

Allotaxonomy

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Principles of Complex Systems,
Vols. 1, 2, 3D, 4 forever, V for Vendetta
CSYS/MATH 6701, 6713, 2025–2026

Prof. Peter Sheridan Dodds

Computational Story Lab | Vermont Complex Systems Institute
Santa Fe Institute | University of Vermont



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A plenitude of
distances

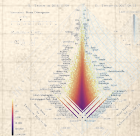
Rank-turbulence
divergence

Probability-
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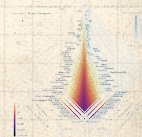
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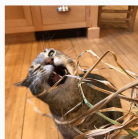
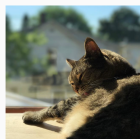
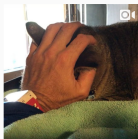
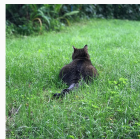
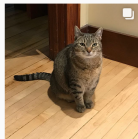
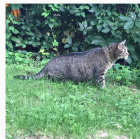
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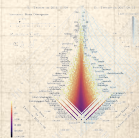
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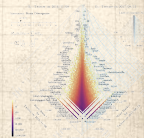
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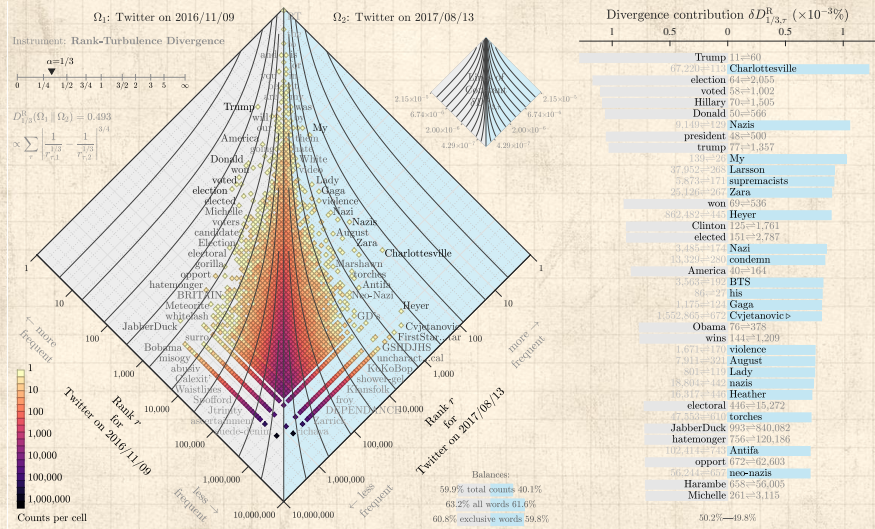
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Goal—Understand this:



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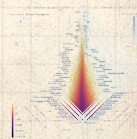
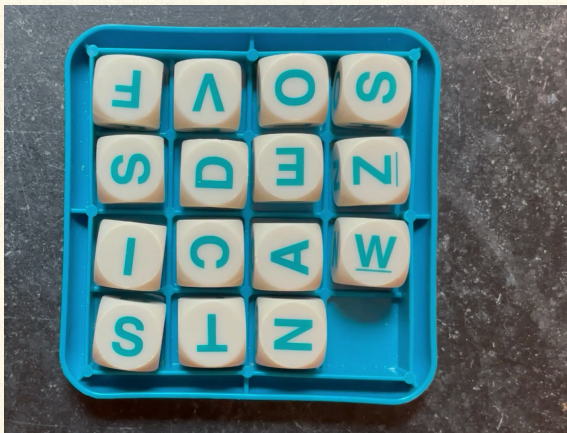
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The Boggoracle Speaks:



Site (papers, examples, code):

<http://compstorylab.org/allotaxonomy/> 

Foundational papers:



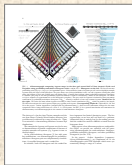
“Allotaxonomy and rank-turbulence divergence:
A universal instrument for comparing complex
systems” 

Dodds et al.,

EPJ Data Science, **12**, 1–42, 2023. ^[5]

EPJ Data Science version 

arXiv version 



“Probability-turbulence divergence: A tunable
allotaxonomic instrument for comparing
heavy-tailed type-size distributions” 

Dodds et al.,

, 2020. ^[6]

Basic science = Describe + Explain:



Dashboards of single scale instruments helps us understand, monitor, and control systems.

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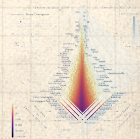
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Archetype: Cockpit dashboard for flying a plane

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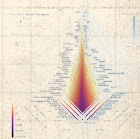
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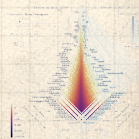
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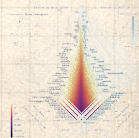
 Dashboards of single scale instruments helps us understand, monitor, and control systems.

 Archetype: Cockpit dashboard for flying a plane

 Okay if comprehensible.

 Complex systems present two problems for dashboards:

1. Scale with internal diversity of components: We need meters for every species, every company, every word.
2. Tracking change: We need to re-arrange meters on the fly.



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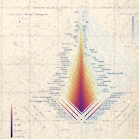
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 Goal—Create comprehensible, dynamically-adjusting, differential dashboards showing two pieces:¹

1. ‘Big picture’ map-like overview,
2. A tunable ranking of components.



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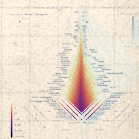
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¹See the [lexicocalorimeter](#) 

Baby names, much studied: ^[12]

just a decade or so. If you were born in the United States around this year, these are names that are more likely to seem common and generic to you, but are distinctive generational markers.

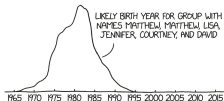
1880 Will, Maude, Minnie, May, Caro, Ida, Lela, Hattie, Jessie, Ada
 1885 Grecco, Maude, Will, Minnie, Lela, Effie, May, Caro, Lela, Nettie
 1890 Maude, May, Minnie, Effie, Maude, Jessie, Nettie, Hattie, Lela, Caro
 1895 Maude, Maude, Minnie, Jessie, Minnie, Hattie, Pearl, Ethel, Bertha
 1900 Maude, Myrtle, Jessie, Mamie, Pearl, Blanche, Gertrude, Ethel, Minnie, Gladys
 1905 Gladys, Viola, Maude, Myrtle, Gertrude, Pearl, Jessie, Blanche, Mamie, Ethel
 1910 Thelma, Gladys, Viola, Mildred, Beatrice, Lucille, Gertrude, Agnes, Hazel, Ethel
 1915 Mildred, Lucille, Thelma, Helen, Bernice, Pauline, Eleanor, Beatrice, Ruth, Dorothy
 1920 Marjorie, Dorothy, Mildred, Lucille, Warren, Thelma, Bernice, Virginia, Helen, June
 1925 Doris, June, Betty, Marjorie, Dorothy, Lorraine, Lela, Norma, Virginia, Joanne
 1930 Dolores, Betty, Jean, Billie, Doris, Norma, Lela, Billy, Jean, Marilyn
 1935 Shirley, Marlene, Joan, Dolores, Marilyn, Bobby, Betty, Billy, Joyce, Beverly
 1940 Corde, Judith, Judy, Carol, Joyce, Barbara, Joan, Carolyn, Shirley, Jerry
 1945 Judy, Judith, Linda, Carol, Sharon, Sandra, Carolyn, Larry, Janice, Dennis
 1950 Linda, Deborah, Gail, Judy, Gary, Larry, Diane, Dennis, Brenda, Annie
 1955 Debra, Deborah, Cathy, Kathy, Pamela, Randy, Kim, Cynthia, Diane, Cheryl
 1960 Debbie, Kim, Terri, Cindy, Kathy, Cathy, Laurie, Lori, Debra, Ricky
 1965 Lisa, Tammy, Lori, Todd, Kim, Mikaela, Tracy, Tina, Susan, Michele, Lisa
 1970 Tammy, Tracy, Tracy, Todd, Dawn, Tina, Sherry, Stacy, Michele, Lisa
 1975 Chad, Jason, Tanya, Heather, Jennifer, Amy, Stacy, Shannon, Sherry, Tara
 1980 Brendy, Crystal, April, Jason, Jeremy, Erin, Tiffany, Jamie, Melissa, Jennifer
 1985 Crystal, Lindsay, Ashley, Lindsey, Doreen, Austin, Amanda, Tiffany, Crystal, Amber
 1990 Brittany, Chelsea, Kelsey, Cindy, Ashley, Courtney, Kayla, Kyle, Megan, Jessica
 1995 Taylor, Kelsey, Dakota, Austin, Haley, Cindy, Tyler, Shelby, Brittany, Kayla
 2000 Destiny, Madison, Hadley, Sydney, Kaitlyn, Hailey, Brianna, Hannah, Alyssa
 2005 Aidan, Diego, Gavin, Hailey, Ethan, Madison, Alex, Isabella, Ayden, Aiden
 2010 Jayden, Aiden, Neveah, Addison, Brayden, London, Peyton, Isabella, Ava, Liam
 2015 Ariu, Harper, Scarlett, Jason, Gracyn, Liviana, Hudson, Liam, Zoey, Layla

If kids in your class were named Jeff, Lisa, Michael, Karen, and David, then you were probably born in the mid-1960s. If they were named Jayden, Isabella, Sophia, Ava, and Ethan, then you were probably born somewhere around 2010.

But names can reveal things about age in other ways.

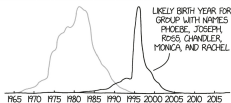
The mid-1990s TV show *Friends* featured six roommates, played by actors, named Matthew, Jennifer, Courtney, Lisa, David, and another Matthew. Each of those names has its own popularity curve; if we combine them all, we can guess what years the group of actors was likely born:

HOW TO: ABSURD SCIENTIFIC ADVICE FOR COMMON REAL-WORLD PROBLEMS



The actors were actually born in the late 1960s, on the very early edge of the popularity of their names. In other words, the actors all have names that were a little before their time. Courtney Cox and Jennifer Aniston had names that didn't really become popular until a decade later. (Maybe people with trendy parents are more likely to wind up in acting.) But the names are generally consistent with their era, if a little ahead of the curve.

We get something very different if we look at the names of their characters—Phoebe, Joseph, Ross, Chandler, Rachel, and Monica:



The show debuted in 1994. There's a clear spike in popularity of the names in 1995 and 1996, which can probably be attributed to the show putting the names in the minds of new parents. But it's not just the show—that name combination was clearly on the rise in the years before *Friends* premiered. It's possible that parents looking for good names for their children are influenced by some of the same cultural trends as TV writers looking for good names for their characters.

How to build a dynamical dashboard that helps sort through a massive number of interconnected time series?

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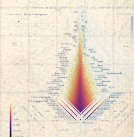
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Probability-
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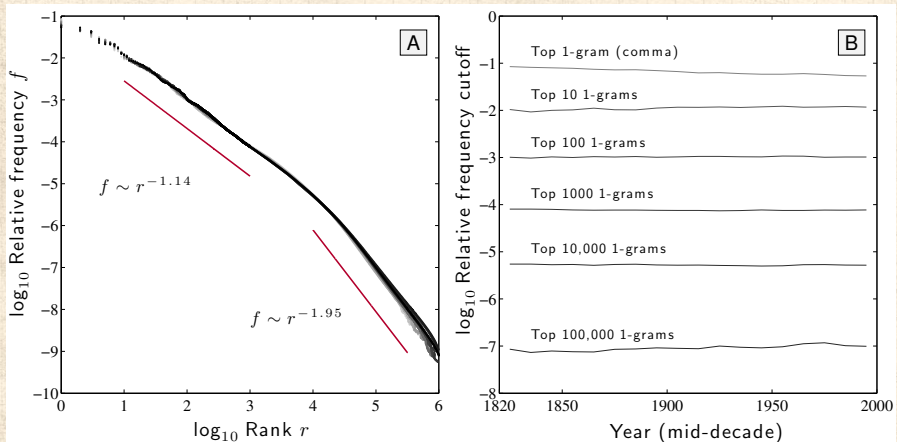
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“Is language evolution grinding to a halt? The scaling of lexical turbulence in English fiction suggests it is not” [↗](#)
Pechenick, Danforth, Dodds, Alshaabi, Adams, Reagan, Danforth, Frank, Reagan, and Danforth.
Journal of Computational Science, **21**, 24–37, 2017. ^[14]



For language, Zipf's law has two scaling regimes: ^[19]

$$f \sim \begin{cases} r^{-\alpha} & \text{for } r \ll r_b, \\ r^{-\alpha'} & \text{for } r \gg r_b, \end{cases}$$

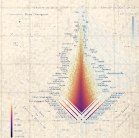
When comparing two texts, define Lexical turbulence as flux of words across a frequency threshold:

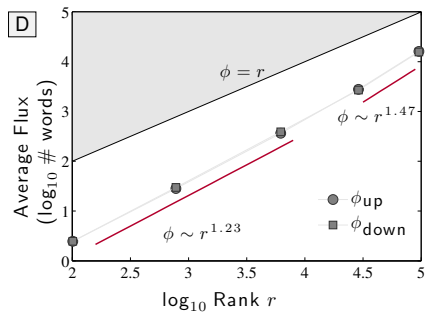
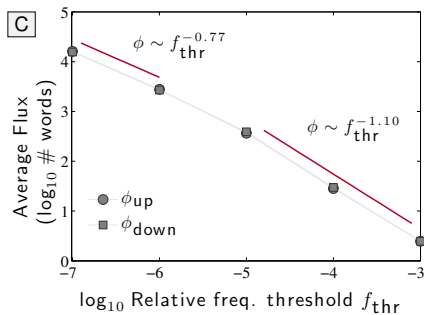
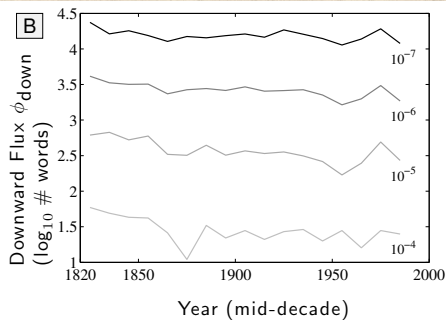
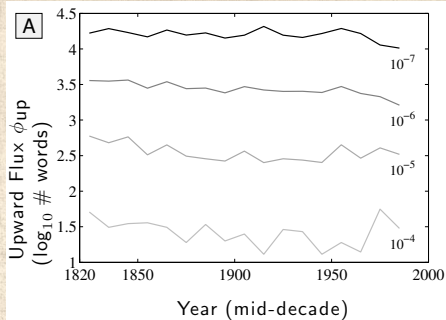
$$\phi \sim \begin{cases} f_{\text{thr}}^{-\mu} & \text{for } f_{\text{thr}} \ll f_b, \\ f_{\text{thr}}^{-\mu'} & \text{for } f_{\text{thr}} \gg f_b, \end{cases}$$

Estimates: $\mu \simeq 0.77$ and $\mu' \simeq 1.10$, and f_b is the scaling break point.

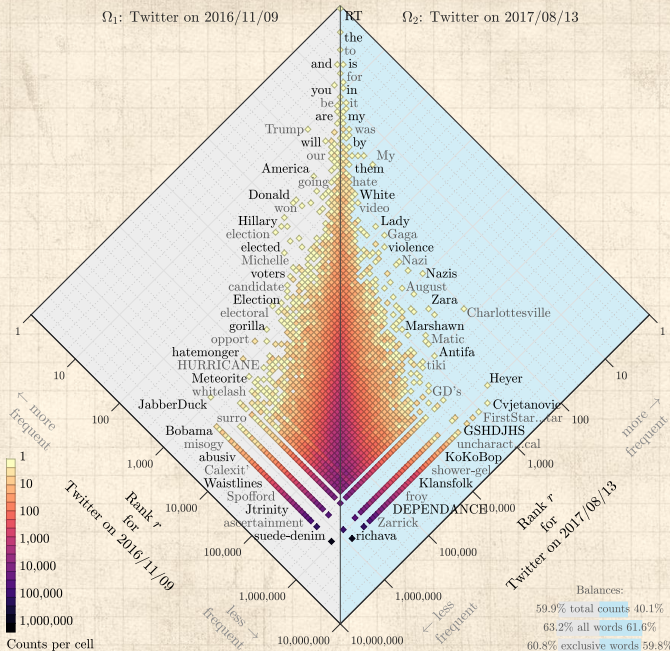
$$\phi \sim \begin{cases} r^\nu = r^{\alpha\mu'} & \text{for } r \ll r_b, \\ r^{\nu'} = r^{\alpha'\mu} & \text{for } r \gg r_b. \end{cases}$$

Estimates: Lower and upper exponents $\nu \simeq 1.23$ and $\nu' \simeq 1.47$.

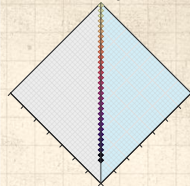




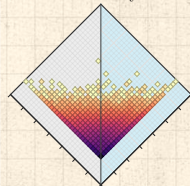
A. Rank-turbulence histogram:

 Ω_1 : Twitter on 2016/11/09 Ω_2 : Twitter on 2017/08/13

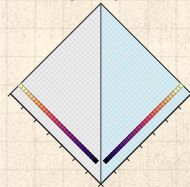
B. Identical systems:



C. Randomized systems:



D. Disjoint systems:



Balances:

59.9% total counts 40.1%

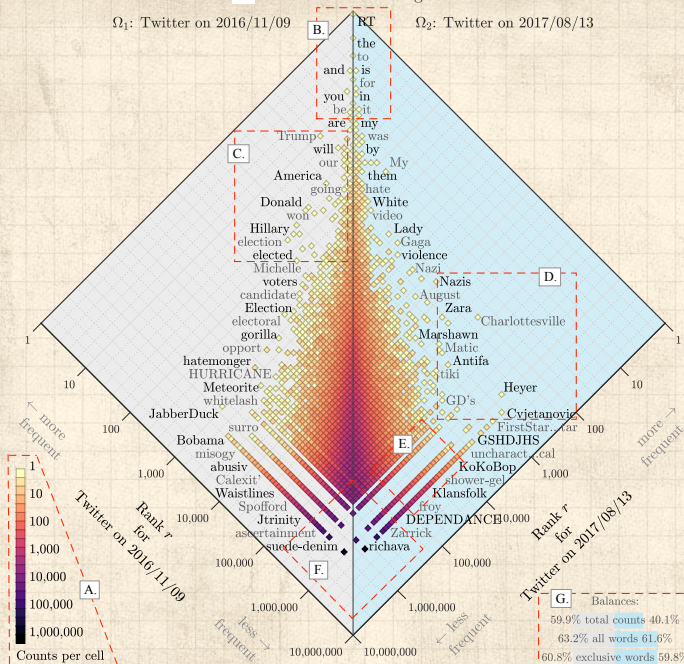
63.2% all words 61.6%

60.8% exclusive words 59.8%

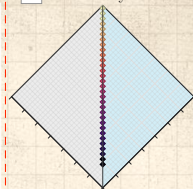
Rank-turbulence histogram:

Ω_1 : Twitter on 2016/11/09

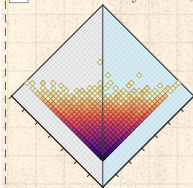
Ω_2 : Twitter on 2017/08/13



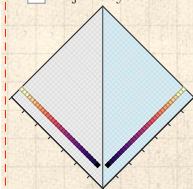
H. Identical systems:



I. Randomized systems:



J. Disjoint systems:



G. Balances:

59.9% total counts 40.1%

63.2% all words 61.6%

60.8% exclusive words 59.8%

Balances:

Top bar (optional)—Total size:



Relative balance of system sizes.



Examples: Total number of words in a book, total number of individuals in an ecology.

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Top bar (optional)—Total size:

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-  Examples: Total number of words in a book, total number of individuals in an ecology.

Middle bar—Types:

-  Fraction of types in each system as a percentage of the union of types from both systems.

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-  Examples: Total number of words in a book, total number of individuals in an ecology.

Middle bar—Types:

-  Fraction of types in each system as a percentage of the union of types from both systems.

Bottom bar—Exclusive types:

-  Types that are present in one system only are ‘exclusive types’.
-  $\Omega^{(1)}$ -exclusive and $\Omega^{(2)}$ -exclusive indicate which system an exclusive type belongs to.
-  Percentage of exclusive types in a system relative to that system’s total number of types.

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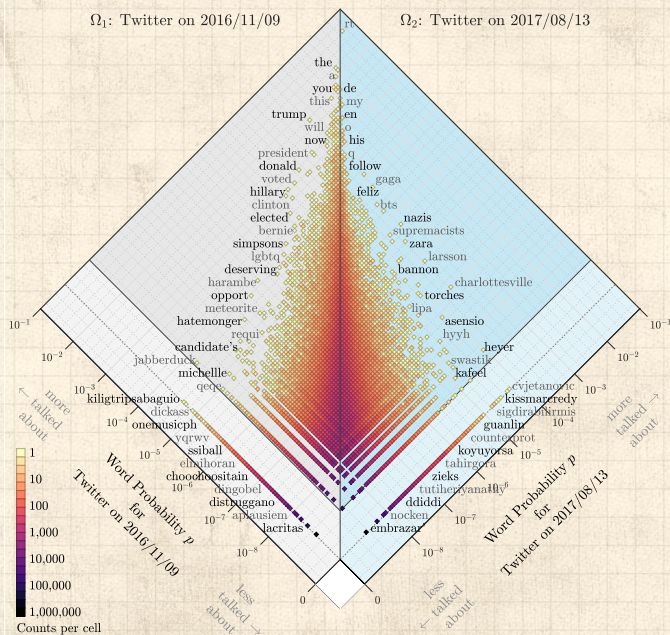
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Probability-turbulence histogram:



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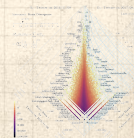
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So, so many ways to compare probability distributions:

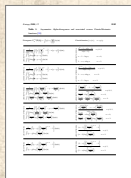


Table 1. Summary of various distance measures

Measure	Formula	Properties
Euclidean	$\sqrt{\sum (x_i - y_i)^2}$	Non-negative, Symmetric, Triangle inequality
Manhattan	$\sum x_i - y_i $	Non-negative, Symmetric, Triangle inequality
Chebyshev	$\max_i x_i - y_i $	Non-negative, Symmetric, Triangle inequality
Hamming	$\sum \delta_{ij}$	Non-negative, Symmetric, Triangle inequality
Jaccard	$1 - \frac{ A \cap B }{ A \cup B }$	Non-negative, Symmetric, Triangle inequality
Sørensen	$1 - \frac{ A \cap B }{ A + B }$	Non-negative, Symmetric, Triangle inequality
Overlap	$1 - \frac{ A \cap B }{\min(A , B)}$	Non-negative, Symmetric, Triangle inequality
Levenshtein	$\min \sum \delta_{ij}$	Non-negative, Symmetric, Triangle inequality
Dynamic Time Warping	$\min \sum \delta_{ij}$	Non-negative, Symmetric, Triangle inequality

“Families of Alpha- Beta- and Gamma- Divergences:
Flexible and Robust Measures of Similarities” 

Cichocki and Amari,

Entropy, **12**, 1532-1568, 2010. ^[2]

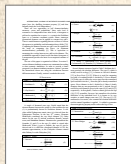


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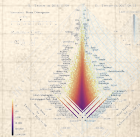
“Comprehensive survey on distance/similarity
measures between probability density functions” 

Sung-Hyuk Cha,

International Journal of Mathematical Models and
Methods in Applied Sciences, **1**, 300–307, 2007. ^[1]



Comparisons are distances, divergences , similarities, inner products, fidelities ...



So, so many ways to compare probability distributions:



Table 1. Families of Alpha- Beta- and Gamma- Divergences: Flexible and Robust Measures of Similarities

Family	Measure	Formula	Properties
Alpha	α -Divergence	$\frac{1}{\alpha} \ln \left(\frac{\sum_i p_i q_i^\alpha}{\sum_i p_i^\alpha} \right)$	...
Beta	β -Divergence	$\frac{1}{\beta} \ln \left(\frac{\sum_i p_i^\beta q_i}{\sum_i p_i^\beta} \right)$	...
Gamma	γ -Divergence	$\frac{1}{\gamma} \ln \left(\frac{\sum_i p_i^\gamma q_i}{\sum_i p_i^\gamma} \right)$	...

“Families of Alpha- Beta- and Gamma- Divergences:
Flexible and Robust Measures of Similarities” ↗

Cichocki and Amari,
Entropy, **12**, 1532-1568, 2010. [2]

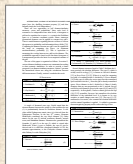


Table 1. Comprehensive survey on distance/similarity measures between probability density functions

Measure	Formula	Properties
...	...	...

“Comprehensive survey on distance/similarity
measures between probability density functions” ↗

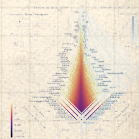
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Comparisons are distances, divergences ↗, similarities, inner products, fidelities ...



60ish kinds of comparisons grouped into 10 families



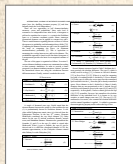
So, so many ways to compare probability distributions:



Test	Family	Properties
Chi-squared	Alpha	...
Kullback-Leibler	Beta	...
...	...	...

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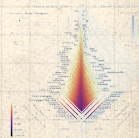
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 60ish kinds of comparisons grouped into 10 families

 A worry: Subsampled distributions with very heavy tails



Quite the festival:

Table 1. L_p Minkowski family		
1. Euclidean L_2	$d_{L_2} = \sqrt{\frac{1}{2} P - Q ^2}$	(1)
2. City block L_1	$d_{L_1} = \frac{1}{2} P - Q $	(2)
3. Minkowski L_p	$d_{L_p} = \sqrt[p]{\frac{1}{2} P - Q ^p}$	(3)
4. Chebyshev L_∞	$d_{L_\infty} = \min(P - Q)$	(4)

Table 2. L_1 family		
5. Sørensen	$d_{L_1} = \frac{\sum_i P_i - Q_i }{\sum_i (P_i + Q_i)}$	(5)

6. Gower	$d_{Gower} = \frac{1}{2} \sum_i \frac{ P_i - Q_i }{R_i}$	(6)
	$+ \frac{1}{2} \sum_i P_i - Q_i $	(7)

7. Soergel	$d_{Soergel} = \frac{\sum_i P_i - Q_i }{\sum_i \max(P_i, Q_i)}$	(8)
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8. Kulczynski d	$d_{Kul} = \frac{\sum_i P_i - Q_i }{\sum_i \max(P_i, Q_i)}$	(9)
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9. Canberra	$d_{Can} = \frac{\sum_i P_i - Q_i }{P_i + Q_i}$	(10)
10. Lorentzian	$d_{Lor} = \sum_i \ln(1 + P_i - Q_i)$	(11)

* L_1 family $\rightarrow$ Intersection (13), Wave Hodges (15), Czekanowski (16), Ruzicka (21), Tanimoto (23), etc.

Table 3. Intersection family		
11. Intersection	$x_{int} = \sum_i \min(P_i, Q_i)$	(12)
	$d_{int} = 1 - x_{int} = \frac{1}{2} \sum_i P_i - Q_i $	(13)
12. Wave Hodges	$d_{WH} = \sum_i \frac{\min(P_i, Q_i)}{\max(P_i, Q_i)}$	(14)
	$= \sum_i \frac{ P_i - Q_i }{\max(P_i, Q_i)}$	(15)
13. Czekanowski	$x_{czk} = \frac{2 \sum_i \min(P_i, Q_i)}{\sum_i (P_i + Q_i)}$	(16)
	$d_{czk} = 1 - x_{czk} = \frac{\sum_i P_i - Q_i }{\sum_i (P_i + Q_i)}$	(17)

14. Motyka	$x_{mot} = \frac{\sum_i \min(P_i^2, Q_i^2)}{\sum_i (P_i^2 + Q_i^2)}$	(18)
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	$d_{mot} = 1 - x_{mot} = \frac{\sum_i \max(P_i^2, Q_i^2)}{\sum_i (P_i^2 + Q_i^2)}$	(19)
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15. Kulczynski r	$x_{kulr} = \frac{1}{d_{kul}} \sum_i \min(P_i, Q_i)$	(20)
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16. Ruzicka	$x_{ruz} = \frac{\sum_i \min(P_i, Q_i)}{\sum_i \max(P_i, Q_i)}$	(21)
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17. Tanimoto	$d_{tan} = \frac{\sum_i P_i + \sum_i Q_i - 2 \sum_i \min(P_i, Q_i)}{\sum_i P_i + \sum_i Q_i}$	(22)
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	$= \frac{\sum_i \max(P_i, Q_i) - \min(P_i, Q_i)}{\sum_i \max(P_i, Q_i)}$	(23)
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Table 4. Inner Product family		
18. Inner Product	$x_{ip} = P \cdot Q = \sum_i P_i Q_i$	(24)
19. Harmonic mean	$x_{hm} = \frac{2 \sum_i P_i Q_i}{\sum_i P_i + \sum_i Q_i}$	(25)
20. Cosine	$x_{cos} = \frac{\sum_i P_i Q_i}{\sqrt{\sum_i P_i^2} \sqrt{\sum_i Q_i^2}}$	(26)

21. Kumar-Hausbrook (PCE)	$x_{khu} = \frac{\sum_i P_i Q_i}{\sum_i P_i^2 + \sum_i Q_i^2 - \sum_i P_i Q_i}$	(27)
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22. Jaccard	$x_{jac} = \frac{\sum_i P_i Q_i}{\sum_i P_i^2 + \sum_i Q_i^2 - \sum_i P_i Q_i}$	(28)
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	$d_{jac} = 1 - x_{jac} = \frac{\sum_i (P_i - Q_i)^2}{\sum_i P_i^2 + \sum_i Q_i^2 - \sum_i P_i Q_i}$	(39)
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23. Dice	$x_{dic} = \frac{2 \sum_i P_i Q_i}{\sum_i P_i^2 + \sum_i Q_i^2}$	(40)
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	$d_{dic} = 1 - x_{dic} = \frac{\sum_i (P_i - Q_i)^2}{\sum_i P_i^2 + \sum_i Q_i^2}$	(31)
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Table 5. Fidelity family or Squared-chord family		
24. Fidelity	$x_{fid} = \sum_i \sqrt{P_i Q_i}$	(32)

25. Bhattacharyya	$d_b = -\ln \sum_i \sqrt{P_i Q_i}$	(33)
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26. Hellinger	$d_h = \sqrt{\frac{1}{2} \sum_i P_i - Q_i }$	(34)
	$= \sqrt{\frac{1}{2} \sum_i \sqrt{P_i Q_i}}$	(35)

27. Mutanata	$d_m = \sqrt{\frac{1}{2} \sum_i P_i - Q_i }$	(36)
	$= \sqrt{\frac{1}{2} \sum_i \sqrt{P_i Q_i}}$	(37)

28. Squared-chord	$d_{sc} = \sqrt{\frac{1}{2} \sum_i P_i - Q_i ^2}$	(38)
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$x_{sc} = 1 - d_{sc}$	$x_{sc} = 2 \sum_i \sqrt{P_i Q_i} - 1$	(39)
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Table 6. Squared L_2 family or χ^2 family		
29. Squared Euclidean	$d_{sq} = \sum_i (P_i - Q_i)^2$	(40)

30. Pearson χ^2	$d_p(P, Q) = \sum_i \frac{(P_i - Q_i)^2}{Q_i}$	(41)
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31. Neyman χ^2	$d_n(P, Q) = \sum_i \frac{(P_i - Q_i)^2}{P_i}$	(42)
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32. Squared χ^2	$d_{sq\chi^2} = \sum_i \frac{(P_i - Q_i)^2}{P_i + Q_i}$	(43)
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33. Probabilistic Symmetric χ^2	$d_{ps\chi^2} = \sum_i \frac{(P_i - Q_i)^2}{P_i + Q_i}$	(44)
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34. Divergence	$d_{div} = 2 \sum_i \frac{(P_i - Q_i)^2}{(P_i + Q_i)^2}$	(45)
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35. Clark	$d_{cl} = \sqrt{\sum_i \left(\frac{ P_i - Q_i }{P_i + Q_i} \right)^2}$	(46)
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36. Additive Symmetric χ^2	$d_{as\chi^2} = \sum_i \frac{(P_i - Q_i)^2}{P_i Q_i}$	(47)
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* Squared L_1 family $\rightarrow$ Jaccard (29), Dice (31)

Table 7. Shannon's entropy family		
37. Kullback-Leibler	$d_{kl} = \sum_i P_i \ln \frac{P_i}{Q_i}$	(48)

38. Jeffreys	$d_j = \sum_i (P_i - Q_i) \ln \frac{P_i}{Q_i}$	(49)
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39. K. divergence	$d_{kdiv} = \sum_i P_i \ln \frac{2P_i}{P_i + Q_i}$	(50)
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40. Topsoe	$d_{top} = \sum_i P_i \ln \left(\frac{2P_i}{P_i + Q_i} + Q_i \ln \left(\frac{2Q_i}{P_i + Q_i} \right) \right)$	(51)
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41. Jensen-Shannon	$d_{js} = \frac{1}{2} \sum_i P_i \ln \left(\frac{2P_i}{P_i + Q_i} \right) + \frac{1}{2} \sum_i Q_i \ln \left(\frac{2Q_i}{P_i + Q_i} \right)$	(52)
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42. Jensen difference	$d_{jd} = \frac{1}{2} \sum_i \left[P_i \ln \frac{P_i + Q_i}{2} + Q_i \ln \left(\frac{P_i + Q_i}{2} \right) \right] \ln \left(\frac{P_i + Q_i}{2} \right)$	(53)
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Table 8. Combinations		
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43. Taneja	$d_{tan} = \frac{\sum_i (P_i + Q_i)}{2} \ln \left(\frac{P_i + Q_i}{2 \sqrt{P_i Q_i}} \right)$	(54)
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44. Kumar-Johnson	$d_{kj} = \sum_i \left(\frac{ P_i - Q_i ^2}{2 P_i Q_i} \right)$	(55)
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45. Avg(L ₁ , L _∞)	$d_{av} = \frac{\sum_i P_i - Q_i + \max_i P_i - Q_i }{2}$	(56)
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Table 10. Vicinostade

Vicin-Wave Hodges	$d_{vwh} = \sum_i \frac{ P_i - Q_i }{\max(P_i, Q_i)}$	(60)
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Vicin-Symmetric χ^2	$d_{vsc\chi^2} = \sum_i \frac{(P_i - Q_i)^2}{\max(P_i, Q_i)}$	(61)
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Vicin-Symmetric χ^2	$d_{vsc\chi^2} = \sum_i \frac{(P_i - Q_i)^2}{\max(P_i, Q_i)}$	(63)
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max-Symmetric	$d_s = \max \left(\sum_i \frac{(P_i - Q_i)^2}{P_i}, \sum_i \frac{(P_i - Q_i)^2}{Q_i} \right)$	(64)
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min-symmetric	$d_{ms} = \min \left(\sum_i \frac{(P_i - Q_i)^2}{P_i}, \sum_i \frac{(P_i - Q_i)^2}{Q_i} \right)$	(65)
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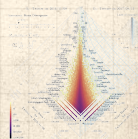
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Shannon tried to slow things down in 1956:



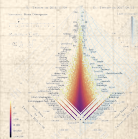
“The bandwagon” 

Claude E Shannon,

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“Information theory has ... become something of a scientific bandwagon.”



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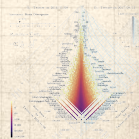
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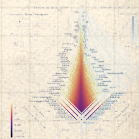
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“While ... information theory is indeed a valuable tool ... [it] is certainly no panacea for the communication engineer or ... for anyone else.



“A few first rate research papers are preferable to a large number that are poorly conceived or half-finished.”





We want two main things:

1. A measure of difference between systems
2. A way of sorting which types/species/words contribute to that difference

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1. Euclidean L_2	$d_{Euc} = \sqrt{\sum_{i=1}^d P_i - Q_i ^2}$	(1)
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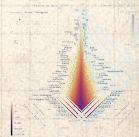
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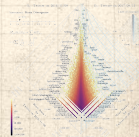
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For sorting, many comparisons give the same ordering.



A few basic building blocks:

-  $|P_i - Q_i|$ (dominant)
-  $\max(P_i, Q_i)$
-  $\min(P_i, Q_i)$
-  $P_i Q_i$
-  $|P_i^{1/2} - Q_i^{1/2}|$ (Hellinger)

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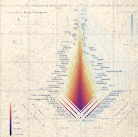
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Information theoretic
sortings are more opaque

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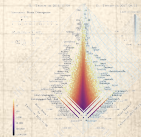
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Information theoretic
sortings are more opaque



No tunability

Table 1. L_p Minkowski family

1. Euclidean L_2	$d_{Euc} = \sqrt{\sum_{i=1}^d P_i - Q_i ^2}$	(1)
2. City block L_1	$d_{CB} = \sum_{i=1}^d P_i - Q_i $	(2)
3. Minkowski L_p	$d_{Mk} = \sqrt[p]{\sum_{i=1}^d P_i - Q_i ^p}$	(3)
4. Chebyshev L_∞	$d_{Cheb} = \max_i P_i - Q_i $	(4)

Table 2. L_1 family

5. Sørensen	$d_{sor} = \frac{\sum_{i=1}^d P_i - Q_i }{\sum_{i=1}^d (P_i + Q_i)}$	(5)
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6. Gower	$d_{gow} = \frac{1}{d} \sum_{i=1}^d \frac{ P_i - Q_i }{R_i}$	(6)
	$= \frac{1}{d} \sum_{i=1}^d P_i - Q_i $	(7)

7. Soergel	$d_{sg} = \frac{\sum_{i=1}^d P_i - Q_i }{\sum_{i=1}^d \max(P_i, Q_i)}$	(8)
------------	---	-----

8. Kulczynski d	$d_{kul} = \frac{\sum_{i=1}^d P_i - Q_i }{\sum_{i=1}^d \min(P_i, Q_i)}$	(9)
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9. Canberra	$d_{can} = \sum_{i=1}^d \frac{ P_i - Q_i }{P_i + Q_i}$	(10)
-------------	--	------

10. Lorentzian	$d_{lor} = \sum_{i=1}^d \ln(1 + P_i - Q_i)$	(11)
----------------	---	------

* L_1 family $\supset$ {Intersectoin (13), Wave Hedges (15), Czekanowski (16), Ruzicka (21), Tanimoto (23), etc}.

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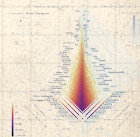
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Shannon's Entropy:

$$H(P) = \langle \log_2 \frac{1}{p_\tau} \rangle = \sum_{\tau \in R_{1,2;\alpha}} p_\tau \log_2 \frac{1}{p_\tau} \quad (1)$$

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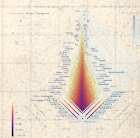
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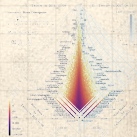
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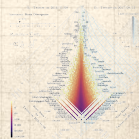


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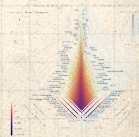
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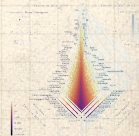
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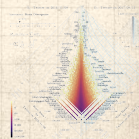
New problem: Re-read solution.



 Jensen-Shannon divergence (JSD): [9, 7, 13, 1]

$$\begin{aligned}
 D^{\text{JS}}(P_1 \parallel P_2) &= \frac{1}{2} D^{\text{KL}}\left(P_1 \parallel \frac{1}{2}[P_1 + P_2]\right) + \frac{1}{2} D^{\text{KL}}\left(P_2 \parallel \frac{1}{2}[P_1 + P_2]\right) \\
 &= \frac{1}{2} \sum_{\tau \in R_{1,2;\alpha}} \left(p_{1,\tau} \log_2 \frac{p_{1,\tau}}{\frac{1}{2}[p_{1,\tau} + p_{2,\tau}]} + p_{2,\tau} \log_2 \frac{p_{2,\tau}}{\frac{1}{2}[p_{1,\tau} + p_{2,\tau}]} \right). \quad (3)
 \end{aligned}$$

 Involving a third intermediate averaged system means JSD is now finite:
 $0 \leq D^{\text{JS}}(P_1 \parallel P_2) \leq 1.$



🔗 Jensen-Shannon divergence (JSD): [9, 7, 13, 1]

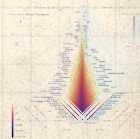
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 &= \frac{1}{2} \sum_{\tau \in R_{1,2;\alpha}} \left(p_{1,\tau} \log_2 \frac{p_{1,\tau}}{\frac{1}{2}[p_{1,\tau} + p_{2,\tau}]} + p_{2,\tau} \log_2 \frac{p_{2,\tau}}{\frac{1}{2}[p_{1,\tau} + p_{2,\tau}]} \right).
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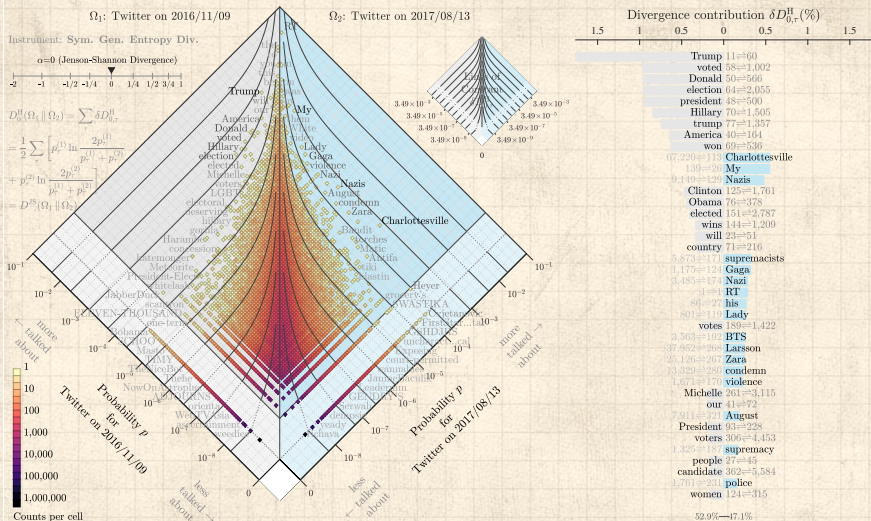
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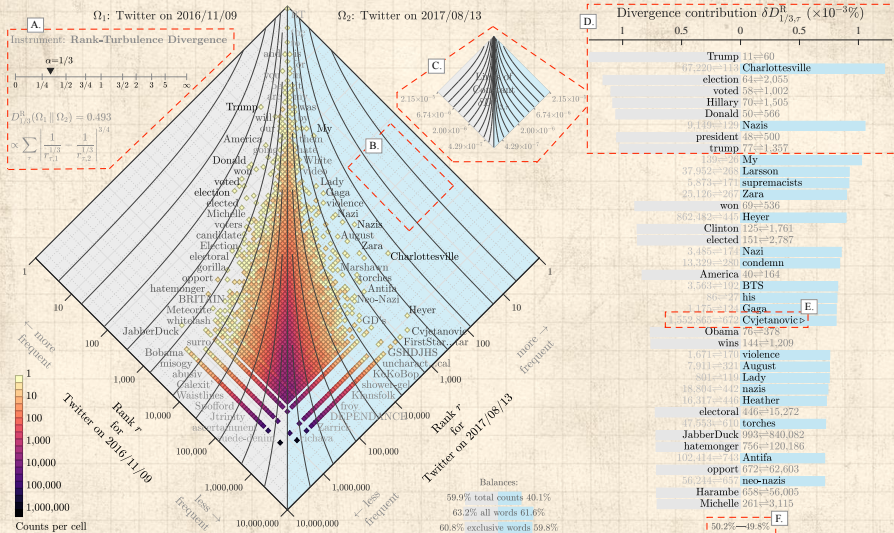
🔗 Generalized entropy divergence: [2]

$$\begin{aligned}
 D_{\alpha}^{\text{AS2}}(P_1 \parallel P_2) &= \\
 &= \frac{1}{\alpha(\alpha-1)} \sum_{\tau \in R_{1,2;\alpha}} \left[\left(p_{\tau,1}^{1-\alpha} + p_{\tau,2}^{1-\alpha} \right) \left(\frac{p_{\tau,1} + p_{\tau,2}}{2} \right)^{\alpha} - (p_{\tau,1} + p_{\tau,2}) \right].
 \end{aligned} \tag{4}$$

Produces JSD when $\alpha \rightarrow 0$.







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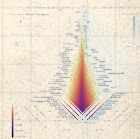
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Desirable rank-turbulence divergence features:

1. Rank-based.



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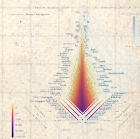
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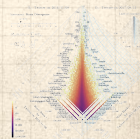
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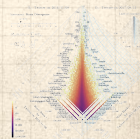
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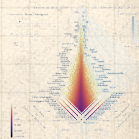
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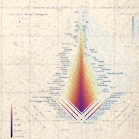
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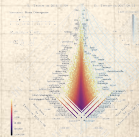
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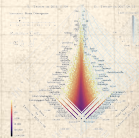
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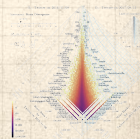
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9. Story-finding: Features 1–8 combine to show which component types are most ‘important’



Some good things about ranks:

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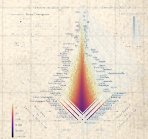
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Working with ranks is intuitive

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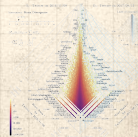
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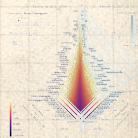
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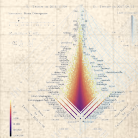
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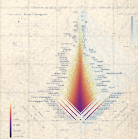
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$$\left| \frac{1}{r_{\tau,1}} - \frac{1}{r_{\tau,2}} \right|. \quad (5)$$

- Inverse of rank gives an increasing measure of 'importance'
- High rank means closer to rank 1
- We assign tied ranks for components of equal 'size'



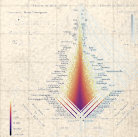
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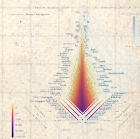
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- We assign tied ranks for components of equal 'size'
- Issue: Biases toward high rank components



We introduce a tuning parameter:

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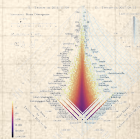


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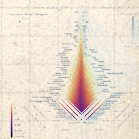
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For words in texts, for example, the weight of common words and rare words move increasingly closer together.



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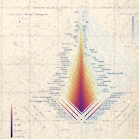
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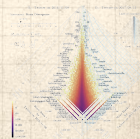
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For texts, the contributions of rare words will vanish.



Trouble:



The limit of $\alpha \rightarrow 0$ does not behave well for

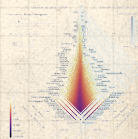
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The leading order term is:

$$\left(1 - \delta_{r_{\tau,1} r_{\tau,2}}\right) \alpha^{1/\alpha} \left| \ln \frac{r_{\tau,1}}{r_{\tau,2}} \right|^{1/\alpha}, \quad (7)$$

which heads toward ∞ as $\alpha \rightarrow 0$.



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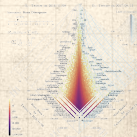
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Oops.



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$$\left(1 - \delta_{r_{\tau,1} r_{\tau,2}}\right) \alpha^{1/\alpha} \left| \ln \frac{r_{\tau,1}}{r_{\tau,2}} \right|^{1/\alpha}, \quad (7)$$

which heads toward ∞ as $\alpha \rightarrow 0$.



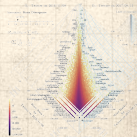
Oops.



But the insides look nutritious:

$$\left| \ln \frac{r_{\tau,1}}{r_{\tau,2}} \right|$$

is a nicely interpretable log-ratio of ranks.



Some reworking:

$$\delta D_{\alpha,\tau}^R(R_1 \parallel R_2) \propto \frac{\alpha+1}{\alpha} \left| \frac{1}{[r_{\tau,1}]^\alpha} - \frac{1}{[r_{\tau,2}]^\alpha} \right|^{1/(\alpha+1)}. \quad (8)$$

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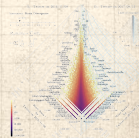
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Some reworking:

$$\delta D_{\alpha,\tau}^R(R_1 || R_2) \propto \frac{\alpha + 1}{\alpha} \left| \frac{1}{[r_{\tau,1}]^\alpha} - \frac{1}{[r_{\tau,2}]^\alpha} \right|^{1/(\alpha+1)}. \quad (8)$$



Keeps the core structure.

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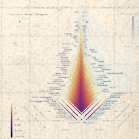
Rank-turbulence
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Some reworking:

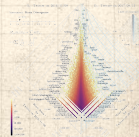
$$\delta D_{\alpha,\tau}^R(R_1 || R_2) \propto \frac{\alpha + 1}{\alpha} \left| \frac{1}{[r_{\tau,1}]^\alpha} - \frac{1}{[r_{\tau,2}]^\alpha} \right|^{1/(\alpha+1)}. \quad (8)$$



Keeps the core structure.



Large α limit remains the same.



Some reworking:

$$\delta D_{\alpha,\tau}^R(R_1 \parallel R_2) \propto \frac{\alpha+1}{\alpha} \left| \frac{1}{[r_{\tau,1}]^\alpha} - \frac{1}{[r_{\tau,2}]^\alpha} \right|^{1/(\alpha+1)}. \quad (8)$$



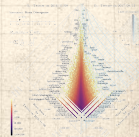
Keeps the core structure.



Large α limit remains the same.



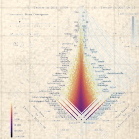
$\alpha \rightarrow 0$ limit now returns log-ratio of ranks.



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$$\delta D_{\alpha,\tau}^R(R_1 || R_2) \propto \frac{\alpha + 1}{\alpha} \left| \frac{1}{[r_{\tau,1}]^\alpha} - \frac{1}{[r_{\tau,2}]^\alpha} \right|^{1/(\alpha+1)}. \quad (8)$$

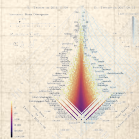
- Keeps the core structure.
- Large α limit remains the same.
- $\alpha \rightarrow 0$ limit now returns log-ratio of ranks.
- Next: Sum over τ to get divergence.



Some reworking:

$$\delta D_{\alpha,\tau}^R(R_1 \parallel R_2) \propto \frac{\alpha+1}{\alpha} \left| \frac{1}{[r_{\tau,1}]^\alpha} - \frac{1}{[r_{\tau,2}]^\alpha} \right|^{1/(\alpha+1)}. \quad (8)$$

-  Keeps the core structure.
-  Large α limit remains the same.
-  $\alpha \rightarrow 0$ limit now returns log-ratio of ranks.
-  Next: Sum over τ to get divergence.
-  Still have an option for normalization.



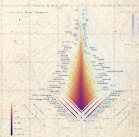
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-  Keeps the core structure.
-  Large α limit remains the same.
-  $\alpha \rightarrow 0$ limit now returns log-ratio of ranks.
-  Next: Sum over τ to get divergence.
-  Still have an option for normalization.

Rank-turbulence divergence:

$$D_{\alpha}^R(R_1 \parallel R_2) = \frac{1}{\mathcal{N}_{1,2;\alpha}} \sum_{\tau \in R_{1,2;\alpha}} \delta D_{\alpha,\tau}^R(R_1 \parallel R_2) \quad (9)$$



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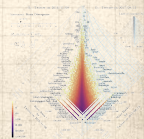
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Normalization:

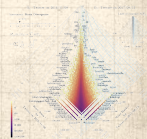


Take a data-driven rather than analytic approach to determining $\mathcal{N}_{1,2;\alpha}$.



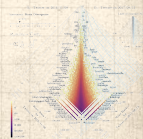
Normalization:

- Take a data-driven rather than analytic approach to determining $\mathcal{N}_{1,2;\alpha}$.
- Compute $\mathcal{N}_{1,2;\alpha}$ by taking the two systems to be disjoint while maintaining their underlying Zipf distributions.



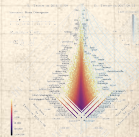
Normalization:

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- Ensures: $0 \leq D_{\alpha}^R(R_1 \parallel R_2) \leq 1$



Normalization:

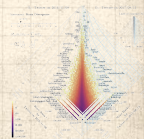
- Take a data-driven rather than analytic approach to determining $\mathcal{N}_{1,2;\alpha}$.
- Compute $\mathcal{N}_{1,2;\alpha}$ by taking the two systems to be disjoint while maintaining their underlying Zipf distributions.
- Ensures: $0 \leq D_{\alpha}^R(R_1 \parallel R_2) \leq 1$
- Limits of 0 and 1 correspond to the two systems having identical and disjoint Zipf distributions.



Rank-turbulence divergence:

Summing over all types, dividing by a normalization prefactor $\mathcal{N}_{1,2;\alpha}$ we have our prototype:

$$D_{\alpha}^R(R_1 \parallel R_2) = \frac{1}{\mathcal{N}_{1,2;\alpha}} \frac{\alpha + 1}{\alpha} \sum_{\tau \in R_{1,2;\alpha}} \left| \frac{1}{[r_{\tau,1}]^{\alpha}} - \frac{1}{[r_{\tau,2}]^{\alpha}} \right|^{1/(\alpha+1)} \quad (10)$$



General normalization:

✖ If the Zipf distributions are disjoint, then in $\Omega^{(1)}$'s merged ranking, the rank of all $\Omega^{(2)}$ types will be $r = N_1 + \frac{1}{2}N_2$, where N_1 and N_2 are the number of distinct types in each system.

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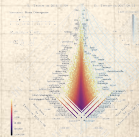
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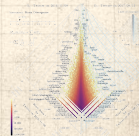
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General normalization:

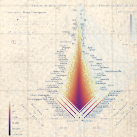
- ❏ If the Zipf distributions are disjoint, then in $\Omega^{(1)}$'s merged ranking, the rank of all $\Omega^{(2)}$ types will be $r = N_1 + \frac{1}{2}N_2$, where N_1 and N_2 are the number of distinct types in each system.
- ❏ Similarly, $\Omega^{(2)}$'s merged ranking will have all of $\Omega^{(1)}$'s types in last place with rank $r = N_2 + \frac{1}{2}N_1$.



General normalization:

- ✖ If the Zipf distributions are disjoint, then in $\Omega^{(1)}$'s merged ranking, the rank of all $\Omega^{(2)}$ types will be $r = N_1 + \frac{1}{2}N_2$, where N_1 and N_2 are the number of distinct types in each system.
- ✖ Similarly, $\Omega^{(2)}$'s merged ranking will have all of $\Omega^{(1)}$'s types in last place with rank $r = N_2 + \frac{1}{2}N_1$.
- ✖ The normalization is then:

$$\begin{aligned} \mathcal{N}_{1,2;\alpha} = & \frac{\alpha+1}{\alpha} \sum_{\tau \in R_1} \left| \frac{1}{[r_{\tau,1}]^\alpha} - \frac{1}{[N_1 + \frac{1}{2}N_2]^\alpha} \right|^{1/(\alpha+1)} \\ & + \frac{\alpha+1}{\alpha} \sum_{\tau \in R_2} \left| \frac{1}{[N_2 + \frac{1}{2}N_1]^\alpha} - \frac{1}{[r_{\tau,2}]^\alpha} \right|^{1/(\alpha+1)}. \end{aligned} \quad (11)$$



Limit of $\alpha \rightarrow 0$:

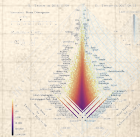
$$D_0^R(R_1 \parallel R_2) = \sum_{\tau \in R_{1,2;\alpha}} \delta D_{0,\tau}^R = \frac{1}{\mathcal{N}_{1,2;0}} \sum_{\tau \in R_{1,2;\alpha}} \left| \ln \frac{r_{\tau,1}}{r_{\tau,2}} \right|, \quad (12)$$

where

$$\mathcal{N}_{1,2;0} = \sum_{\tau \in R_1} \left| \ln \frac{r_{\tau,1}}{N_1 + \frac{1}{2}N_2} \right| + \sum_{\tau \in R_2} \left| \ln \frac{r_{\tau,2}}{\frac{1}{2}N_1 + N_2} \right|. \quad (13)$$



Largest rank ratios dominate.



Limit of $\alpha \rightarrow \infty$:

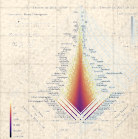
$$\begin{aligned} D_{\infty}^R(R_1 \parallel R_2) &= \sum_{\tau \in R_{1,2;\alpha}} \delta D_{\infty,\tau}^R \\ &= \frac{1}{\mathcal{N}_{1,2;\infty}} \sum_{\tau \in R_{1,2;\alpha}} \left(1 - \delta_{r_{\tau,1} r_{\tau,2}}\right) \max_{\tau} \left\{ \frac{1}{r_{\tau,1}}, \frac{1}{r_{\tau,2}} \right\}. \quad (14) \end{aligned}$$

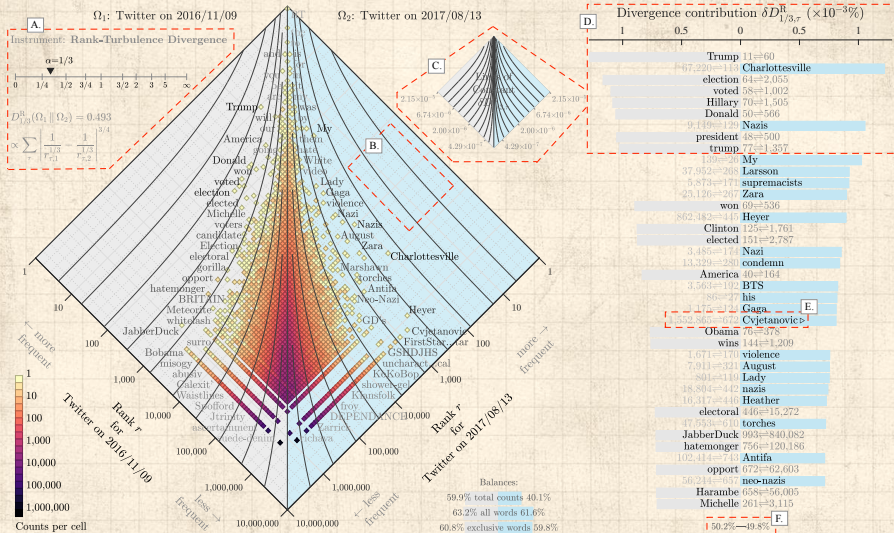
where

$$\mathcal{N}_{1,2;\infty} = \sum_{\tau \in R_1} \frac{1}{r_{\tau,1}} + \sum_{\tau \in R_2} \frac{1}{r_{\tau,2}}. \quad (15)$$



Highest ranks dominate.





Probability-turbulence divergence:

$$D_{\alpha}^p(P_1 \parallel P_2) = \frac{1}{\mathcal{N}_{1,2;\alpha}^p} \frac{\alpha + 1}{\alpha} \sum_{\tau \in R_{1,2;\alpha}} \left| [p_{\tau,1}]^{\alpha} - [p_{\tau,2}]^{\alpha} \right|^{1/(\alpha+1)}. \quad (16)$$



For the unnormalized version ($\mathcal{N}_{1,2;\alpha}^p=1$), some troubles return with 0 probabilities and $\alpha \rightarrow 0$.

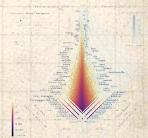


Weep not: $\mathcal{N}_{1,2;\alpha}^p$ will save the day.

Normalization:

With no matching types, the probability of a type present in one system is zero in the other, and the sum can be split between the two systems' types:

$$\mathcal{N}_{1,2;\alpha}^P = \frac{\alpha + 1}{\alpha} \sum_{\tau \in R_1} [p_{\tau,1}]^{\alpha/(\alpha+1)} + \frac{\alpha + 1}{\alpha} \sum_{\tau \in R_2} [p_{\tau,2}]^{\alpha/(\alpha+1)} \quad (17)$$

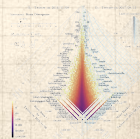


Limit of $\alpha = 0$ for probability-turbulence divergence

if both $p_{\tau,1} > 0$ and $p_{\tau,2} > 0$ then

$$\lim_{\alpha \rightarrow 0} \frac{\alpha + 1}{\alpha} \left| [p_{\tau,1}]^{\alpha} - [p_{\tau,2}]^{\alpha} \right|^{1/(\alpha+1)} = \left| \ln \frac{p_{\tau,2}}{p_{\tau,1}} \right|. \quad (18)$$

But if $p_{\tau,1} = 0$ or $p_{\tau,2} = 0$, limit diverges as $1/\alpha$.

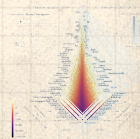


Limit of $\alpha=0$ for probability-turbulence divergence

 Normalization:

$$\mathcal{N}_{1,2;\alpha}^p \rightarrow \frac{1}{\alpha} (N_1 + N_2) . \quad (19)$$

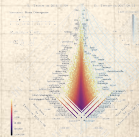
 Because the normalization also diverges as $1/\alpha$, the divergence will be zero when there are no exclusive types and non-zero when there are exclusive types.



Combine these cases into a single expression:

$$D_0^p(P_1 \parallel P_2) = \frac{1}{(N_1 + N_2)} \sum_{\tau \in R_{1,2;0}} (\delta_{p_{\tau,1},0} + \delta_{0,p_{\tau,2}}). \quad (20)$$

-  The term $(\delta_{p_{\tau,1},0} + \delta_{0,p_{\tau,2}})$ returns 1 if either $p_{\tau,1} = 0$ or $p_{\tau,2} = 0$, and 0 otherwise when both $p_{\tau,1} > 0$ and $p_{\tau,2} > 0$.
-  Ratio of types that are exclusive to one system relative to the total possible such types,

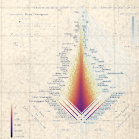


Type contribution ordering for the limit of $\alpha=0$

✉ In terms of contribution to the divergence score, all exclusive types supply a weight of $1/(N_1 + N_2)$. We can order them by preserving their ordering as $\alpha \rightarrow 0$, which amounts to ordering by descending probability in the system in which they appear.

✉ And while types that appear in both systems make no contribution to $D_0^P(P_1 \parallel P_2)$, we can still order them according to the log ratio of their probabilities.

✉ The overall ordering of types by divergence contribution for $\alpha=0$ is then: (1) exclusive types by descending probability and then (2) types appearing in both systems by descending log ratio.

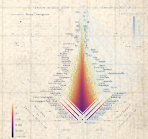


Limit of $\alpha=\infty$ for probability-turbulence divergence

$$D_{\infty}^p(P_1 \parallel P_2) = \frac{1}{2} \sum_{\tau \in R_{1,2;\infty}} (1 - \delta_{p_{\tau,1}, p_{\tau,2}}) \max(p_{\tau,1}, p_{\tau,2}) \quad (21)$$

where

$$\mathcal{N}_{1,2;\infty}^p = \sum_{\tau \in R_{1,2;\infty}} (p_{\tau,1} + p_{\tau,2}) = 1 + 1 = 2. \quad (22)$$



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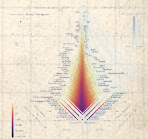
Connections for PTD:

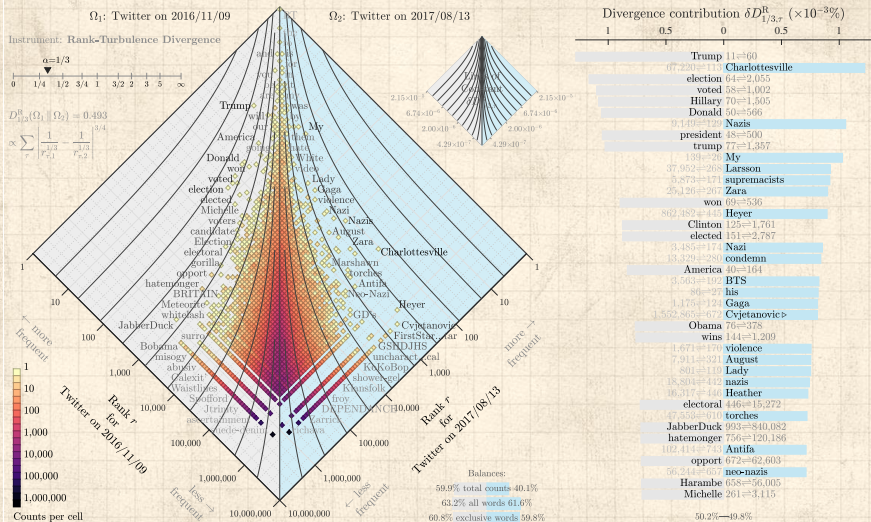
 $\alpha = 0$: Similarity measure Sørensen-Dice coefficient ^[4, 17, 10], F_1 score of a test's accuracy ^[18, 15].

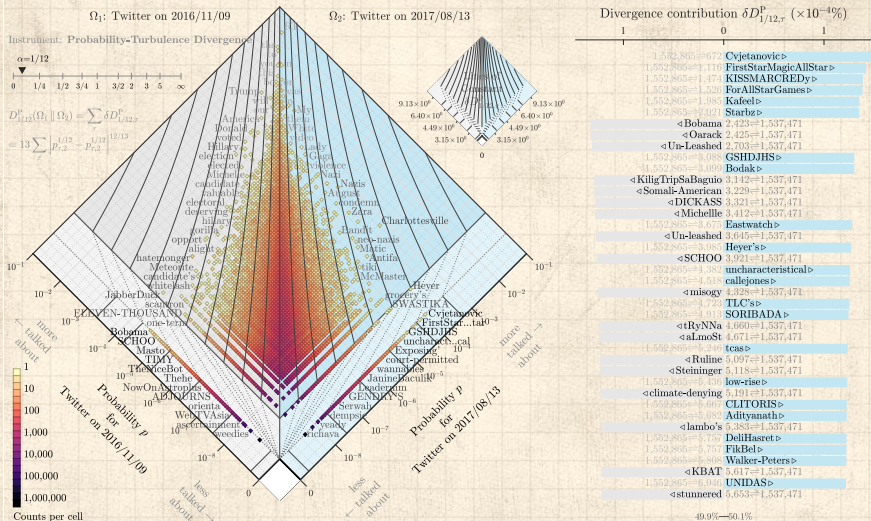
 $\alpha = 1/2$: Hellinger distance ^[8] and Mautusita distance ^[11].

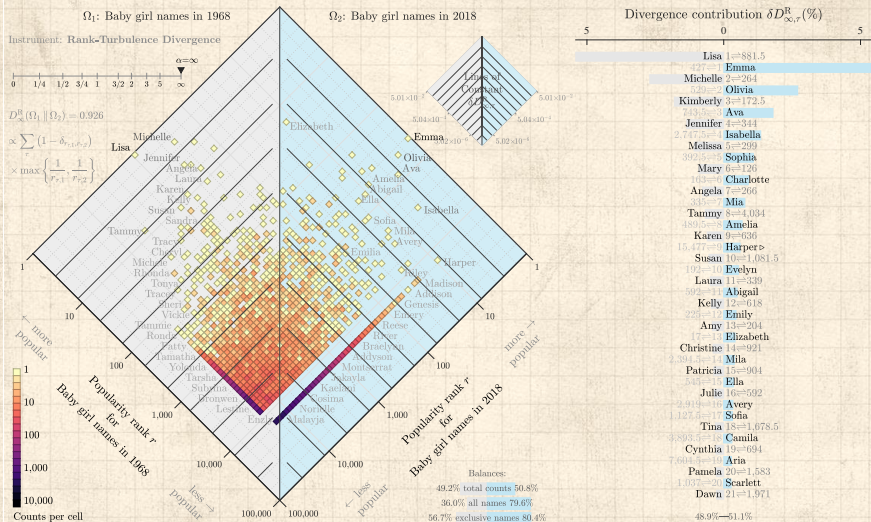
 $\alpha = 1$: Many including all $L^{(p)}$ -norm type constructions.

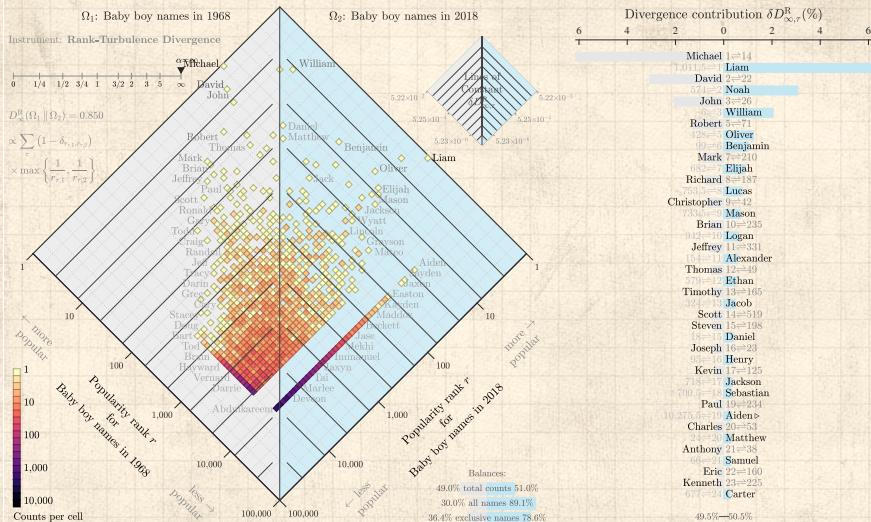
 $\alpha = \infty$: Motyka distance ^[3].

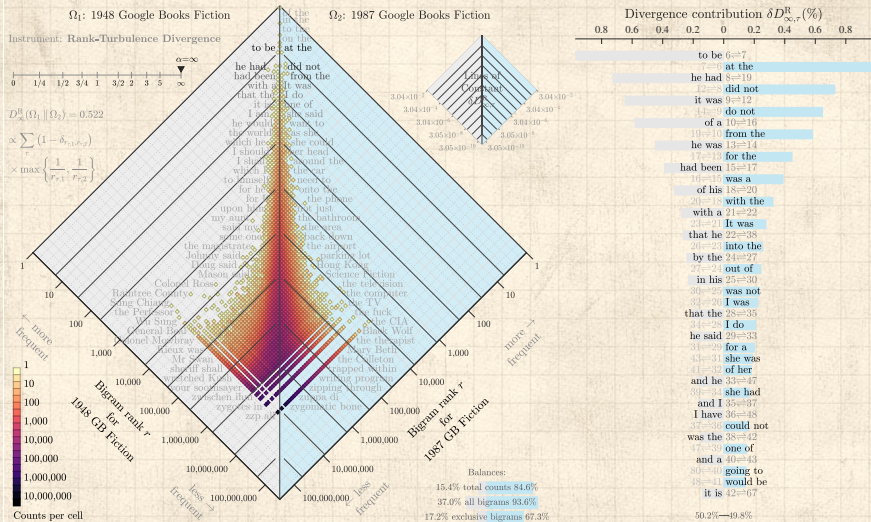




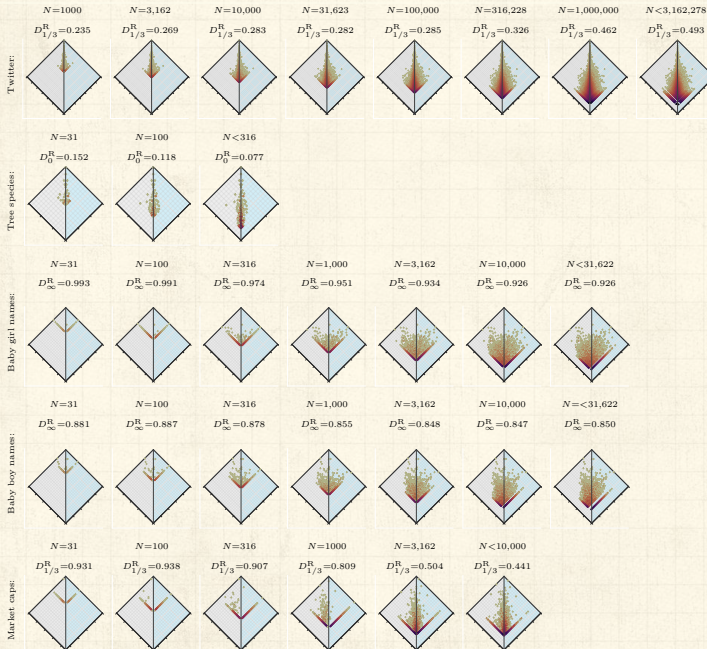








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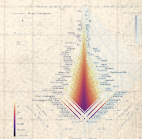
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divergence

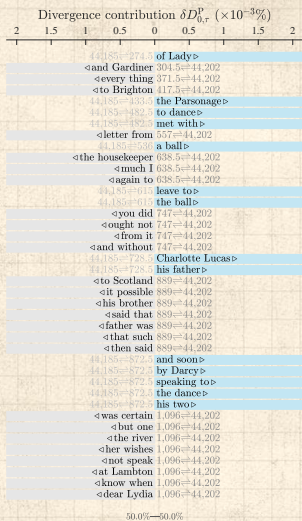
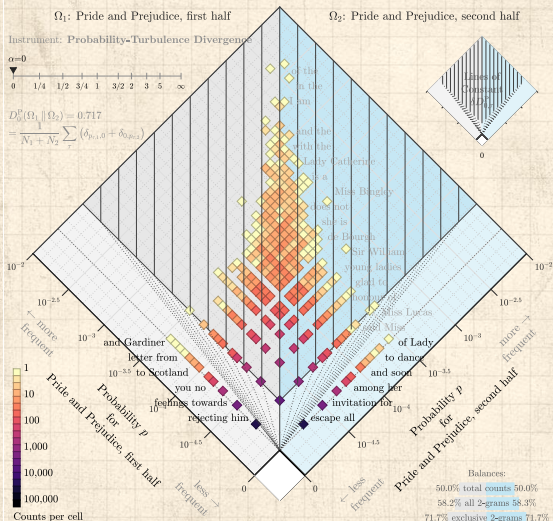
Probability-
turbulence divergence

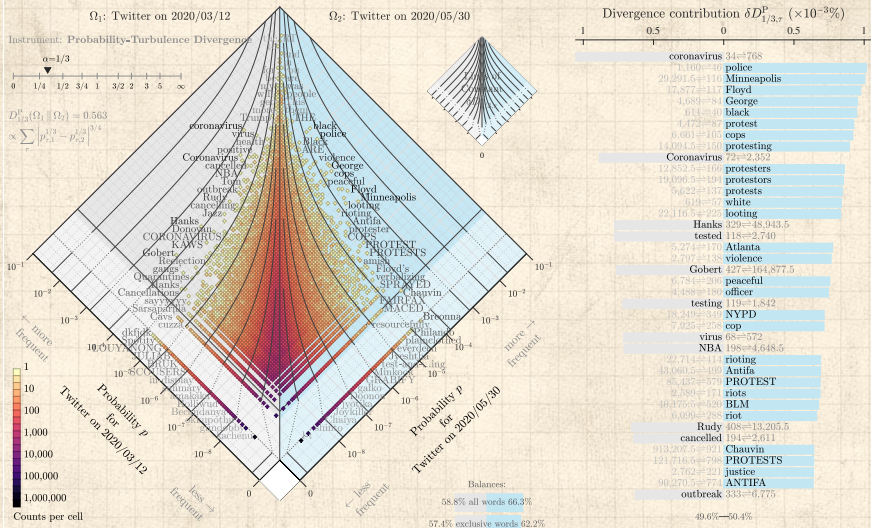
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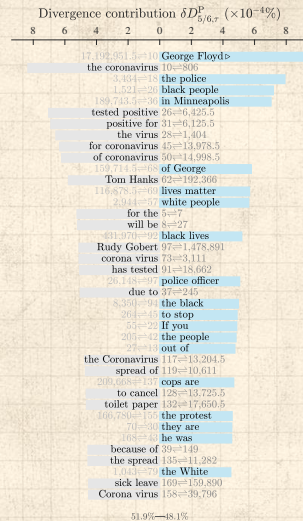
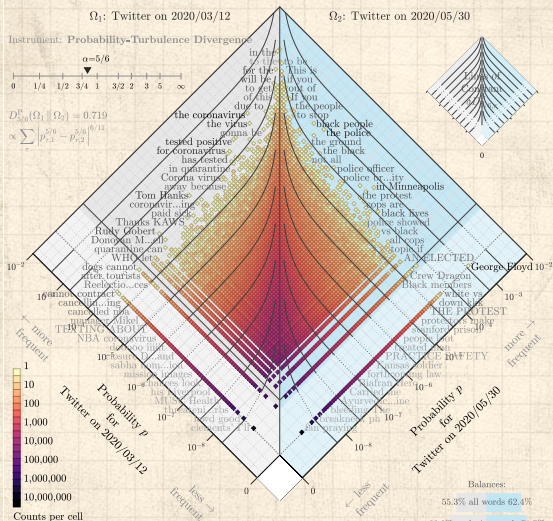
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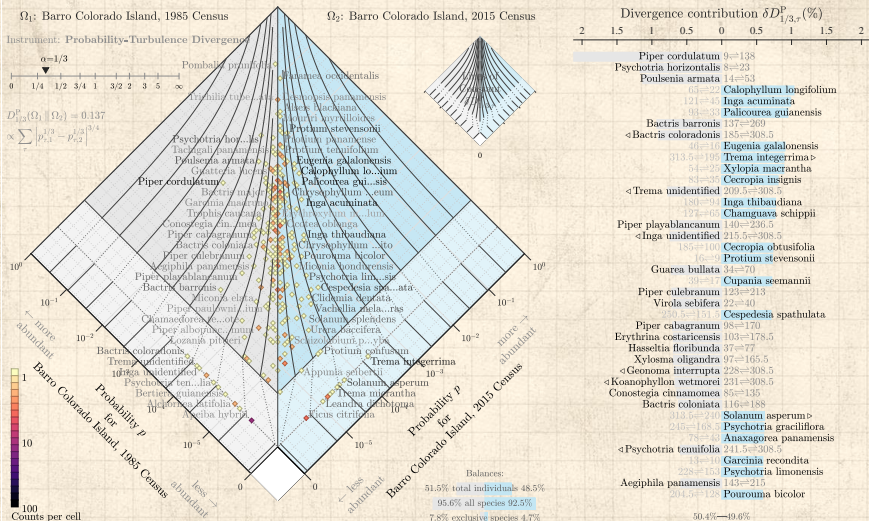
References











Flipbooks for RTD:



Twitter:

[allotaxonometer-flipbook-1-rank-div.pdf](#)  

[allotaxonometer-flipbook-2-probability-div.pdf](#)  

[allotaxonometer-flipbook-3-gen-entropy-div.pdf](#)  



Market caps:

[allotaxonometer-flipbook-4-marketcaps-6years-rank-div.pdf](#)  



Baby names:

[allotaxonometer-flipbook-5-babynames-girls-50years-rank-div.pdf](#)  

[allotaxonometer-flipbook-6-babynames-boys-50years-rank-div.pdf](#)  

[Baby girl names over time relative to 1950](#)  

[Baby boy names over time relative to 1950](#)  



Google books:

[allotaxonometer-flipbook-7-google-books-onegrams-rank-div.pdf](#)  

[allotaxonometer-flipbook-8-google-books-bigrams-rank-div.pdf](#)  

[allotaxonometer-flipbook-9-google-books-trigrams-rank-div.pdf](#)  

Flipbooks for PTD:



Jane Austen:

Pride and Prejudice, 1-grams  

Pride and Prejudice, 2-grams  

Pride and Prejudice, 3-grams  



Social media:

Twitter, 1-grams  

Twitter, 2-grams  

Twitter, 3-grams  



Ecology:

Barro Colorado Island  

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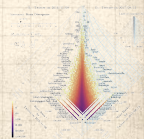
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Code:

<https://gitlab.com/compstorylab/allotaxonometer>



Claims, exaggerations, reminders:



Needed for comparing large-scale complex systems:
Comprehensible, dynamically-adjusting, differential
dashboards.

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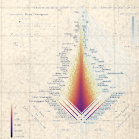
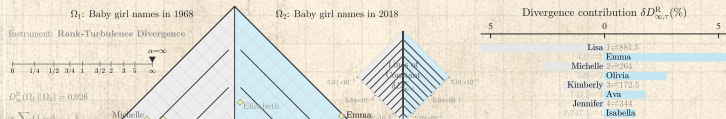
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Comprehensible, dynamically-adjusting, differential
dashboards.



Many measures seem poorly motivated and largely
unexamined (e.g., JSD).

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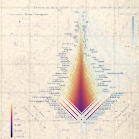
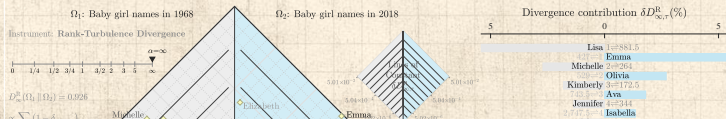
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Claims, exaggerations, reminders:



Needed for comparing large-scale complex systems:
Comprehensible, dynamically-adjusting, differential
dashboards.



Many measures seem poorly motivated and largely
unexamined (e.g., JSD).



Of value: Combining big-picture maps with ranked lists.

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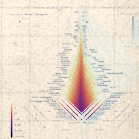
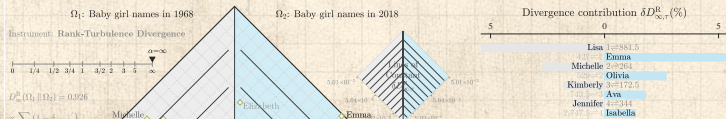
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Online tunable versions of rank-turbulence divergence now
exist:



App version: <https://allotax.vercel.app/>



Observable version:

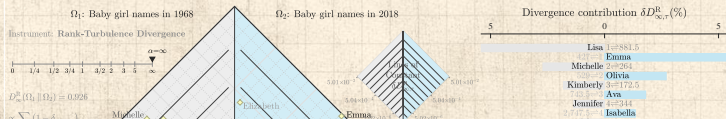
<https://observablehq.com/@jstonge/allotaxonometer-4-all>



Github: <https://github.com/jstonge/allotaxp>



Future: Probability-turbulence divergence plus many other
instruments.



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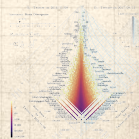
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References



References I

- [1] S.-H. Cha.
Comprehensive survey on distance/similarity measures
between probability density functions.
International Journal of Mathematical Models and Methods
in Applied Sciences, 1:300–307, 2007. pdf ↗
- [2] A. Cichocki and S.-i. Amari.
Families of Alpha- Beta- and Gamma- divergences: Flexible
and robust measures of similarities.
Entropy, 12:1532–1568, 2010. pdf ↗
- [3] M.-M. Deza and E. Deza.
Dictionary of Distances.
Elsevier, 2006.

The PoCverse
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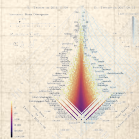
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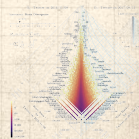
Nutshell

References



References II

- [4] L. R. Dice.
Measures of the amount of ecologic association between species.
Ecology, 26:297–302, 1945.
- [5] P. S. Dodds, J. R. Minot, M. V. Arnold, T. Alshaabi, J. L. Adams, D. R. Dewhurst, T. J. Gray, M. R. Frank, A. J. Reagan, and C. M. Danforth.
Allotaxonomy and rank-turbulence divergence: A universal instrument for comparing complex systems.
EPJ Data Science, 12(37):1–42, 2023.
arXiv version: <https://arxiv.org/abs/2002.09770>. pdf ↗



References III

- [6] P. S. Dodds, J. R. Minot, M. V. Arnold, T. Alshaabi, J. L. Adams, A. J. Reagan, and C. M. Danforth.

Probability-turbulence divergence: A tunable allotaxonomic instrument for comparing heavy-tailed type-size distributions, 2020.

Available online at <https://arxiv.org/abs/2008.13078>. pdf ↗

- [7] D. M. Endres and J. E. Schindelin.

A new metric for probability distributions.

IEEE Transactions on Information Theory, 49(7):1858–1860, 2003. pdf ↗

- [8] E. Hellinger.

Neue begründung der theorie quadratischer formen von unendlichvielen veränderlichen.

Journal für die reine und angewandte Mathematik (Crelles Journal), 1909(136):210–271, 1909. pdf ↗

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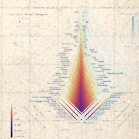
Rank-turbulence
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References IV

[9] J. Lin.

Divergence measures based on the Shannon entropy.

IEEE Transactions on Information theory, 37(1):145–151,
1991. pdf ↗

[10] J. Looman and J. B. Campbell.

Adaptation of Sørensen's k (1948) for estimating unit
affinities in prairie vegetation.

Ecology, 41(3):409–416, 1960. pdf ↗

[11] K. Matusita et al.

Decision rules, based on the distance, for problems of fit, two
samples, and estimation.

The Annals of Mathematical Statistics, 26(4):631–640, 1955.
pdf ↗

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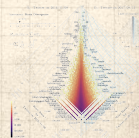
Rank-turbulence
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References



References V

[12] R. Munroe.

How To: Absurd Scientific Advice for Common Real-World Problems.

Penguin, 2019.

[13] F. Osterreicher and I. Vajda.

A new class of metric divergences on probability spaces and its applicability in statistics.

Annals of the Institute of Statistical Mathematics,
55(3):639–653, 2003.

[14] E. A. Pechenick, C. M. Danforth, and P. S. Dodds.

Is language evolution grinding to a halt? The scaling of lexical turbulence in English fiction suggests it is not.

Journal of Computational Science, 21:24–37, 2017. pdf 

[15] Y. Sasaki.

The truth of the f -measure, 2007.

The PoCverse
Allotaxonomy
71 of 73

A plenitude of
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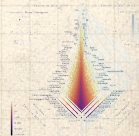
Rank-turbulence
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References



References VI

- [16] C. E. Shannon.
The bandwagon.
[IRE Transactions on Information Theory, 2\(1\):3, 1956.](#)
[pdf](#) 
- [17] T. Sorensen.
A method of establishing groups of equal amplitude in plant sociology based on similarity of species content and its application to analyses of the vegetation on Danish commons.
[Videnski Selskab Biologiske Skrifter, 5:1–34, 1948.](#)
- [18] C. J. Van Rijsbergen.
[Information retrieval.](#)
Butterworth-Heinemann, 2nd edition, 1979.

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A plenitude of
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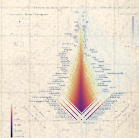
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References

[19] J. R. Williams, J. P. Bagrow, C. M. Danforth, and P. S. Dodds.

Text mixing shapes the anatomy of rank-frequency
distributions.

Physical Review E, 91:052811, 2015. pdf ↗

