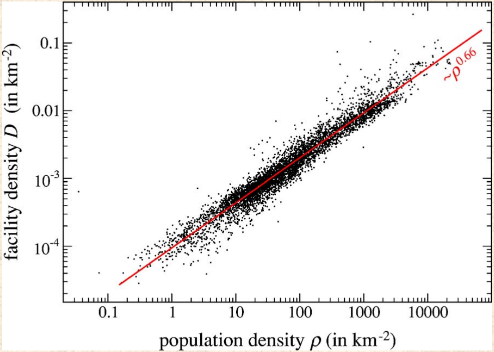
"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



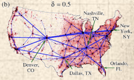
- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.



Optimal source allocation



- Optimal facility density ρ_{fac} vs. population density ρ_{pop}.



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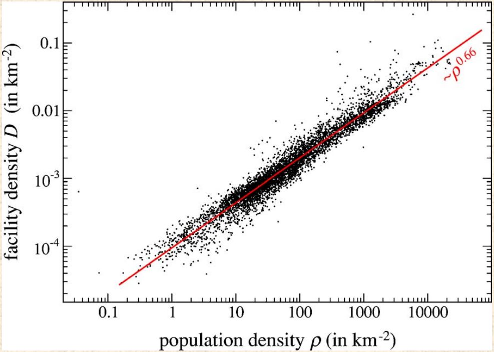

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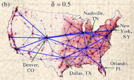
- Fit is $\rho_{\text{fac}} \propto \rho_{\text{pop}}^{0.66}$ with $r^2 = 0.94$.



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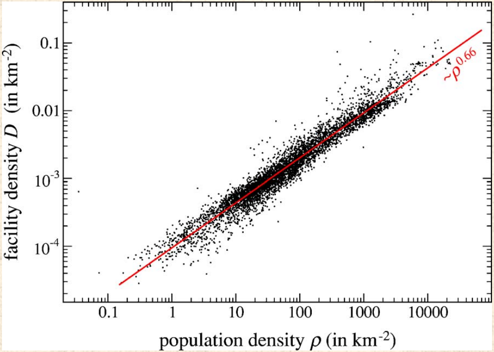

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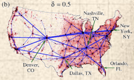
- Looking good for a 2/3 power ...



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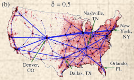
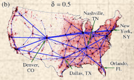


Optimal source allocation

Size-density law:

$$\boxed{\rho_{\text{fac}} \propto \rho_{\text{pop}}^{2/3}}$$



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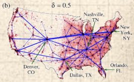
- Why?



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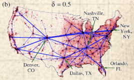
- Again: Different story to branching networks where there was either one source or one sink.



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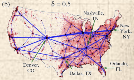
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region.



Optimal source allocation


Territorial Divisions: The Least-Time Constraint Behind the Formation of Subnational Boundaries

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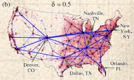
G. Edward Stephan,
Science, **196**, 523–524, 1977. [4]

- We first examine Stephan's treatment (1977) [4, 5]



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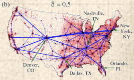

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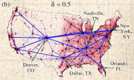
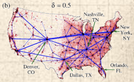
- Zipf-like approach: invokes principle of minimal effort.
- Also known as the Homer Simpson principle.



Optimal source allocation

- Consider a region of area A and population P with a single functional center that everyone needs to access every day.



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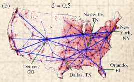
- Build up a general cost function based on time expended to access and maintain center.



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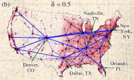

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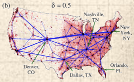

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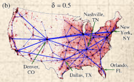
- Assume isometry: average travel distance $\langle d \rangle$ will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\langle d \rangle / \langle v \rangle = cA^{1/2} / \langle v \rangle$$

where c is an unimportant shape factor.



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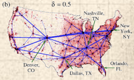

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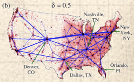
- Call this quantity τ.



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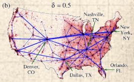

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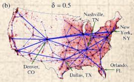
- Replace P by $\rho_{\text{pop}} A$ where ρ_{pop} is density.



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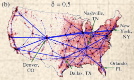

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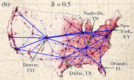
- Total average time cost per person:

$$T = \langle d \rangle / \langle v \rangle + \tau / (\rho_{\text{pop}} A)$$



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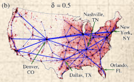

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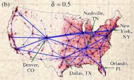
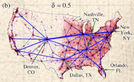
- Now Minimize with respect to A ...



Optimal source allocation

- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\left\langle v \right\rangle + \tau/(\rho_{\text{pop}}A)\right)$$

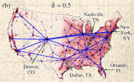


Optimal source allocation

- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\langle v \rangle + \tau/(\rho_{\text{pop}}A)\right)$$

$$= \frac{c}{2\langle v \rangle A^{1/2}} - \frac{\tau}{\rho_{\text{pop}}A^2}$$

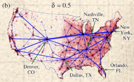


Optimal source allocation

- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\left\langle v\right\rangle + \tau/(\rho_{\text{pop}}A)\right)$$

$$= \frac{c}{2\left\langle v\right\rangle A^{1/2}} - \frac{\tau}{\rho_{\text{pop}}A^{2}} = 0$$



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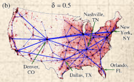

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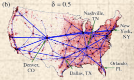

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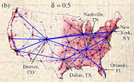

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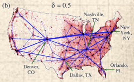
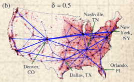


Optimal source allocation

An issue:

- Maintenance (τ) is assumed to be independent of population and area (P and A)

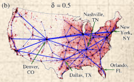


Optimal source allocation

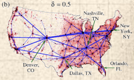
An issue:

- Maintenance (τ) is assumed to be independent of population and area (P and A)

- Stephan's online book "The Division of Territory in Society" is here.
- (It used to be here.)
- The Readme is well worth reading (1995).

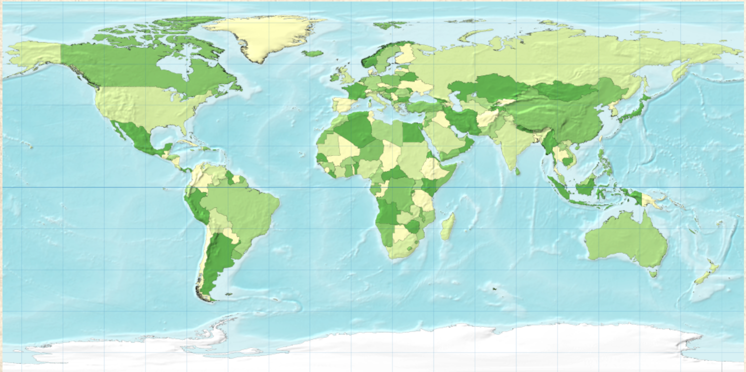


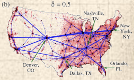
Outline



Cartograms

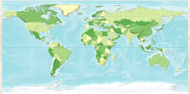
Standard world map:

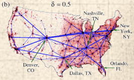




Cartograms

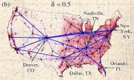
Cartogram of countries 'rescaled' by population:





Cartograms

Diffusion-based cartograms:



Cartograms

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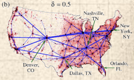
- Idea of cartograms is to distort areas to more accurately represent some local density ρ_{pop} (e.g. population).



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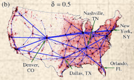

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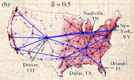

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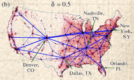

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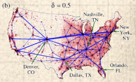
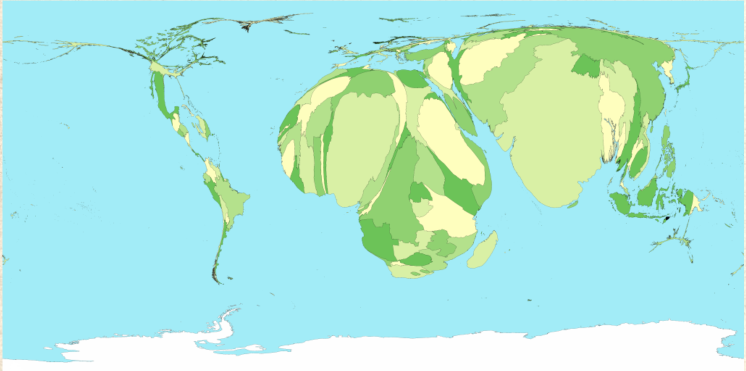
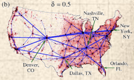
- Allow density to diffuse and trace the movement of individual elements and boundaries.
- Diffusion is constrained by boundary condition of surrounding area having density $\langle \rho \rangle_{\text{pop}}$.



Cartograms

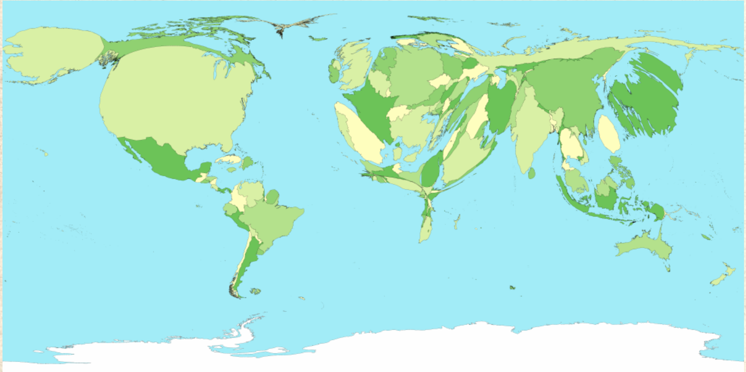
Child mortality:

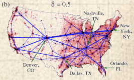




Cartograms

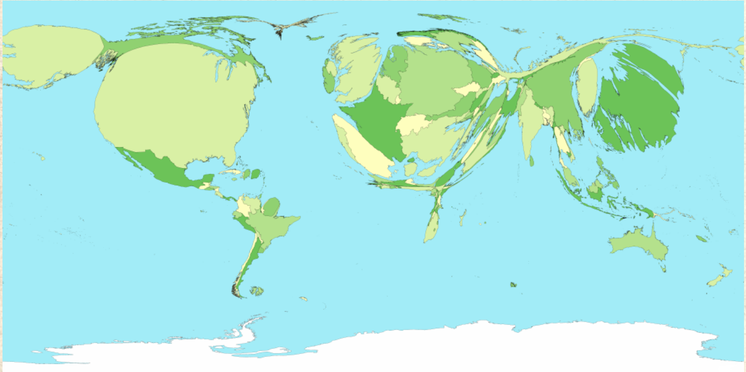
Energy consumption:

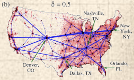




Cartograms

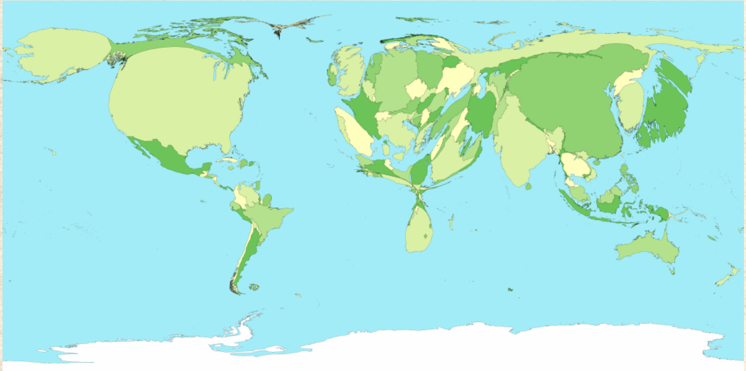
Gross domestic product:

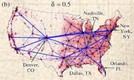




Cartograms

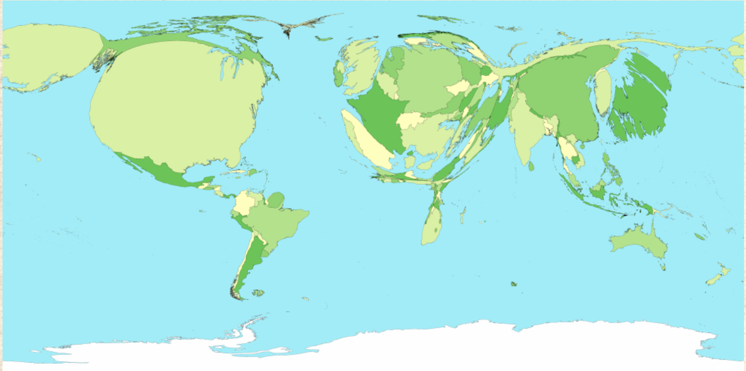
Greenhouse gas emissions:

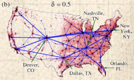




Cartograms

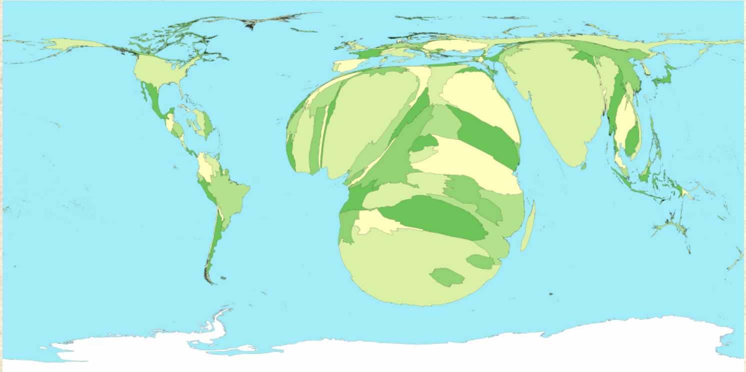
Spending on healthcare:

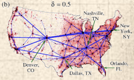




Cartograms

People living with HIV:

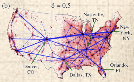




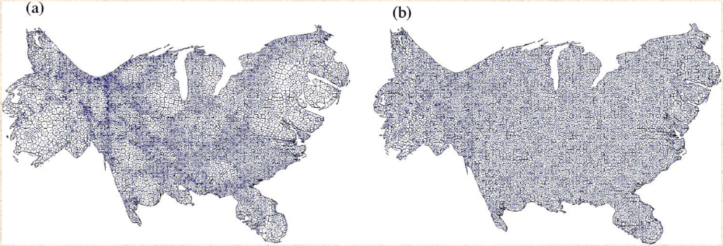
Cartograms

- The preceding sampling of Gastner & Newman's cartograms lives here.
- A larger collection can be found at worldmapper.org.

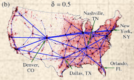
WORLDMAPPER The world as you've never seen it before



"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



- Left: population density-equalized cartogram.



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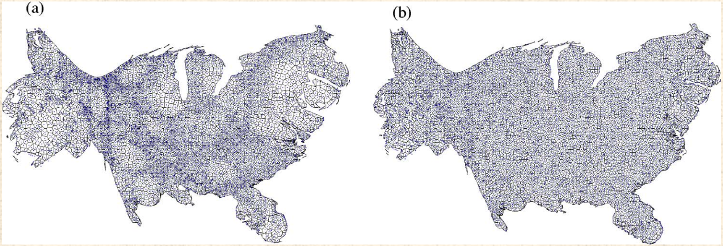
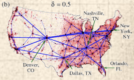


- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.



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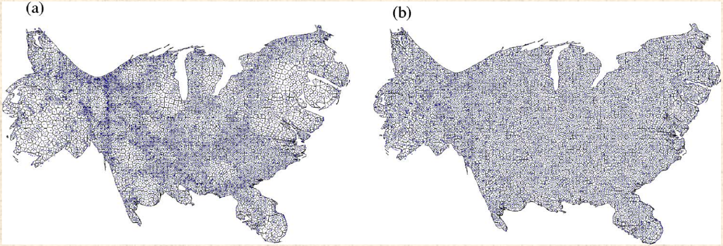
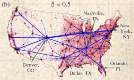
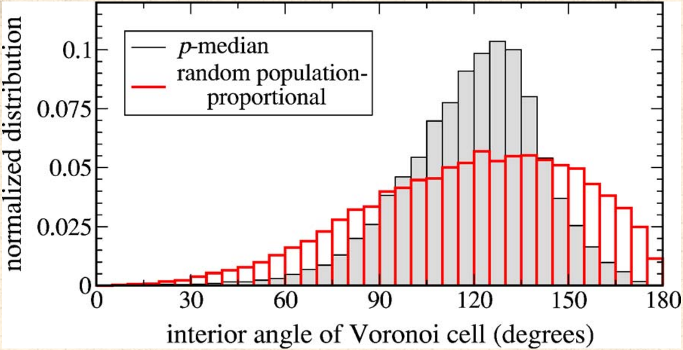


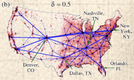
- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho_{\text{pop}}^{2/3}$ cartogram.





From Gastner and Newman (2006) [2]

- Cartogram's Voronoi cells are somewhat hexagonal.



Better cartograms

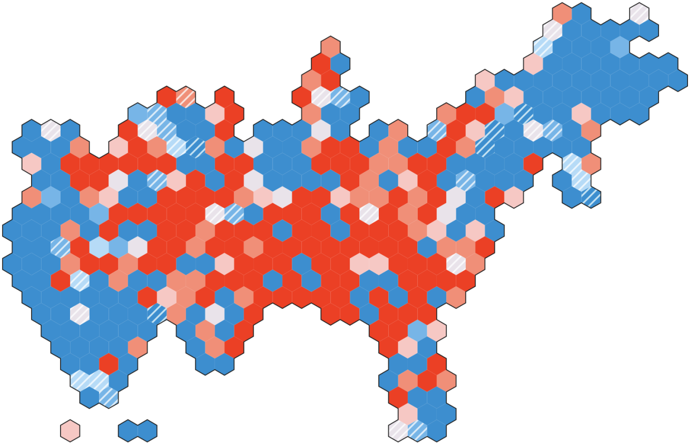
Our forecast for every district

Cartogram | Map

The chance of each candidate winning, with all 435 House districts shown at the same size

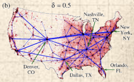
Solid D	Likely D	Lean D	Toss-up	Lean R	Likely R	Solid R	Party flip
≥95% D	≥75% D	≥60% D	<60% both	≥60% R	≥75% R	≥95% R	>50% nonincumbent party

= one district



District totals by category

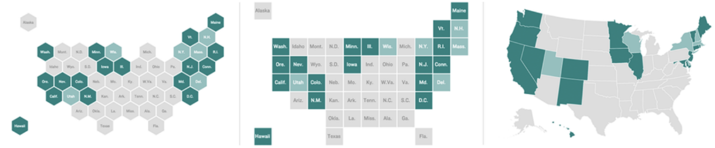
190 18 8 20 24 48 127



Better cartograms:

Let's tesselate: Hexagons for tile grid maps

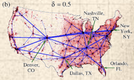
By Danny DeBelius | May 11, 2015



A hexagon tile grid, square tile grid and geographic choropleth map. Maps by Danny DeBelius and Alyson Hurt.

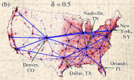
As the saying goes, nothing is certain in this life but death, taxes and requests for geographic data to be represented on a map.

For area data, the choropleth map is a tried and true visualization technique, but not without significant dangers depending on the nature of the data and map areas represented. Clarity of mapped state-level data, for instance, is frequently complicated by the reality that most states in the western U.S. carry far more visual weight than the northeastern states.

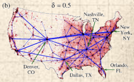


HARB: Hexagons Are The Bestagons

Outline

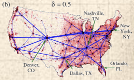


Deriving the optimal source distribution:



Deriving the optimal source distribution:

- **Basic idea:** Minimize the average distance from a random individual to the nearest facility. [2]



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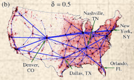
- Assume given a fixed population density ρ_{pop} defined on a spatial region Ω.



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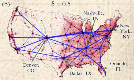
- Formally, we want to find the locations of n sources $\{\vec{x}_1, \ldots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \ldots, \vec{x}_n\}) = \int_{\Omega} \rho_{\text{pop}}(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x} \, .$$



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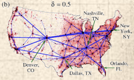
- Also known as the p-median problem, and connected to cluster analysis.



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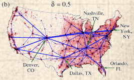
- Also known as the p-median problem, and connected to cluster analysis.
- Not easy ...in fact this one is an NP-hard problem. [2]



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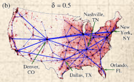
- Basic idea: Minimize the average distance from a random individual to the nearest facility. [2]
- Assume given a fixed population density ρ_{pop} defined on a spatial region Ω.
- Formally, we want to find the locations of n sources $\{\vec{x}_1, \dots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_{\Omega} \rho_{\mathrm{pop}}(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x} \, .$$

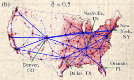
- Also known as the p-median problem, and connected to cluster analysis.
- Not easy ...in fact this one is an NP-hard problem. [2]
- Approximate solution originally due to Gusein-Zade [3].



Size-density law

Approximations:

- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells, one per source.



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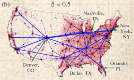

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$$c_i A(\vec{x})^{1/2}$$

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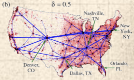

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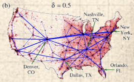
- Approximate c_i as a constant c.



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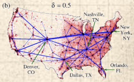
Carrying on:

- The cost function is now

$$F = c \int_{\Omega} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} \, .$$



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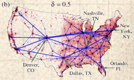

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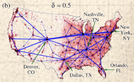
$$\int_{\Omega} \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = n.$$



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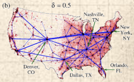

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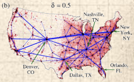
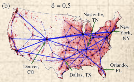
- So ...integral over each of the n cells equals 1.



Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, \ldots, \vec{x}_n\}$, minimize

$$G(A) = c \int_\Omega \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} - \lambda \left(n - \int_\Omega \left[A(\vec{x})\right]^{-1} \mathrm{d}\vec{x} \right)$$



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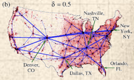

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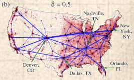

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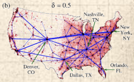

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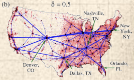
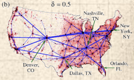


Size-density law

Now a Lagrange multiplier story:

- Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\text{pop}}^{-2/3}.$$



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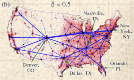

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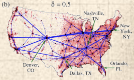

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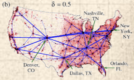
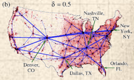
- Normalizing (or solving for λ):

$$\rho_{\text{fac}}(\vec{x}) = n \frac{[\rho_{\text{pop}}(\vec{x})]^{2/3}}{\int_{\Omega} [\rho_{\text{pop}}(\vec{x})]^{2/3} \mathrm{d}\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/3}.$$



Outline

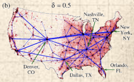


(b)
$\delta = 0.5$
Nashville, TN
New York, NY
Denver, CO
Dallas, TX
Orlando, FL

Global redistribution networks

One more thing:

- How do we supply these facilities?



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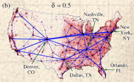

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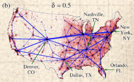

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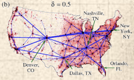

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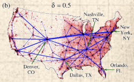

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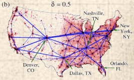
- When $\delta = 1$, only number of hops matters.



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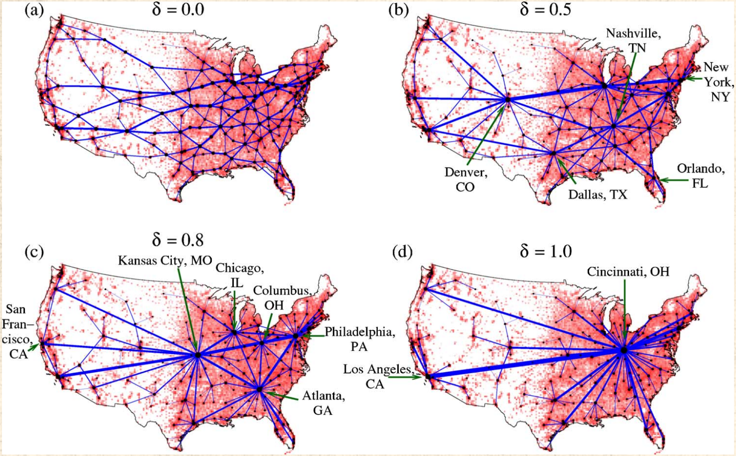
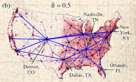


From Gastner and Newman (2006) [2]




THE JOHN DORY
THE SUN
THE NETWORKS COMPLEX
"I am the Monarch of the Sea."
THE ARK OF THE RANDOM NETWORK
THE NETWORKS FREE OF SCALE
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE LANDSCAPES OF FORKING PATHS
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THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
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THE FAMILY OF PANCAKE COWS
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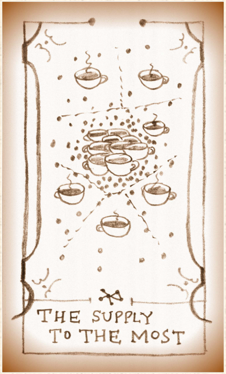
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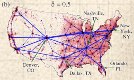
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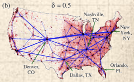
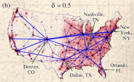


Public versus private facilities

Beyond minimizing distances:



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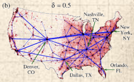

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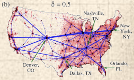

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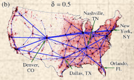
- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;



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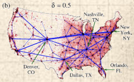
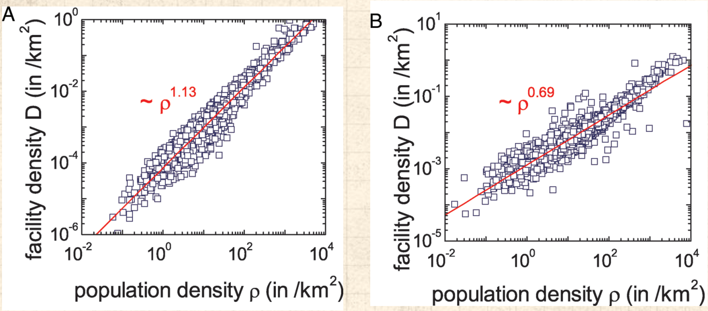
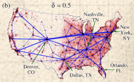
- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;
  2. Pro-social, public facilities: $\alpha = 2/3$.
- Um *et al.* investigate facility locations in the United States and South Korea.



Public versus private facilities: evidence



- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.



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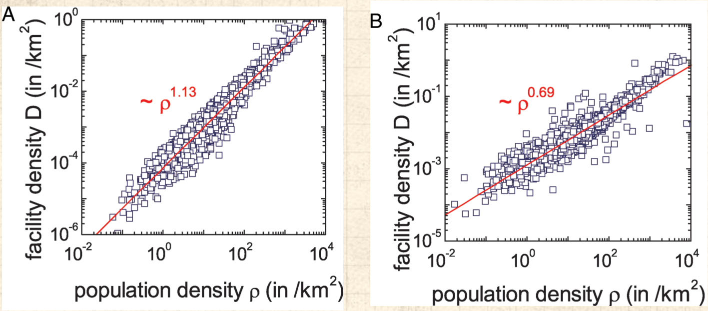
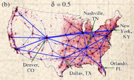


- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho_{\text{pop}} \simeq 100$.



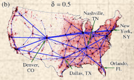
Public versus private facilities: evidence

US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87

SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition between public and private at $\alpha \simeq 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.



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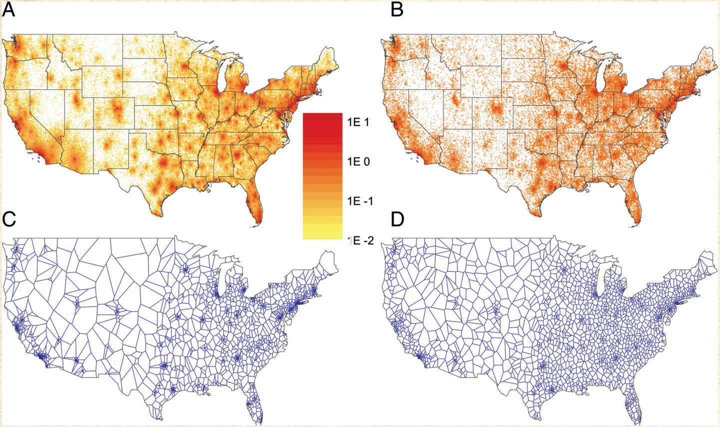
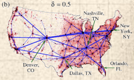


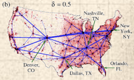
A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.



Public versus private facilities: the story

So what's going on?

- Social institutions seek to minimize distance of travel.



Public versus private facilities: the story

So what's going on?

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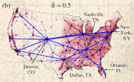
- Commercial institutions seek to maximize the number of visitors.



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- Defns: For the ith facility and its Voronoi cell V_i, define
  - n_i = population of the ith cell;
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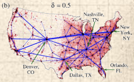

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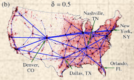
$$v_i = n_i \langle r_i \rangle^{\beta} \text{ with } 0 \leq \beta \leq 1.$$



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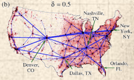
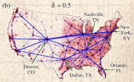
- Limits:
  - $\beta = 0$: purely commercial.
  - $\beta = 1$: purely social.



Public versus private facilities: the story

- Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um *et al.* do, observing that the cost for each cell should be the same, we have:

$$\rho_{\text{fac}}(\vec{x}) = n\frac{[\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}}{\int_{\Omega}[\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}\mathrm{d}\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/(\beta+2)}.$$



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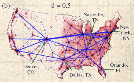
- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.



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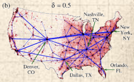
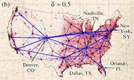
- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.
- For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.



Return to minimizing average cost in time/resources for facility allocation:

- Facility locations: $\{\vec{x}_1, \ldots, \vec{x}_n\}$.



Return to minimizing average cost in time/resources for facility allocation:

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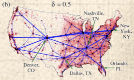
- Vary locations to minimize cost function:

$$F = c \int_{\Omega} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} \,.$$



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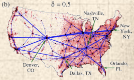

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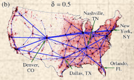

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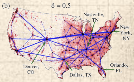


[1] No.

Generalize to m constraints:

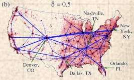
- We line things up more cleanly:

$$G(A) = c \int_{\Omega} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} \; + \lambda \left[A(\vec{x}) \right]^{-1} \mathrm{d}\vec{x} \, .$$

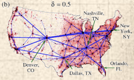
$$G(A) = \int_{\Omega} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{\alpha} \mathrm{d}\vec{x} \; + \lambda_1 \left[A(\vec{x}) \right]^{-\beta_1} \mathrm{d}\vec{x} +$$

$$\lambda_2 \left[A(\vec{x}) \right]^{-\beta_2} \mathrm{d}\vec{x} \; + \cdots + \lambda_m \left[A(\vec{x}) \right]^{-\beta_m} \mathrm{d}\vec{x} \, .$$

- Absorb constants into λs, ignore additive constants.
- $\left[A(\vec{x}) \right]^{-1/d}$: Partition boundary is cost constraint
- $\left[A(\vec{x}) \right]^{-1}$: Number of partitions fixed
- Only constraint with dominant scaling matters (lowest β)



System type:	Dominant cost/benefit scaling:	Dominant constraint scaling:	Scaling of number of events per partition:	Density scaling:	Quantity equalized across partitions:
General form	$\rho_{\text{event}} V^{\alpha}$ $0 < \alpha \leq 1$	$V^{-\beta}$ $1 - \alpha \leq \beta \leq 1$	$N \propto V^{1-\alpha-\beta}$	$\rho_{\text{partition}} \propto \rho_{\text{event}}^{1/(\alpha+\beta)}$	$NV^{\alpha+\beta-1}$
I. Event rate equalizing with partition number constrained (for-profit)	$\sim \rho_{\text{event}} \ln V$	V^{-1}	$N \propto V^{0}$	$\rho_{\text{partition}} \propto \rho_{\text{event}}^{1}$	N
II. Minimizing average event access time with partition number constrained (*p*-median problem, pro-social)	$\rho_{\text{event}} V^{1/d}$	V^{-1}	$N \propto V^{-1/d}$	$\rho_{\text{partition}} \propto \rho_{\text{event}}^{d/(d+1)}$	$NV^{1/d}$
III. System under stochastic threat with partition boundary constrained (HOT model)	$\rho_{\text{event}} V^{1}$	$V^{-1/d}$	$N \propto V^{-1/d}$	$\rho_{\text{partition}} \propto \rho_{\text{event}}^{d/(d+1)}$	$NV^{1/d}$
IV. System under stochastic threat with partition number constrained	$\rho_{\text{event}} V^{1}$	V^{-1}	$N \propto V^{-1}$	$\rho_{\text{partition}} \propto \rho_{\text{event}}^{1/2}$	NV



Paper to be finished.

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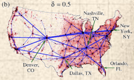

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