Scale-free networks

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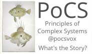

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Outline

Scale-free networks
Main story
Model details
Analysis
A more plausible mechanism
Robustness
Krapivsky & Redner's model
Generalized model
Analysis
Universality?
Sublinear attachment kernels
Superlinear attachment kernels
Nutshell
References


THE SUN
THE GOLDEN AGE OF REDUCTIONISM
THE MANIFESTO
WELL OF SCALES
THE LAWS OF DECAYING POWER
THE UNEXPECTED TRANSFIGURATION
MINISTRY OF RANDOM WALKS
THE RICH GET RICHER
THE DREAM OF EFFICIENT LANGUAGE
THE LAW OF DIGITS FIRST
THE SURPRISE OF BEING ROBUST YET FRAGILE
THE MISREADING OF THE BOOK OF SAND
THE MANY TAILS OF MANY TAILS
THE MEASURING
THE DATA POOR
THE POURING DATA
THE CHARMS OF THE MANY WRONG PATHS
THE BOOKS THAT SHOULD NOT BE OPENED
THE KNOWLEDGE OF DEAD MONKS
THE EMERGENCE OF STRUCTURE
THE EMERGENCE OF DESTRUCTION
THE EMERGENCE OF THINKING
THE EMERGENCE OF STORIES
THE MECHANICS STATISTICAL
THE THRESHOLDS OF PERCOLATION
THE NETWORKS COMPLEX
THE ELEMENTS OF WEBS
THE BIRDS OF A FEATHER
THE BIRDS OF NOT A FEATHER
THE ARK OF THE RANDOM NETWORK
THE STRANGENESS OF FRIENDS
THE SMALLNESS OF WORLDS
THE THEORY OF SIX DEGREES
THE CONTAGION
THE CONFUSION OF CONTAGIONS
THE DISEASE OF RANDOM MIXING
THE UNENDING PANDEMICS
THE THRESHOLDS OF THE MIND
THE SOCIAL WILD
THE SOCIAL CONTAGION
THE RATIO OF GAIN
THE MECHANICS OF FAME
THE TALK OF FAME
THE STORIES CONTAGIOUS
THE JOHN DORY
THE ADJACENT NARRATIVES
THE STORYTELLERS
THE FATED ASTRAL PATH
THE PATHS OF UNIVERSALITY
THE DEPENDENT PATHS
THE TROUSERS OF BIFURCATING REALITIES
THE UNKNOWN MECHANISM
THE LANDSCAPES OF FORKING PATHS
THE NETWORKS OF BLOOD
THE TREES OF REALITY
THE ORDERS OF STREAMS
THE LAWS OF BRANCHING
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THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS


THE NETWORKS FREE OF SCALE

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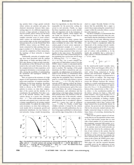

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Times cited: $\sim 43,853$ (as of May 19, 2023)

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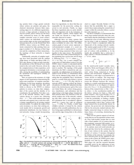

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- Somewhat misleading nomenclature ...

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- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...

Some real data (we are feeling brave):

From Barabási and Albert's original paper [2]:

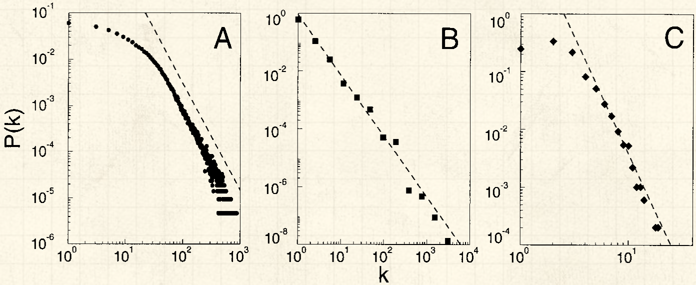
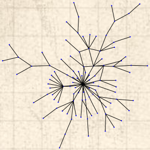
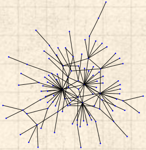


Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

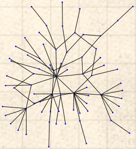
Random networks: largest components



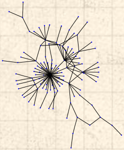
$\gamma = 2.5$
$\langle k \rangle = 1.8$



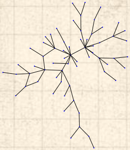
$\gamma = 2.5$
$\langle k \rangle = 2.05333$



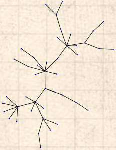
$\gamma = 2.5$
$\langle k \rangle = 1.66667$



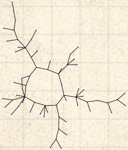
$\gamma = 2.5$
$\langle k \rangle = 1.92$



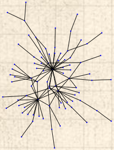
$\gamma = 2.5$
$\langle k \rangle = 1.6$



$\gamma = 2.5$
$\langle k \rangle = 1.50667$



$\gamma = 2.5$
$\langle k \rangle = 1.62667$



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A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

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- Yes, we've seen this all before in Simon's model.

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where $N(t) = m_0 + t$ is # nodes at time t
and $N_k(t)$ is # degree k nodes at time t.

Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.

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- Next find c_i ...

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- Clearly, a Ponzi scheme.

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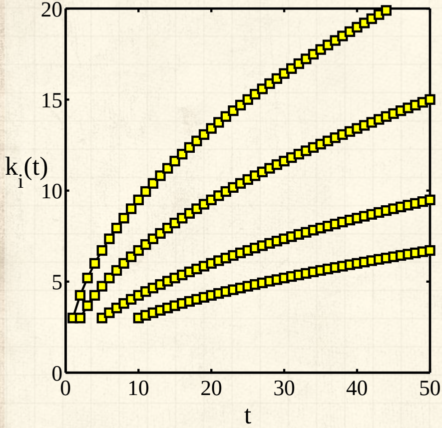
- Our connection $\alpha = 1/(\gamma - 1)$ or $\gamma = 1 + 1/\alpha$ then gives

$$\boxed{\gamma = 1 + 1/(1/2) = 3.}$$



- $m = 3$
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Back to that real data:

From Barabási and Albert's original paper [2]:

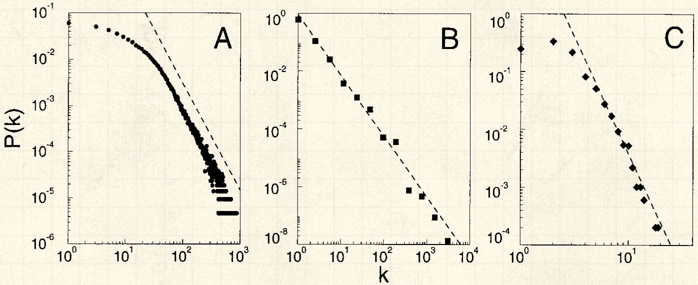

Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\mathrm{actor}} = 2.3$, (B) $\gamma_{\mathrm{www}} = 2.1$ and (C) $\gamma_{\mathrm{power}} = 4$.

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The Internets is a different business...

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- So rich-gets-richer scheme can now be seen to work in a natural way.

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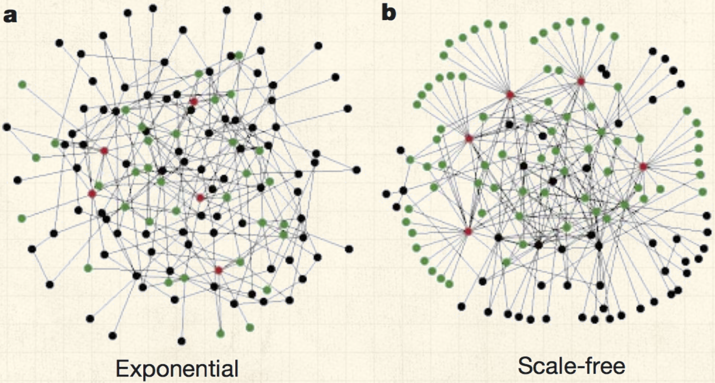
Robustness

- Albert et al., Nature, 2000:
  "Error and attack tolerance of complex networks" [1]
- Standard random networks (Erdős-Rényi)
  versus Scale-free networks:



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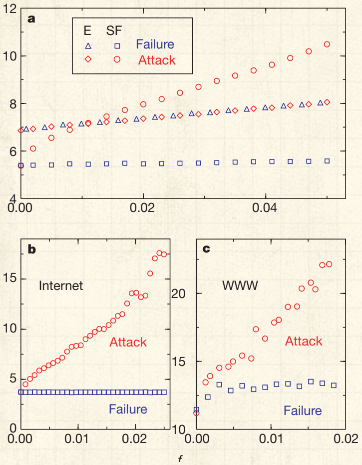

from Albert et al., 2000

- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

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- Need to explore cost of various targeting schemes.

Robustness

Not a robust paper:

"The "Robust yet Fragile" nature of the Internet"
Doyle et al.,
Proc. Natl. Acad. Sci., **2005**, 14497–14502, 2005. [3]

- HOT networks versus scale-free networks
- Same degree distributions, different arrangements.
- Doyle *et al.* take a look at the actual Internet.

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- KR also looked at changing the details of the attachment kernel.

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- Detail: we are ignoring initial seed network's edges.

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- We replace $\mathrm{d}N_k/\mathrm{d}t$ with $\mathrm{d}n_k t/\mathrm{d}t = n_k$.
- We arrive at a difference equation:

$$n_k = \frac{1}{2\cancel{t}}\left[(k-1)n_{k-1}\cancel{t} - kn_k\cancel{t}\right] + \delta_{k1}$$

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- Keep A_k linear in k but tweak details.
- Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

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- As before, also assume $N_k(t) = n_k t$.

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- We can solve for μ in some cases.
- Our assumption that $A = \mu t$ looks to be not too horrible.

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- And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.

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- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- For $\nu > 2$, all but a finite # of nodes connect to one node.

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Overview Key Points for Models of Networks:

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Neural reboot (NR):

Turning the corner:

https://www.you

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