Optimal Supply Networks III: Redistribution

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Principles of Complex Systems,
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Outline

Many sources, many sinks

How do we distribute sources?

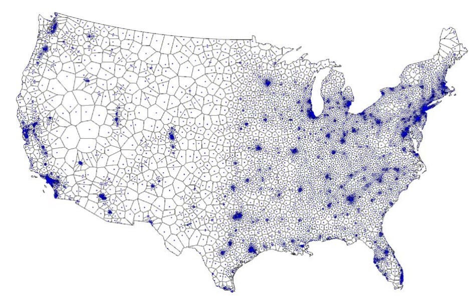
- Focus on 2-d (results generalize to higher dimensions).
- Sources = hospitals, post offices, pubs, ...
- Key problem: How do we cope with uneven population densities?
- Obvious: if density is uniform then sources are best distributed uniformly.
- Which lattice is optimal? The hexagonal lattice
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan (1977, 1984) [4, 5], Gastner and Newman (2006) [2], Um *et al.* (2009) [6], and work cited by them.

Optimal source allocation: Size-density law

Solidifying the basic problem

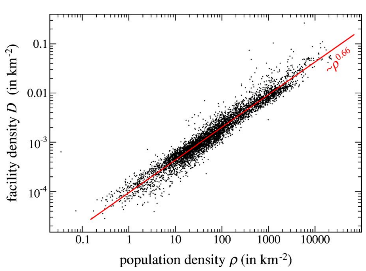
- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

Optimal source allocation



- Optimal facility density ρ_{fac} vs. population density ρ_{pop}.
- Fit is $\rho_{\mathrm{fac}} \propto \rho_{\mathrm{pop}}^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power ...

Optimal source allocation

Size-density law:

- $$\rho_{\mathrm{fac}} \propto \rho_{\mathrm{pop}}^{2/3}$$
- Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region.

Optimal source allocation

"Territorial division: The least-time constraint behind the formation of subnational boundaries"
G. Edward Stephan,
Science, **196**, 523–524, 1977. [4]

- We first examine Stephan's treatment (1977) [4, 5]
- Zipf-like approach: invokes principle of minimal effort.
- Also known as the Homer Simpson principle.

Optimal source allocation

- Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.
- Write average travel distance to center as $\langle d \rangle$ and assume average speed of travel is $\langle v \rangle$.
- Assume isometry: average travel distance $\langle d \rangle$ will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\langle d \rangle / \langle v \rangle = cA^{1/2} / \langle v \rangle$$

where c is an unimportant shape factor.

Optimal source allocation

- Next assume facility requires regular maintenance (person-hours per day).
- Call this quantity τ.
- If burden of mainenance is shared then average cost per person is τ / P where P = population.
- Replace P by $\rho_{\text{pop}} A$ where ρ_{pop} is density.
- Important assumption: uniform density.
- Total average time cost per person:

$$T = \langle d \rangle / \langle v \rangle + \tau / (\rho_{\text{pop}} A) = cA^{1/2} / \langle v \rangle + \tau / (\rho_{\text{pop}} A).$$

- Now Minimize with respect to A ...

Optimal source allocation

- Differentiating ...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A} \left(cA^{1/2} / \langle v \rangle + \tau / (\rho_{\text{pop}} A) \right)$$
$$= \frac{c}{2 \langle v \rangle A^{1/2}} - \frac{\tau}{\rho_{\text{pop}} A^2} = 0$$

- Rearrange:

$$A = \left(\frac{2 \langle v \rangle \tau}{c \rho_{\text{pop}}} \right)^{2/3} \propto \rho_{\text{pop}}^{-2/3}$$

- # facilities per unit area ρ_{fac}:

$$\rho_{\text{fac}} \propto A^{-1} \propto \rho_{\text{pop}}^{2/3}$$

- Groovy ...

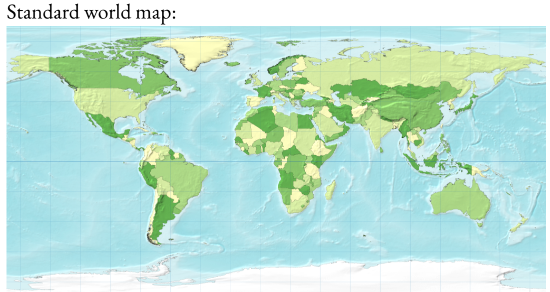
Optimal source allocation

An issue:

- Maintenance (τ) is assumed to be independent of population and area (P and A)
- Stephan's online book "The Division of Territory in Society" is here.
- (It used to be here.)
- The Readme is well worth reading (1995).

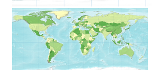
Cartograms

Standard world map:



Cartograms

Cartogram of countries 'rescaled' by population:



Cartograms

Diffusion-based cartograms:

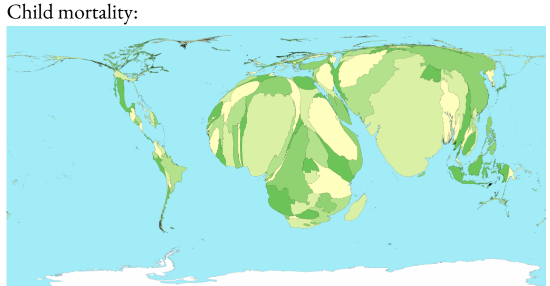
- Idea of cartograms is to distort areas to more accurately represent some local density ρ_{pop} (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- Algorithm due to Gastner and Newman (2004) [1] is based on standard diffusion:

$$\nabla^2 \rho_{\text{pop}} - \frac{\partial \rho_{\text{pop}}}{\partial t} = 0.$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.
- Diffusion is constrained by boundary condition of surrounding area having density $\langle \rho \rangle_{\text{pop}}$.

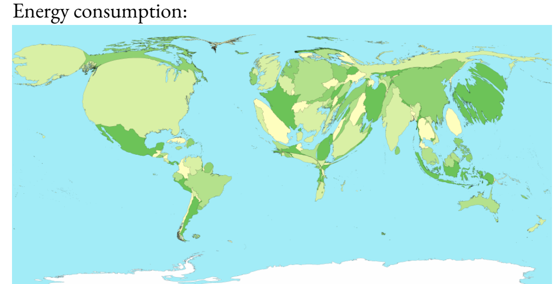
Cartograms

Child mortality:



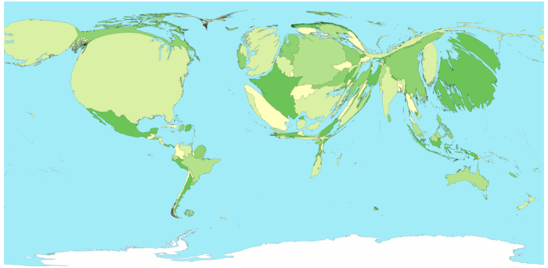
Cartograms

Energy consumption:



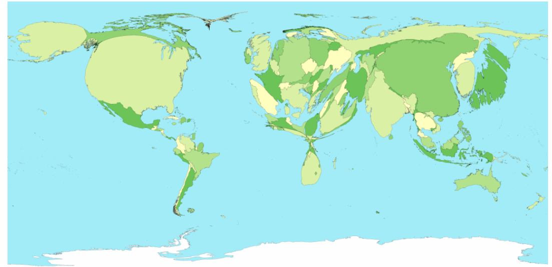
Cartograms

Gross domestic product:



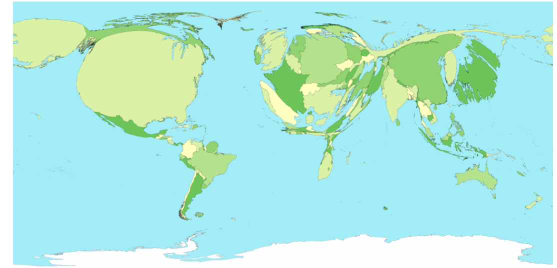
Cartograms

Greenhouse gas emissions:



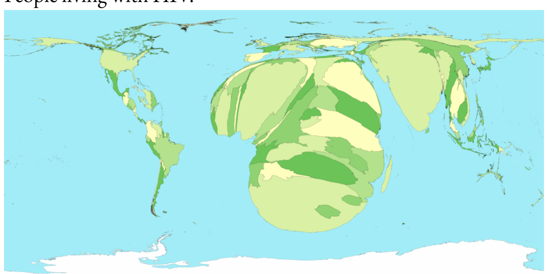
Cartograms

Spending on healthcare:



Cartograms

People living with HIV:

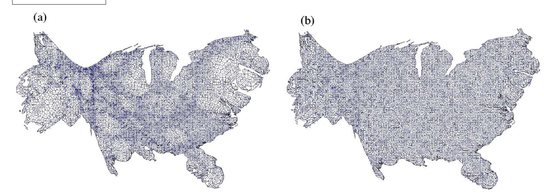


Cartograms

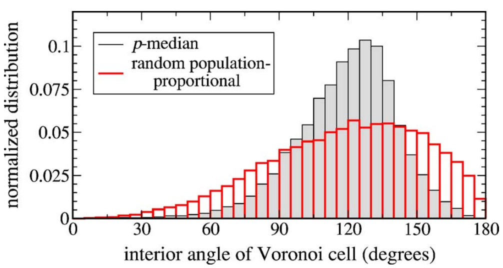
- The preceding sampling of Gastner & Newman's cartograms lives here.
- A larger collection can be found at worldmapper.org.

WORLDMAPPER The world as you've never seen it before

"Optimal design of spatial distribution networks"
Gastner and Newman,
Phys. Rev. E, **74**, 016117, 2006. [2]



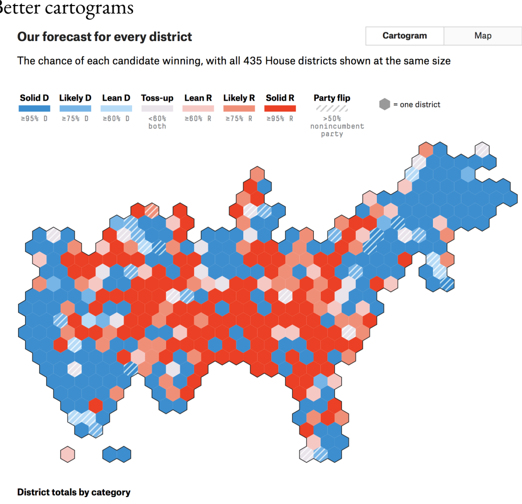
- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho_{\rm pop}^{2/3}$ cartogram.



From Gastner and Newman (2006) [2]

- Cartogram's Voronoi cells are somewhat hexagonal.

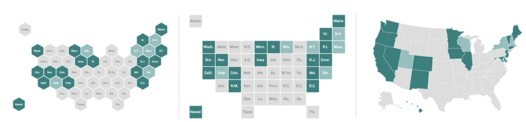
Better cartograms



Better cartograms:

Let's tesselate: Hexagons for tile grid maps

By Danny DeBelius | May 11, 2015



A hexagon tile grid, square tile grid and geographic choropleth map. Maps by Danny DeBelius and Alyson Hurt.

As the saying goes, nothing is certain in this life but death, taxes and requests for geographic data to be represented on a map.

For area data, the choropleth map is a tried and true visualization technique, but not without significant dangers depending on the nature of the data and map areas represented. Clarity of mapped state-level data, for instance, is frequently complicated by the reality that most states in the western U.S. carry far more visual weight than the northeastern states.

HARB: Hexagons Are The Bestagons

Deriving the optimal source distribution:

- Basic idea: Minimize the average distance from a random individual to the nearest facility. [2]
- Assume given a fixed population density ρ_{pop} defined on a spatial region Ω.
- Formally, we want to find the locations of n sources $\{\vec{x}_1, \dots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_\Omega \rho_{\text{pop}}(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \text{d}\vec{x} \, .$$

- Also known as the p-median problem, and connected to cluster analysis.
- Not easy ...in fact this one is an NP-hard problem. [2]
- Approximate solution originally due to Gusein-Zade [3].

Size-density law

Approximations:

- For a given set of source placements $\{\vec{x}_1, \dots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells, one per source.
- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.
- As per Stephan's calculation, estimate typical distance from $\vec{x}$ to the nearest source (say i) as

$$c_i A(\vec{x})^{1/2}$$

where c_i is a shape factor for the ith Voronoi cell.

- Approximate c_i as a constant c.

Size-density law

Carrying on:

- The cost function is now

$$F = c \int_\Omega \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} \text{d}\vec{x} \, .$$

- We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^n A(\vec{x}_i) = A_\Omega$.
- Sneakily turn this into an integral constraint:

$$\int_\Omega \frac{\text{d}\vec{x}}{A(\vec{x})} = n.$$

- Within each cell, $A(\vec{x})$ is constant.
- So ...integral over each of the n cells equals 1.

Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, \dots, \vec{x}_n\}$, minimize

$$G(A) = c \int_\Omega \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{1/2} \text{d}\vec{x} - \lambda \left(n - \int_\Omega [A(\vec{x})]^{-1} \, \text{d}\vec{x} \right)$$

- I Can Haz Calculus of Variations?
- Compute $\delta G / \delta A$, the functional derivative of the functional $G(A)$.
- This gives

$$\int_\Omega \left[\frac{c}{2} \rho_{\text{pop}}(\vec{x}) A(\vec{x})^{-1/2} - \lambda [A(\vec{x})]^{-2} \right] \text{d}\vec{x} \; = 0.$$

- Setting the integrand to be zilch, we have:

$$\rho_{\text{pop}}(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$

Size-density law

Now a Lagrange multiplier story:

- Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho_{\text{pop}}^{-2/3}.$$

- Finally, we indentify $1/A(\vec{x})$ as $\rho_{\text{fac}}(\vec{x})$, an approximation of the local source density.
- Substituting $\rho_{\text{fac}} = 1/A$, we have

$$\rho_{\text{fac}}(\vec{x}) = \left(\frac{c}{2\lambda} \rho_{\text{pop}} \right)^{2/3}.$$

- Normalizing (or solving for λ):

$$\rho_{\text{fac}}(\vec{x}) = n \frac{[\rho_{\text{pop}}(\vec{x})]^{2/3}}{\int_\Omega [\rho_{\text{pop}}(\vec{x})]^{2/3} \text{d}\vec{x}} \propto [\rho_{\text{pop}}(\vec{x})]^{2/3}.$$

Global redistribution networks

One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gastner and Newman model: cost is a function of basic maintenance and travel time:

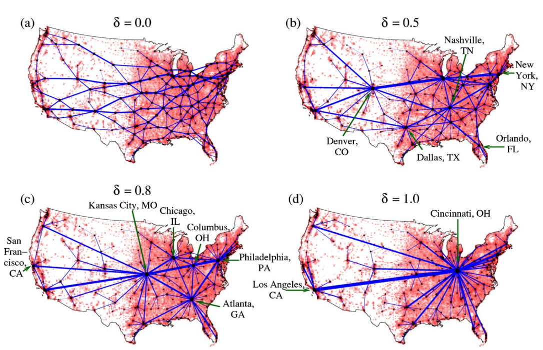
$$C_{\text{maint}} + \gamma C_{\text{travel}}.$$

- Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

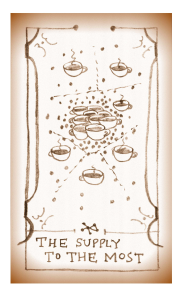
$$(1 - \delta)\ell_{ij} + \delta(\#\text{hops}).$$

- When $\delta = 1$, only number of hops matters.

Global redistribution networks



From Gastner and Newman (2006) [2]



Public versus private facilities

Beyond minimizing distances:

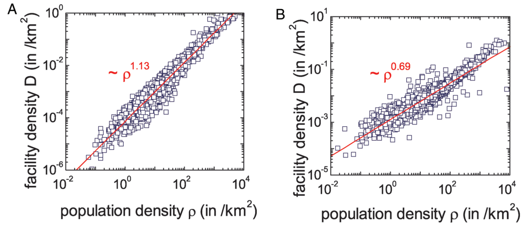
- "Scaling laws between population and facility densities" by Um *et al.*, Proc. Natl. Acad. Sci., 2009. [6]
- Um *et al.* find empirically and argue theoretically that the connection between facility and population density

$$\rho_{\text{fac}} \propto \rho_{\text{pop}}^\alpha$$

does not universally hold with $\alpha = 2/3$.

- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;
  2. Pro-social, public facilities: $\alpha = 2/3$.
- Um *et al.* investigate facility locations in the United States and South Korea.

Public versus private facilities: evidence



- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho_{\rm pop} \simeq 100$.

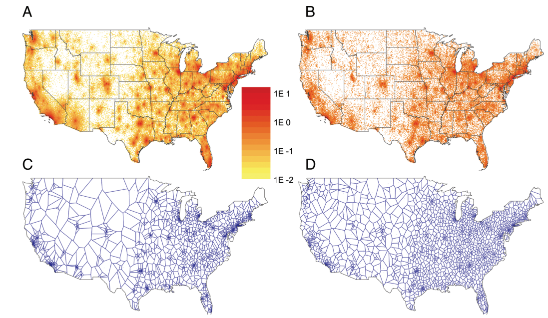
Public versus private facilities: evidence

US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87
SK facility	**α (SE)**	**R^2**
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition between public and private at $\alpha \simeq 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.

Public versus private facilities: evidence



A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

Public versus private facilities: the story

So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- Defns: For the ith facility and its Voronoi cell V_i, define
  - n_i = population of the ith cell;
  - $\langle r_i \rangle$ = the average travel distance to the ith facility.
  - A_i = area of ith cell (s_i in Um *et al.* [6])
- Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^\beta \text{ with } 0 \le \beta \le 1.$$

- Limits:
  - $\beta = 0$: purely commercial.
  - $\beta = 1$: purely social.

Public versus private facilities: the story

- Either proceeding as per the Gastner-Newman-Gusein-Zade calculation or, as Um *et al.* do, observing that the cost for each cell should be the same, we have:

$$\rho_{\rm fac}(\vec{x}) = n \frac{[\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)}}{\int_\Omega [\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)} \mathrm{d}\vec{x}} \propto [\rho_{\rm pop}(\vec{x})]^{2/(\beta+2)}.$$

- For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.
- For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.

Return to minimizing average cost in time/resources for facility allocation:

- Facility locations: $\{\vec{x}_1, \ldots, \vec{x}_n\}$.
- Vary locations to minimize cost function:

$$F = c \int_\Omega \rho_{\rm pop}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x}\,.$$

- Constraint function is on the number of facilities:

$$n = \int_\Omega \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = \int_\Omega [A(\vec{x})]^{-1}\, \mathrm{d}\vec{x}$$

- Issue: Will these facilities all be the same size?[1]

[1] No.

Generalize to m constraints:

- We line things up more cleanly:

$$G(A) = c \int_\Omega \rho_{\rm pop}(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} \;+ \lambda\, [A(\vec{x})]^{-1}\, \mathrm{d}\vec{x}\,.$$

$$G(A) = \int_\Omega \rho_{\rm pop}(\vec{x}) A(\vec{x})^{\alpha} \mathrm{d}\vec{x} \;+ \lambda_1\, [A(\vec{x})]^{-\beta_1}\, \mathrm{d}\vec{x} \;+$$

$$\lambda_2\, [A(\vec{x})]^{-\beta_2}\, \mathrm{d}\vec{x} \;+ \cdots + \lambda_m\, [A(\vec{x})]^{-\beta_m}\, \mathrm{d}\vec{x}\,.$$

- Absorb constants into λs, ignore additive constants.
- $[A(\vec{x})]^{-1/d}$: Partition boundary is cost constraint
- $[A(\vec{x})]^{-1}$: Number of partitions fixed
- Only constraint with dominant scaling matters (lowest β)

System type:	Dominant cost/benefit scaling:	Dominant constraint scaling:	Scaling of number of events per partition:	Density scaling:	Quantity equalized across partitions:
General form	$\rho_{\rm event} V^\alpha$ $0 < \alpha \le 1$	$V^{-\beta}$ $1-\alpha \le \beta \le 1$	$N \propto V^{1-\alpha-\beta}$	$\rho_{\rm partition} \propto \rho_{\rm event}^{1/(\alpha+\beta)}$	$NV^{\alpha+\beta-1}$
I. Event rate equalizing with partition number constrained (for-profit)	$\sim \rho_{\rm event} \ln V$	V^{-1}	$N \propto V^0$	$\rho_{\rm partition} \propto \rho_{\rm event}^{1}$	N
II. Minimizing average event access time with partition number constrained (*p*-median problem, pro-social)	$\rho_{\rm event} V^{1/d}$	V^{-1}	$N \propto V^{-1/d}$	$\rho_{\rm partition} \propto \rho_{\rm event}^{d/(d+1)}$	$NV^{1/d}$
III. System under stochastic threat with partition boundary constrained (HOT model)	$\rho_{\rm event} V^1$	$V^{-1/d}$	$N \propto V^{-1/d}$	$\rho_{\rm partition} \propto \rho_{\rm event}^{d/(d+1)}$	$NV^{1/d}$
IV. System under stochastic threat with partition number constrained	$\rho_{\rm event} V^1$	V^{-1}	$N \propto V^{-1}$	$\rho_{\rm partition} \propto \rho_{\rm event}^{1/2}$	NV

Paper to be finished.

References I

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