Optimal Supply Networks I: Branching

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Outline

Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people, ...
- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
  3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems

Single source optimal supply

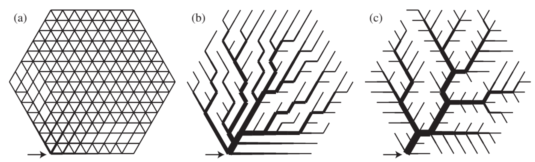
Basic question for distribution/supply networks:

- How does flow behave given cost:

$$C = \sum_j I_j^\gamma Z_j$$

  where
  I_j = current on link j
  and
  Z_j = link j's impedance.
- Example: $\gamma = 2$ for electrical networks.

Single source optimal supply



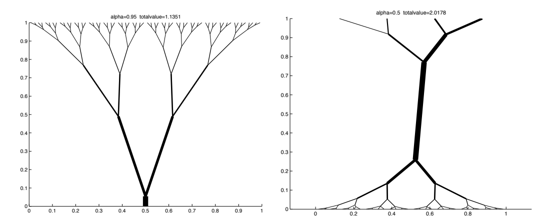
(a) $\gamma > 1$: Braided (bulk) flow
(b) $\gamma < 1$: Local minimum: Branching flow
(c) $\gamma < 1$: Global minimum: Branching flow

- Note: This is a single source supplying a region.

From Bohn and Magnasco [3]
See also Banavar *et al.* [1]: "Topology of the Fittest Transportation Network"; focus is on presence or absence of loops—same story

Single source optimal supply

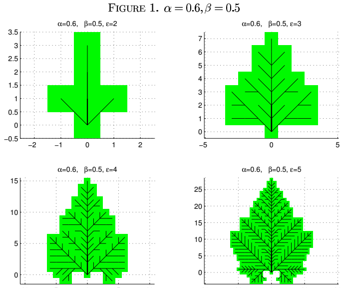
Optimal paths related to transport (Monge) problems:



"Optimal paths related to transport problems"
Qinglan Xia,
Communications in Contemporary Mathematics, **5**, 251–279, 2003. [20]

Growing networks—two parameter model: [21]

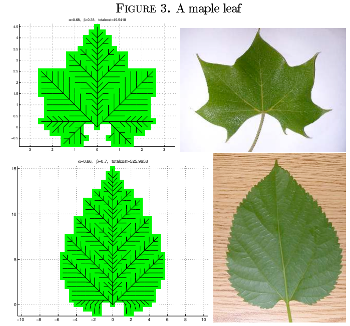
FIGURE 1. $\alpha = 0.6, \beta = 0.5$



- Parameters control impedance ($0 \leq \alpha < 1$) and angles of junctions ($0 < \beta$)
- For this example: $\alpha = 0.6$ and $\beta = 0.5$

Growing networks: [21]

FIGURE 3. A maple leaf



- Top: $\alpha = 0.66$, $\beta = 0.38$; Bottom: $\alpha = 0.66$, $\beta = 0.70$

Single source optimal supply

An immensely controversial issue ...

- The form of natural branching networks: Random, optimal, or some combination? [6, 19, 2, 5, 4]
- River networks, blood networks, trees, ...

Two observations:

- Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- Real networks differ in details of scaling but reasonably agree in scaling relations.

River network models

Optimality:

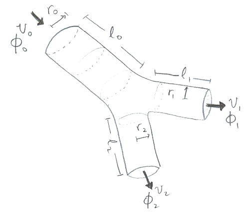
- Optimal channel networks [13]
- Thermodynamic analogy [14]

versus …

Randomness:

- Scheidegger's directed random networks
- Undirected random networks

Optimization—Murray's law



- Murray's law (1926) connects branch radii at forks: [11, 10, 12, 7, 17]

$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$

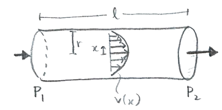
where r_{parent} = radius of 'parent' branch, and $r_{\text{offspring1}}$ and $r_{\text{offspring2}}$ are radii of the two 'offspring' sub-branches.

- Holds up well for outer branchings of blood networks [15].
- Also found to hold for trees [12, 8] when xylem is not a supporting structure [9].
- See D'Arcy Thompson's "On Growth and Form" for background and general inspiration [16, 17].

- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux.



- Fluid mechanics: Poiseuille impedance for smooth Poiseuille flow in a tube of radius r and length ℓ:

$$Z = \frac{8\eta\ell}{\pi r^4}$$

- η = dynamic viscosity (units: $ML^{-1}T^{-1}$).
- Power required to overcome impedance:

$$P_{\text{drag}} = \Phi\Delta p = \Phi^2 Z.$$

- Also have rate of energy expenditure in maintaining blood given metabolic constant c:

$$P_{\text{metabolic}} = cr^2\ell$$

Optimization—Murray's law

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- Δp = Pressure differential = Force per unit area
- Φ = Volume flow per unit time (current) = cross-sectional area · velocity
- S o $\Phi\Delta p$ = Force · velocity

Optimization—Murray's law

Murray's law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2\frac{8\eta\ell}{\pi r^4} + cr^2\ell$$

- Observe power increases linearly with ℓ
- But r's effect is nonlinear:
  - increasing r makes flow easier but increases metabolic cost (as r^2)
  - decreasing r decrease metabolic cost but impedance goes up (as r^{-4})

Optimization—Murray's law

Murray's law:

- Minimize P with respect to r:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2\frac{8\eta\ell}{\pi r^4} + cr^2\ell\right)$$

- Flow rates at each branching have to add up (else our organism is in serious trouble …):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

Optimization—Murray's law

Murray's law:

- Find:

$$\Phi = kr^3$$

- Insert assignment question
- All of this means we have a groovy cube-law:

$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$

Murray meets Tokunaga:

- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = kr_\omega^3$

$$(r_\omega)^3 = 2\left(r_{\omega-1}\right)^3 + \sum_{k=1}^{\omega-1} T_k\left(r_{\omega-k}\right)^3$$

- Same form as:

$$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega}\underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$

Optimization

Murray meets Tokunaga:

- Find Horton ratio for vessel radius $R_r = r_\omega/r_{\omega-1}$.
- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$R_r^3 = R_n = R_v$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

Optimization

Murray meets Tokunaga:

- Isometry: $V_\omega \propto \ell_\omega^3$
- Gives

$$\boxed{R_\ell^3 = R_r^3 = R_n = R_v}$$

- We need one more constraint ...
- West *et al.* (1997) [19] achieve similar results following Horton's laws (but this work is a disaster).
- So does Turcotte *et al.* (1998) [18] using Tokunaga (sort of).

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