

# Power-Law Mechanisms: Variable Transformation

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Principles of Complex Systems, Vols. 1, 2, & 3D  
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The PoCVerse  
Variable  
Transformation  
1 of 22

Variable  
transformation

Basics

Holtsmark's Distribution

PLIPLO

References



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The PoCverse  
Variable  
Transformation  
2 of 22

Variable  
transformation

Basics

Holtzmark's Distribution

PLIPL0

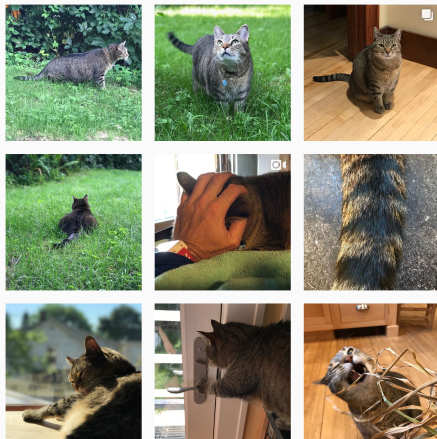
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The PoCVerse  
Variable  
Transformation  
3 of 22

Variable  
transformation

Basics

Holtsmark's Distribution

PLIPLLO

References



# Outline

The PoCSverse  
Variable  
Transformation  
4 of 22

Variable  
transformation

Basics

Holtmark's Distribution

PLIPLO

References

## Variable transformation

Basics

Holtmark's Distribution

PLIPLO

## References



Variable  
transformation

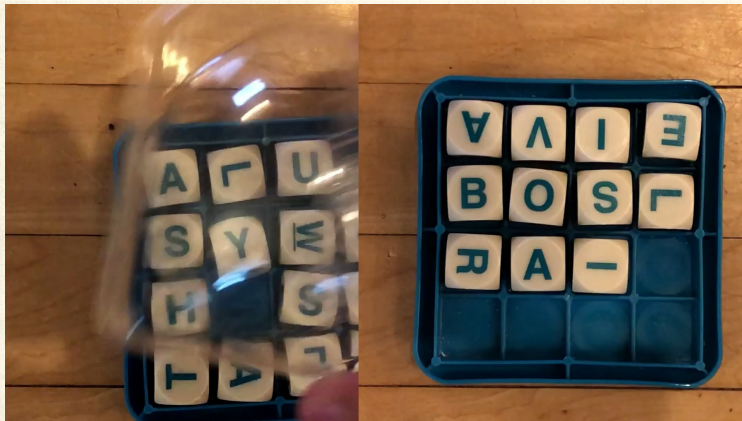
Basics

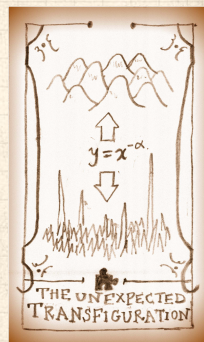
Holtmark's Distribution

PLIPL0

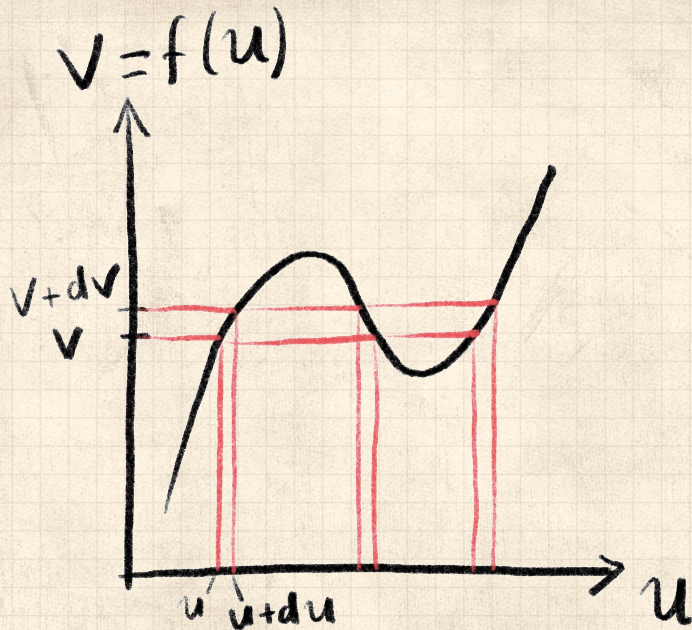
References

# The Boggoracle Speaks:









# Variable Transformation

Understand power laws as arising from:

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

 Random variable  $X$  with known distribution  $P_x$

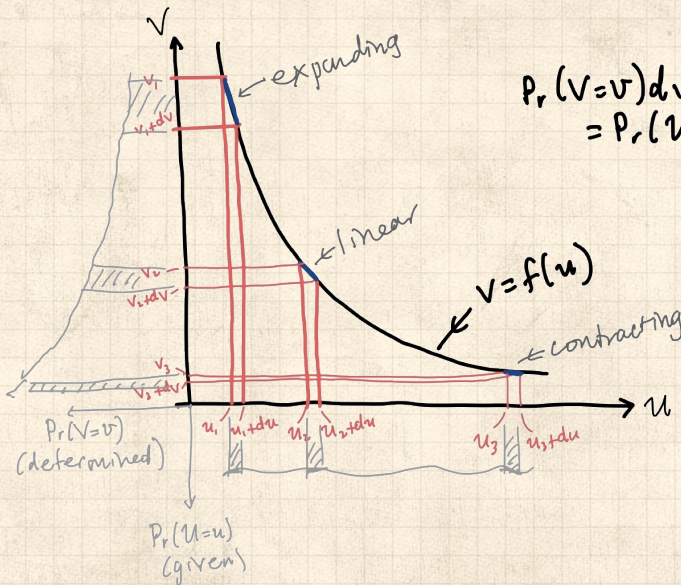
 Second random variable  $Y$  with  $y = f(x)$ .

$$\begin{aligned} \text{ } P_Y(y)dy &= \\ &\sum_{x|f(x)=y} P_X(x)dx \\ &= \\ &\sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|} \end{aligned}$$

 Often easier to do by hand...



$$P_r(V=v)dv = P_r(U=u)du$$



## Example



Assume relationship between  $x$  and  $y$  is 1-1.



Power-law relationship between variables:

$$y = cx^{-\alpha}, \alpha > 0$$



Look at  $y$  large and  $x$  small



$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

$$\text{invert: } dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$





Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left( \overbrace{\left( \frac{y}{c} \right)^{-1/\alpha}}^{(x)} \right) \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy}^{dx}$$

 If  $P_x(x) \rightarrow \text{non-zero constant as } x \rightarrow 0$  then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

 If  $P_x(x) \rightarrow x^\beta \text{ as } x \rightarrow 0$  then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$



# Example

## Exponential distribution

Given  $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$



Exponentials arise from randomness (easy) ...



More later when we cover robustness.



# Gravity

- Select a random point in the universe  $\vec{x}$ .
- Measure the force of gravity  $F(\vec{x})$ .
- Observe that  $P_F(F) \sim F^{-5/2}$ .
- Distribution named after Holtsmark who was thinking about electrostatics and plasma <sup>[1]</sup>.
- Again, the humans naming things after humans, poorly.<sup>1</sup>



The PoCVerse  
Variable  
Transformation  
15 of 22

Variable  
transformation

Basics

Holtsmark's Distribution  
PLIPLO

References



<sup>1</sup>Stigler's Law of Eponymy 

Matter is concentrated in stars: <sup>[2]</sup>

  $F$  is distributed unevenly

 Probability of being a distance  $r$  from a single star at  $\vec{x} = \vec{0}$ :

$$P_r(r)dr \propto r^2 dr$$

 Assume stars are distributed randomly in space (oops?)

 Assume only one star has significant effect at  $\vec{x}$ .

 Law of gravity:

$$F \propto r^{-2}$$

 invert:

$$r \propto F^{-\frac{1}{2}}$$

 Connect differentials:  $dr \propto dF^{-\frac{1}{2}} \propto F^{-\frac{3}{2}} dF$





# Transformation:

Using  $r \propto F^{-1/2}$ ,  $dr \propto F^{-3/2}dF$ , and  $P_r(r) \propto r^2$



$$P_F(F)dF = P_r(r)dr$$



$$\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF$$



$$\propto (F^{-1/2})^2 F^{-3/2}dF$$



$$= F^{-1-3/2}dF$$



$$= F^{-5/2}dF.$$



# Gravity:

The PoCverse  
Variable  
Transformation  
18 of 22

Variable  
transformation

Basics

Holtmark's Distribution  
.....  
PLIPLO

References

$$P_F(F) = F^{-5/2} dF$$



$$\gamma = 5/2$$



Mean is finite.



Variance =  $\infty$ .



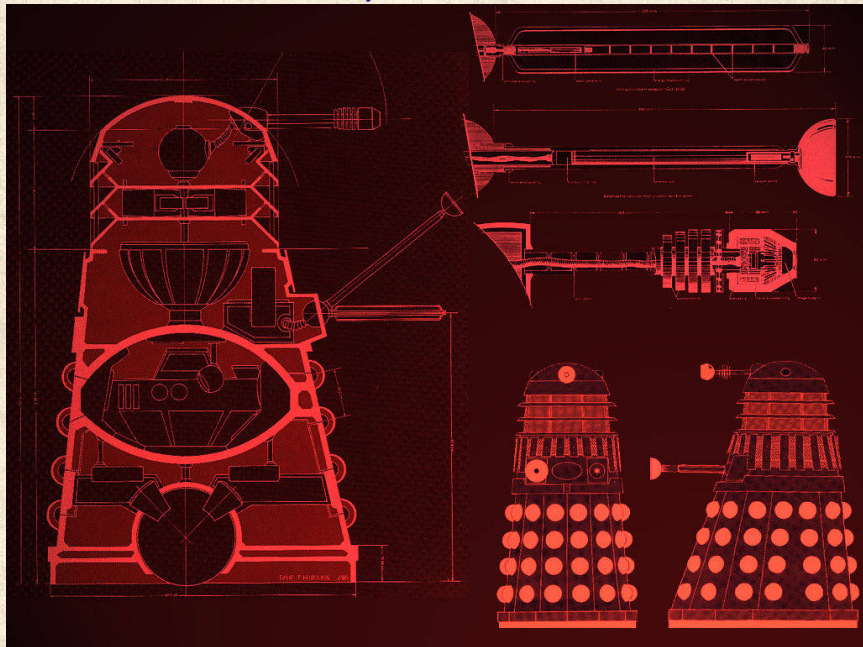
A **wild** distribution.



**Upshot:** Random sampling of space usually safe  
but can end badly...



□ Todo: Build Dalek army.



# Extreme Caution!

 PLIPLO = Power law in, power law out

 Explain a power law as resulting from another unexplained power law.

 Yet another homunculus argument ...

 Don't do this!!! (slap, slap)

 MIWO = Mild in, Wild out is the stuff.

 In general: We need mechanisms!





# References I

The PoCVerse  
Variable  
Transformation  
22 of 22

Variable  
transformation

Basics

Holtzmark's Distribution

PLIPLIO

References

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