# Scale-free networks

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## Outline



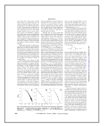


## Scale-free networks

- Real networks with power-law degree distributions became known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

- One of the seminal works in complex networks:



"Emergence of scaling in random networks"
Barabási and Albert,
Science, **286**, 509–511, 1999. [?]

Times cited: $\sim 43,853$ (as of May 19, 2023)

- Somewhat misleading nomenclature ...



## Scale-free networks

- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...



## Some real data (we are feeling brave):

From Barabási and Albert's original paper [?]:

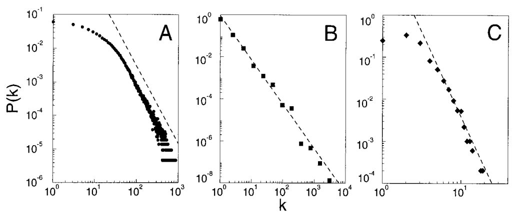
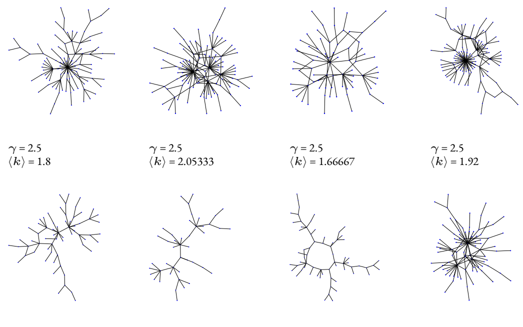



**Fig. 1.** The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$. **(B)** WWW, $N = 325,729$, $\langle k \rangle = 5.46$ (6). **(C)** Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{actor} = 2.3$, (B) $\gamma_{www} = 2.1$ and (C) $\gamma_{power} = 4$.



## Random networks: largest components






## Scale-free networks

### The big deal:

- We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

### A big deal for scale-free networks:

- How does the exponent $\gamma$ depend on the mechanism?
- Do the mechanism details matter?



## BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with $m_0$ disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes $m$ links to nodes already present.
  3. Preferential attachment—Probability of connecting to $i$th node is $\propto k_i$.
- In essence, we have a rich-gets-richer scheme.
- Yes, we've seen this all before in Simon's model.



## BA model

- Definition: $A_k$ is the attachment kernel for a node with degree $k$.
- For the original model:

$$A_k = k$$

- Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\max}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time $t$
and $N_k(t)$ is # degree $k$ nodes at time $t$.

## Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node $i$ is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.



- Deal with denominator: each added node brings $m$ new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

- Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i\, t^{1/2}.}$$

- Next find $c_i$ ...



- Know $i$th node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \quad \text{for } t \ge t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which flattens out growth curve.
- First-mover advantage: Early nodes do best.
- Clearly, a Ponzi scheme.



## We are already at the Zipf distribution:

- Degree of node $i$ is the size of the $i$th ranked node:

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \quad \text{for } t \ge t_{i,\text{start}}.$$

- From before:

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

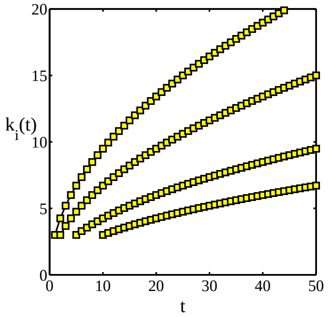
so $t_{i,\text{start}} \sim i$ which is the rank.
- We then have:

$$k_i \propto i^{-1/2} = i^{-\alpha}.$$

- Our connection $\alpha = 1/(\gamma - 1)$ or $\gamma = 1 + 1/\alpha$ then gives

$$\boxed{\gamma = 1 + 1/(1/2) = 3.}$$






- $m = 3$
- $t_{i,\text{start}} = 1, 2, 5,$ and $10.$



## Degree distribution

- So what's the degree distribution at time $t$?
- Use fact that birth time for added nodes is distributed uniformly between time 0 and t:

$$\mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}} \simeq \frac{\mathrm{d}t_{i,\text{start}}}{t}$$

- Also use

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

Transform variables—Jacobian:

$$\frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i} = -2\frac{m^2 t}{k_i(t)^3}.$$



## Degree distribution

$$\mathbf{Pr}(k_i)\mathrm{d}k_i = \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}}$$

$$= \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}k_i \left|\frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i}\right|$$

$$= \frac{1}{t}\mathrm{d}k_i\, 2\frac{m^2 t}{k_i(t)^3}$$

$$= 2\frac{m^2}{k_i(t)^3}\mathrm{d}k_i$$

$$\propto k_i^{-3}\mathrm{d}k_i\,.$$



## Degree distribution

- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- $\gamma > 3$: finite mean and variance (mild)



## Back to that real data:

### From Barabási and Albert's original paper [?]:

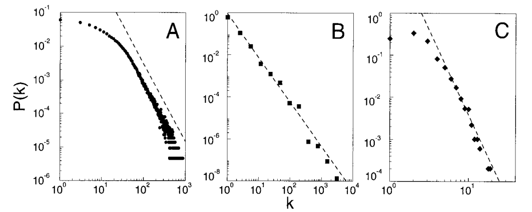



**Fig. 1.** The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

## Examples

| | |
|---|---|
| Web | $\gamma \simeq 2.1$ for in-degree |
| Web | $\gamma \simeq 2.45$ for out-degree |
| Movie actors | $\gamma \simeq 2.3$ |
| Words (synonyms) | $\gamma \simeq 2.8$ |

The Internets is a different business...




## Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect $\gamma$?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter? Maybe ...




## Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- PA is ∴ an outrageous assumption of node capability.
- But a very simple mechanism saves the day...




## Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- Assuming the existing network is random, we know probability of a random friend having degree $k$ is

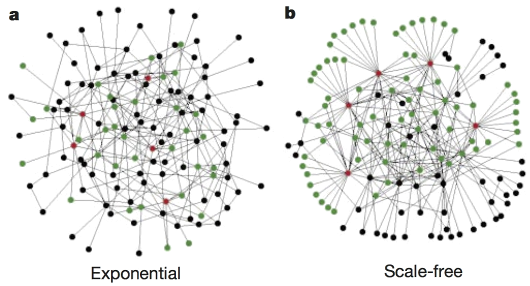
$$Q_k \propto k P_k$$

- So rich-gets-richer scheme can now be seen to work in a natural way.




## Robustness

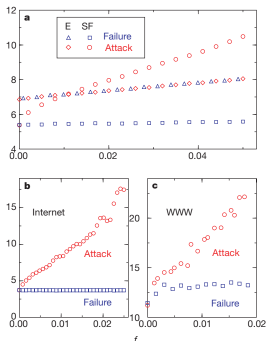
- Albert et al., Nature, 2000:
  "Error and attack tolerance of complex networks" [?]
- Standard random networks (Erdős-Rényi) versus Scale-free networks:




from Albert et al., 2000




## Robustness




from Albert et al., 2000

- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)




## Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'
  2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.




## Robustness

### Not a robust paper:

"The "Robust yet Fragile" nature of the Internet"
Doyle et al.,
Proc. Natl. Acad. Sci., **2005**, 14497–14502, 2005. [?]

- HOT networks versus scale-free networks
- Same degree distributions, different arrangements.
- Doyle *et al.* take a look at the actual Internet.




## Generalized model

### Fooling with the mechanism:

- 2001: Krapivsky & Redner (KR) [?] explored the general attachment kernel:

$$\mathbf{Pr}(\text{attach to node } i) \propto A_k = k_i^{\nu}$$

where $A_k$ is the attachment kernel and $\nu > 0$.
- KR also looked at changing the details of the attachment kernel.

## Generalized model





- We'll follow KR's approach using rate equations.
- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where $N_k$ is the number of nodes of degree $k$.

1. One node with one link is added per unit time.
2. The first term corresponds to degree $k-1$ nodes becoming degree $k$ nodes.
3. The second term corresponds to degree $k$ nodes becoming degree $k-1$ nodes.
4. $A$ is the correct normalization (coming up).
5. Seed with some initial network (e.g., a connected pair)
6. Detail: $A_0 = 0$

## Generalized model





- In general, probability of attaching to a specific node of degree $k$ at time $t$ is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

- Detail: we are ignoring initial seed network's edges.

## Generalized model





- So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- We replace $\mathrm{d}N_k/\mathrm{d}t$ with $\mathrm{d}n_k t/\mathrm{d}t = n_k$.
- We arrive at a difference equation:

$$n_k = \frac{1}{2\cancel{t}}\left[(k-1)n_{k-1}\cancel{t} - kn_k\cancel{t}\right] + \delta_{k1}$$

## Universality?





- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}t \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel $A_k$?
- Again, we're asking if the result $\gamma = 3$ universal?
- KR's natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- But we'll first explore a more subtle modification of $A_k$ made by Krapivsky/Redner [?]
- Keep $A_k$ linear in $k$ but tweak details.
- Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

## Universality?





- Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of $A_k$.

- We assume that $A = \mu t$
- We'll find $\mu$ later and make sure that our assumption is consistent.
- As before, also assume $N_k(t) = n_k t$.

## Universality?





- For $A_k = k$ we had

$$n_k = \frac{1}{2}\left[(k-1)n_{k-1} - kn_k\right] + \delta_{k1}$$

- This now becomes

$$n_k = \frac{1}{\mu}\left[A_{k-1}n_{k-1} - A_k n_k\right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

- Again two cases:

$$k = 1: n_1 = \frac{\mu}{\mu + A_1}; \qquad k > 1: n_k = n_{k-1}\frac{A_{k-1}}{\mu + A_k}.$$

## Universality?





- Time for pure excitement: Find asymptotic behavior of $n_k$ given $A_k \to k$ as $k \to \infty$.
- For large $k$, we find:

$$n_k = \frac{\mu}{A_k}\prod_{j=1}^{k}\frac{1}{1+\frac{\mu}{A_j}} \propto k^{-\mu-1}$$

- Since $\mu$ depends on $A_k$, details matter...

## Universality?





- Now we need to find $\mu$.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now subsitute in our expression for $n_k$:

$$1\cancel{\mu} = \sum_{k=1}^{\infty}\frac{\cancel{\mu}}{\cancel{A_k}}\prod_{j=1}^{k}\frac{1}{1+\frac{\mu}{A_j}}\cancel{A_k}$$

- Closed form expression for $\mu$.
- We can solve for $\mu$ in some cases.
- Our assumption that $A = \mu t$ looks to be not too horrible.

## Universality?





- Consider tunable $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- Again, we can find $\gamma = \mu + 1$ by finding $\mu$.
- Closed form expression for $\mu$:

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

#mathisfun

-

$$\mu(\mu-1) = 2\alpha \Rightarrow \mu = \frac{1+\sqrt{1+8\alpha}}{2}.$$

- Since $\gamma = \mu + 1$, we have

$$0 \leq \alpha < \infty \Rightarrow 2 \leq \gamma < \infty$$

- Craziness...

## Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner: [?]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- Distribution of degree is universal providing $\nu < 1$.



## Sublinear attachment kernels

### Details:

- For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu\left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

- For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

- And for $1/(r+1) < \nu < 1/r$, we have $r$ pieces in exponential.



## Superlinear attachment kernels

- Rich-get-much-richer:

$$A_k \sim k^{\nu} \text{ with } \nu > 1.$$

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- For $\nu > 2$, all but a finite # of nodes connect to one node.



## Nutshell:

### Overview Key Points for Models of Networks:

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  1. Description: Characterizing very large networks
  2. Explanation: Micro story ⇒ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics... #excitement



## References I

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