From
The Book of Strang

Matrixology

(linear algebra)

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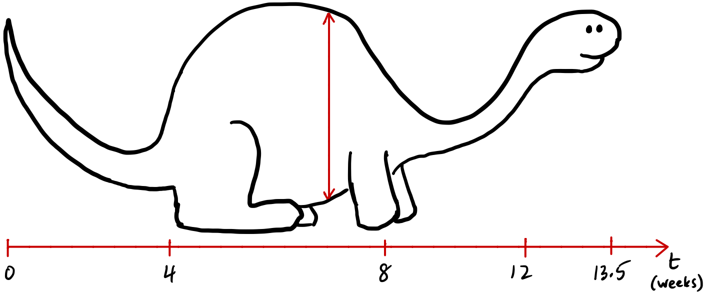
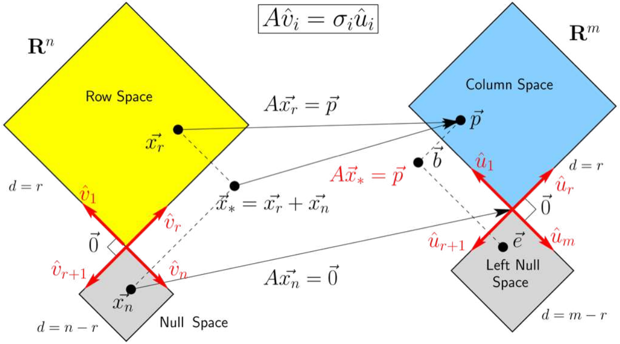
Recorded in 2016

Melvin the Course Difficulty Dinosaur:





The Central problem of Matrixology:

Given a matrix $\underset{m\times n}{\mathbb{A}}$ and a vector $\underset{m\times 1}{\vec{b}}$ find all $\underset{n\times 1}{\vec{x}}$ such that

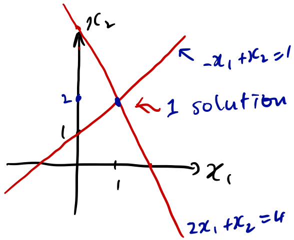
$$\underset{m\times \textcircled{n}}{\mathbb{A}}\ \underset{\textcircled{n}\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$$

$$\left.\begin{aligned} -\overset{1.}{x}_1 + \overset{1.}{x}_2 &= 1 \quad \cdots ① \\ 2x_1 + x_2 &= 4 \quad \cdots ② \end{aligned}\right\}$$ system of linear equations

2×2 system

$\uparrow m$ rows $= \#$ equations

$\uparrow n$ columns $= \#$variables



Row picture

Usual way:

$-x_1 + x_2 = 1 \quad \cdots ①$

$2x_1 + x_2 = 4 \quad \cdots ②$

eq ③ = eq ② + 2 eq ①

$3x_2 = 6 \quad \cdots ③$

$\Rightarrow x_2 = 2$

substitute into eq ①

$-x_1 + 2 = 1$

$\Rightarrow x_1 = 1$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

notes

- Found intersection of two lines
- Both equations are true at this one point

Algebra ⇄ Geometry ← yes!!

Three possibilities:

× a) 1 soln ; // b) no soln ; / c) same line. ∞ly many soln.

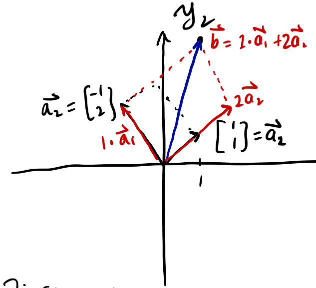
E1ap1

Rewrite system as

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

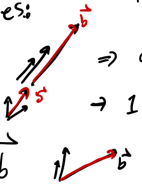
$\vec{a}_1$ ↗ building blocks ↖ $\vec{a}_2$; $\vec{b}$

Column picture



from before:
$x_1 = 1$
$x_2 = 2$

Three possibities:



a) $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$ $\Rightarrow$ ∞ly many solns

b) $\vec{a}_1 \nparallel \vec{a}_2$ → 1 soln (can always make $\vec{b}$ in one way only)

c) $\vec{a}_1 \parallel \vec{a}_2 \nparallel \vec{b}$ → 0 solns.

E1ap2

$$\underset{m\times n}{A}\,\underbrace{\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}}$$

Matrix Picture:

$$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Two ways to multiply matrices.

① dot products of rows and columns

$$\underset{2\times 2}{\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}}\underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} -x_1+x_2 \\ 2x_1+x_2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Row picture: $-x_1 + x_2 = 1$

$2x_1 + x_2 = 4$

②

$$\left[\begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

row vector
1×2

3x3 example: $A\vec{x} = \vec{b}$

find $\vec{x}$ such that A transforms $\vec{x}$ into $\vec{b}$

Matrix picture:

$$m\left\{\overbrace{\begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}}^{n}\right. \underset{\substack{n\times 1\\ 3\times 1}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\substack{m\times 1\\ 3\times 1}}{\begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}}$$

$$\underset{m \quad n}{3\times 3}$$

Multiply out:

Row picture:

$$\begin{aligned} 2x_1 + x_2 \quad\quad &= 2 \\ -x_1 + x_2 + 2x_3 &= 2 \\ 3x_2 + x_3 &= 6 \end{aligned}$$

equations of planes in 3-d

More sneakily

Column picture:

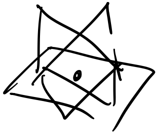
$$\underset{0}{x_1}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \underset{2}{x_2}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + \underset{0}{x_3}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$$

See: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

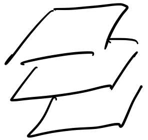
(Always takes more work than this!!!)

E16p1

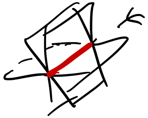
Row picture:



1 soln



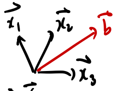
0 soln



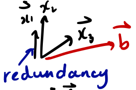
← ∞ly many solns

way too hard in 4-d and above...

Column picture:



1 soln.



0 or ∞ly many soln, } depends on $\vec{b}$.

0 solns or ∞ly many }

← easy in many dimensions

Story: We (people + computers) solve systems of linear equations by "Elimination" ← Gaussian & Gauss-Jordan

Menu:
- Perform Elimination using Row Operations
- Anatomy of Row operations • Triangles !!
- Back Substitution
- key: Pivots D_i, multipliers l_{ij}, upper triangular matrix $\mathcal{U}$, Augmented Matrix
- when things go "wrong"

$$\mathbb{A}\vec{x} = \vec{b}$$

$$\begin{cases} \overset{-1}{-x_1} + 3x_2 = 1 \quad \cdots \cancel{①} \; R_1 \\ \underset{\text{eliminate}}{②x_1} + x_2 = 5 \quad \cdots \cancel{②} \; R_2 \end{cases}$$

2x2 system
$m=2$ = # eqs
$n=2$ = # variables

$D_1 = -1$

$-x_1 + 3x_2 = 1 \quad \cdots \; R_1$

$0x_1 + 7x_2 = 7 \quad \cdots$

upper triangular

D_2

$$R_2' = R_2 - \left(\frac{2}{-1}\right) R_1$$

l_{21}

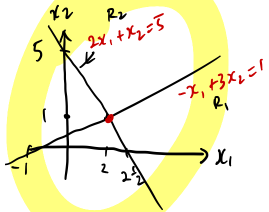
always use this form

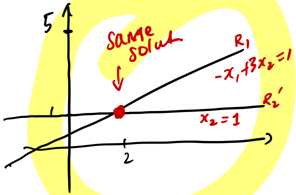
+2

D_1 = first Pivot without division

$D_2 = 7$

R_2 (drop primes): $x_2 = 1$





E2ap1

We have $x_2 = 1$, now solve for x_1 using back substitution:

$$-x_1 + 3x_2 = 1 \Rightarrow -x_1 + 3 = 1$$
$$x_1 = 2$$

$\nwarrow x_2 = 1$

soluti $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓

For later, we can go further and avoid back substition.
Gauss-Jordan elimination:

$$\begin{cases} -x_1 + 3x_2 = 1 \quad \cdots R_1 \\ 0 + 1x_2 = 1 \quad \cdots R_2 \end{cases}$$

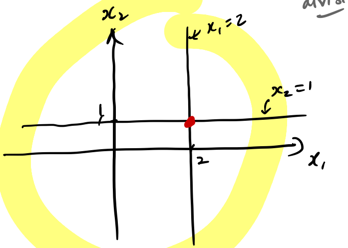
$$R_1' = R_1 - \left(\frac{3}{1}\right) R_2$$

$D_2 = 7$ before division

$$\begin{cases} -x_1 + 0 = -2 \\ 0 + x_2 = 1 \end{cases}$$

$R_2' = R_2$

$$\Rightarrow \; +x_1 = 2$$
$$x_2 = 1$$



E2ap2

Basic Elimination rules:

① Create upper trianglur system by systematic by row operations

② Swap rows if needed when pivots = 0

$$\begin{aligned} 0 + x_2 &= 3 \\ 3x_1 - 7x_2 &= 0 \end{aligned} \quad \underset{R_1 \leftrightarrow R_2}{\sim} \quad \begin{aligned} 3x_1 - 7x_2 &= 0 \\ x_2 &= 3 \end{aligned}$$

Augmented Matrix approach:

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_2 + x_2 &= 5 \end{aligned} \Rightarrow \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Row picture

$A\vec{x} = \vec{b}$, matrix

$$\overset{D_1}{\rightarrow} \left[\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & 1 & 5 \end{array}\right] \sim R_2' = R_2 - \left(\frac{2}{-1}\right) \underset{D_1}{R_1} \quad \left[\begin{array}{cc|c} \overset{D_1}{-1} & 3 & 1 \\ 0 & \underset{D_2}{7} & 7 \end{array}\right]$$

$[A \mid \vec{b}]$

~ means systems have same solut.

E2b p1

Menu:

- 3x3 example of solving $A\vec{x}=\vec{b}$ with Elimination and Row Swaps
- Turn $A\vec{x}=\vec{b}$ into $U\vec{x}=\vec{c}$

↖ upper triangul

Row picture:

$2x_1 - 3x_2 + 0\cdot x_3 = 3$ eq1

$4x_1 - 5x_2 + 1\cdot x_3 = 7$ eq2

$2x_1 - 1x_2 - 3x_3 = 5$ eq3

Three planes

Column Picture:

$$x_1\underbrace{\begin{bmatrix}2\\4\\2\end{bmatrix}}_{\vec{a}_1} + x_2\underbrace{\begin{bmatrix}-3\\-5\\-1\end{bmatrix}}_{\vec{a}_2} + x_3\underbrace{\begin{bmatrix}0\\1\\-3\end{bmatrix}}_{\vec{a}_3} = \underbrace{\begin{bmatrix}3\\7\\5\end{bmatrix}}_{\vec{b}}$$

Matrix Picture

$$\begin{bmatrix}2 & -3 & 0\\4 & -5 & 1\\2 & -1 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\7\\5\end{bmatrix} \qquad A\vec{x}=\vec{b}$$

Augmented Matrix version of row picture:

$$\left[\begin{array}{ccc|c}2 & -3 & 0 & 3\\4 & -5 & 1 & 7\\2 & -1 & -3 & 5\end{array}\right]\begin{array}{l}R_1\\R_2\\R_3\end{array}$$

① ② ③ 0s

$$[\underbrace{\underbrace{A}_{m\times n} \mid \underbrace{\vec{b}}_{m\times 1}}_{m\times(n+1)}]$$

ⓘ = order of eliminatn

multiplier

$$R_2' = R_2 - \left(\frac{4}{2}\right)R_1 \qquad \ell_{21} \qquad D_1$$

$$\left[\begin{array}{ccc|c}2 & -3 & 0 & 3\\0 & 1 & 1 & 1\\2 & -1 & -3 & 5\end{array}\right]$$

$$R_3' = R_3 - \left(\frac{2}{2}\right)R_1 \qquad \ell_{31} \qquad D_1$$

$$\left[\begin{array}{ccc|c}2 & -3 & 0 & 3\\0 & 1 & 1 & 1\\0 & 2 & -3 & 2\end{array}\right] \qquad D_1 \quad D_2$$

$$R_3' = R_3 - \left(\frac{2}{1}\right)R_2 \qquad \ell_{32} \qquad D_2$$

$$\left[\begin{array}{ccc|c}2 & -3 & 0 & 3\\0 & 1 & 1 & 1\\0 & 0 & -5 & 0\end{array}\right]$$

main diagonal

$$\underset{\uparrow\, A}{U}\vec{x} = \underset{\uparrow\, \vec{b}}{\vec{c}} \Rightarrow \begin{bmatrix}2 & -3 & 0\\0 & 1 & 1\\0 & 0 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}$$

↗ easy to solve with back substitution

Back substitution:

Step back out to equations and work upwards:

R_3: $-5x_3 = 0 \Rightarrow x_3 = 0$

R_2: $x_2 + \underset{0}{x_3} = 1 \Rightarrow x_2 = 1$

R_1: $2x_1 - 3\underset{=1}{x_2} = 3 \Rightarrow 2x_1 - 3 = 3$

$2x_1 = 6$

$x_1 = 3$

Solution:

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}$$

Important:

$\ell_{21} = 2$ $\ell_{32} = 2$

$\ell_{31} = 1$ multipliers

Pivots: find in U:

$D_1 = 2$, $D_2 = -1$, $D_3 = 5$

Menu:

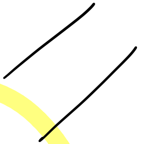
- what can happen when a pivot is zilch....
- Singular system

ex 1

$-x_1 + x_2 = 1$ .... R_1

$\overset{+1}{x_1} - x_2 = 5$ ... R_2

parallel



$$\Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & -1 & 5 \end{array}\right]$$

"has same solution as"

$$R_2' = R_2 - \underbrace{\left(\frac{1}{-1}\right)}_{D_1}^{l_{21}} R_1$$

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 6 \end{array}\right]$$

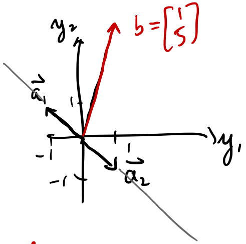
↖ 0 pivot
pivot is "missing"

R_2: $0x_1 + 0x_2 = 6$ not true!

$0 = 6$

Column picture:

$$x_2\underbrace{\begin{bmatrix}-1\\1\end{bmatrix}}_{\vec{a}} + x_2\underbrace{\begin{bmatrix}1\\-1\end{bmatrix}}_{\vec{b}} = \overset{\vec{b}}{\begin{bmatrix}1\\5\end{bmatrix}}$$



example of
Singular System

↘ no unique solutn
may have 0 or ∞ly many

E2cp1

ex 2

$-x_1 + x_2 = 1$ ..... R_1

$2x_1 - 2x_2 = -2$ ..... R_2

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 2 & -2 & -2 \end{array}\right]$$

$A\vec{x} = \vec{b}$

$$R_2' = R_2 - \underbrace{\left(\frac{2}{-1}\right)}_{D_1}^{l_{21} = -2} R_1$$

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$U\vec{x} = \vec{c}$

singular matrix

eqs: $-x_1 + x_2 = 1$ ... R_1

~~$0 = 0$~~ ... R_2

later pivot variable

free variable

Let $x_2 \in \mathbb{R}^1$ ← real numbers → x_1 now depends on x_2

$-x_1 = 1 - x_2$

$x_1 = x_2 - 1$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2 - 1\\x_2\end{bmatrix} = x_2\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix}$$

$1 \cdot x_2 - 1$

$1 \cdot x_2 + 0$

↑ replace pivot variables with free variables

↑ dialable piece

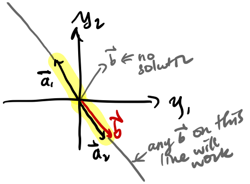
↑ fixed

where $x_2 \in \mathbb{R}^1$

Column pic

$$x_1\underbrace{\begin{bmatrix}-1\\2\end{bmatrix}}_{\vec{a}_1} + x_2\underbrace{\begin{bmatrix}1\\-2\end{bmatrix}}_{\vec{a}_2} = \underbrace{\begin{bmatrix}1\\-2\end{bmatrix}}_{\vec{b}}$$

$\vec{a}_1 = -\vec{a}_2$



E2dp1

- Our task: Solve systems of linear equations
- Three pictures: row, column, & matrix.

row ↑ where solving happens

column ↑ understanding

matrix ↑ deep understanding

2x2 example from Episode 2

$$\begin{matrix} -x_1+3x_2=1 \\ 2x_1+x_2=5 \end{matrix} \Leftrightarrow x_1\overset{\vec{a}_1}{\begin{bmatrix}-1\\2\end{bmatrix}}+x_2\overset{\vec{a}_2}{\begin{bmatrix}3\\1\end{bmatrix}}=\overset{\vec{b}}{\begin{bmatrix}1\\5\end{bmatrix}} \Leftrightarrow \overset{\mathbb{A}}{\begin{bmatrix}-1&3\\2&1\end{bmatrix}}\overset{\vec{x}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}}=\overset{\vec{b}}{\begin{bmatrix}1\\5\end{bmatrix}}$$

Row 1 · Row 2

row pic · column pic · matrix pic

Solve by Gaussian Elimination

Equations (first) ⟷ same ⟷ Augmented Matrix (second) ← tidy

$-x_1+3x_2=1$ ... R1

$2x_1+x_2=5$ ... R2

multiplier ℓ_{21} ↓

⇒ $-x_1+3x_2=1$ ... R1

$0x_1+7x_2=7$ ... $R2'=R2-\left(\frac{2}{-1}\right)R_1$

↑ echelon form

$$\left[\begin{array}{cc|c}-1&3&1\\2&1&5\end{array}\right]$$

$R_2'=R_2-\left(\frac{2}{\boxed{-1}}\right)R_1$ — $D_1=-1=$ first pivot

$\sim$ $\left[\begin{array}{cc|c}-1&3&1\\0&7&7\end{array}\right]$

multiplier $\ell_{21}=-2$

↑ echelon form

Matrix picture: $\mathbb{A}\vec{x}=\vec{b} \Rightarrow \mathbb{U}\vec{x}=\vec{c}$

$$\mathbb{U}=\begin{bmatrix}-1&3\\0&7\end{bmatrix} \qquad \vec{c}=\begin{bmatrix}1\\7\end{bmatrix}$$

The Gaussian Eliminator 9000:

$$\left[\begin{array}{ccc|c}2&-3&0&3\\ \boxed{4}&-5&1&7\\2&-1&-3&5\end{array}\right]$$

$\mathbb{A}$ · $\vec{b}$ · ← Augmented Matrix for $\mathbb{A}\vec{x}=\vec{b}$

① → 4 · ② → 2 · ③ → −1

multiplier $\ell_{21}=2$

$R_2'=R_2-\left(\frac{4}{2}\right)R_1$ ↖ D_1

big tilde → $\sim$ ↑ equivalent to

$$\left[\begin{array}{ccc|c}2&-3&0&3\\0&1&1&1\\2&-1&-3&5\end{array}\right]$$

↓ rowoperatorify ↓

$\ell_{31}=1$

$R_3'=R_3-\left(\frac{2}{2}\right)R_1$ ↖ D_1

$\sim$

$$\left[\begin{array}{ccc|c}2&-3&0&3\\0&1&1&1\\0&2&-3&2\end{array}\right]$$

$\ell_{32}=2$

$R_3'=R_3-\left(\frac{2}{1}\right)R_2$ ↖ D_2

$\sim$

$$\left[\begin{array}{ccc|c}2&-3&0&3\\0&1&1&1\\0&0&-5&0\end{array}\right]$$

D_1 · D_2 · D_3

#eliminated · ← echelon form

$\mathbb{U}\vec{x}=\vec{c}$

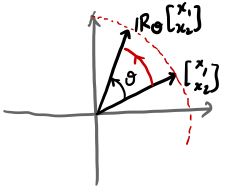
⇓ easy to solve with back substitution

Menu:
- Using Elimination matrices to do the work for us
- Surprising help for our understanding will be possible
- Somehow, elimination makes two triangles.

Observation: Matrices can do sneaky, gadgety things for us

ex Rotate a vector in 2-d through θ radians

$$\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}}_{\mathbb{R}_\theta}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$



ex Permute entries in a vector:

$$\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}x_2\\ x_3\\ x_4\\ x_1\end{bmatrix}$$

← cycle by 1.

Plan: encode row operations as
(1) elimination matrices ← normal elimination steps
& (2) permutation matrices ← row swaps

E3ap1

Augmented Matrix approach:

$$\begin{aligned}-x_1 + 3x_2 &= 1\\ 2x_2 + x_2 &= 5\end{aligned} \Rightarrow \begin{bmatrix}-1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1\\ 5\end{bmatrix}$$

Row picture

$A\vec{x} = \vec{b}$, matrix

$$\underset{[A\,|\,\vec{b}]}{\begin{bmatrix}\overset{D_1}{-1} & 3 & \big| & 1\\ 2 & 1 & \big| & 5\end{bmatrix}} \overset{R_2' = R_2 - \left(\frac{2}{-1}\right)R_1}{\sim} \begin{bmatrix}\overset{D_1}{-1} & 3 & \big| & 1\\ 0 & \underset{D_2}{7} & \big| & 7\end{bmatrix}$$

$l_{21} = -2$

D_1

~ means systems have same solut.

replace w. matrix multiplication

$\mathbb{E}_{21}$ = elimination matrix that removes the $\underset{\text{row}\ \ \text{column}}{2,1}$ entry in A or 1st entry in 2nd row.

here

$$\mathbb{E}_{21} = \begin{bmatrix}1 & 0\\ \underset{-l_{21}}{-\frac{2}{-1}} & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

Let's see how this works:

$$\underset{\mathbb{E}_{21}}{\begin{bmatrix}1&0\\2&1\end{bmatrix}}\underset{\mathbb{A}}{\begin{bmatrix}-1&3\\2&1\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}}=\underset{\mathbb{E}_{21}}{\begin{bmatrix}1&0\\2&1\end{bmatrix}}\underset{\vec{b}}{\begin{bmatrix}1\\5\end{bmatrix}}$$

↑ premultiply both sides

$$\underset{\mathbb{U}}{\begin{bmatrix}-1&3\\0&7\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}}=\begin{bmatrix}1\\7\end{bmatrix}$$

$$= \vec{c}$$

Anatomy of $\mathbb{E}_{21}$:

$$\begin{bmatrix}1&0\\2&1\end{bmatrix}$$

keep copy of first row $R_1' = R_1$

add 2x first row to second row to make new second row

$$R_2' = R_2 - \underset{\nwarrow 2}{\ell_{21}} R_1$$

3×3 example:

E3ap2

We need $\mathbb{E}_{21}$✓, $\mathbb{E}_{31}$, & $\mathbb{E}_{32}$

(ℓ_{21}), (ℓ_{31}), (ℓ_{32})

ex

$$\begin{bmatrix}1&0&-1\\2&3&1\\3&6&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2\\2\\4\end{bmatrix}$$

Row op

$$R_2' = \underset{\uparrow}{1}R_2 - \overset{\ell_{21}}{\left(\frac{2}{\underset{D_1}{1}}\right)} R_1$$

Eliminat matrix

$$\mathbb{E}_{21} = \begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$$

← mostly indentity matrix $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$-\ell_{21}$

$$R_3' = 1\cdot R_3 - \overset{\ell_{31}=3}{\left(\frac{3}{\underset{D_2}{1}}\right)} R_1$$

$$\mathbb{E}_{31} = \begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}$$

$-\ell_{31}$

Must use elimination matrices to get to $\mathbb{E}_{32}$

$$\underset{\mathbb{E}_2}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\underset{\mathbb{A}}{\begin{bmatrix}1&0&-1\\2&3&1\\3&6&2\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}} = \underset{\mathbb{E}_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\underset{\vec{b}}{\begin{bmatrix}-2\\2\\4\end{bmatrix}}$$

$$\Rightarrow \quad \begin{bmatrix}1&0&-1\\0&3&3\\3&6&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\6\\4\end{bmatrix}$$

next: premultiply by $\mathbb{E}_{31} = \begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix}1&0&-1\\0&3&3\\0&6&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\6\\10\end{bmatrix}$$

Important: can now see next row op

$$R_3' = R_3 - \Big(\overset{\ell_{21}=2}{\frac{6}{\underset{D_2}{3}}}\Big) R_2 \quad \Longleftrightarrow \quad \mathbb{E}_{32} = \begin{bmatrix}1&0&0\\0&1&0\\0&\underset{-\ell_{32}}{-2}&1\end{bmatrix}$$

E3ap3

As before
Premultiply by
elimination matrix $\mathbb{E}_{32}$

LHS:

$$\mathbb{E}_{32}\,\mathbb{E}_{31}\,\mathbb{E}_{21}\,\mathbb{A} = \mathbb{U} =$$

$$= \underset{\mathbb{U}}{\begin{bmatrix}1&0&-1\\0&3&3\\0&0&-1\end{bmatrix}}$$

pivots
$D_1 = 1$
$D_2 = 3$
$D_3 = -1$

RHS

$$\mathbb{E}_{32}\,\mathbb{E}_{31}\,\mathbb{E}_{21}\,\vec{b}$$

$$= \begin{bmatrix}-2\\6\\-2\end{bmatrix}$$

To find solution, now
use back substitution

Note: $\mathbb{E}_{ij}$ are always $m \times m$ lower triangular matrices (0's above main diagonal).

Sometimes row swaps are necessary.

ex

$$\underset{\mathbb{P}_{12}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

ex: 3×3 that swaps rows 2 & 3.

$$\mathbb{P}_{23} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{keep: } R_1' = R_1 \\ \leftarrow R_2' = R_3 \\ \leftarrow R_3' = R_2 \end{array}$$

Usually, do row swaps first

3×3 example

$$\mathbb{U} = \mathbb{E}_{32}\mathbb{E}_{31}\mathbb{E}_{21}\underset{\substack{\uparrow \\ \text{row swaps}}}{\mathbb{P}}\mathbb{A}$$

$\|$

$$\vec{c} = \mathbb{E}_{32}\mathbb{E}_{31}\mathbb{E}_{21}\mathbb{P}\vec{b}$$

Menu:
- Matrix operations
- How to add, scale, and multiply
- The Sneakiness of Matrix multiplication

① Scalar multiplication:

$$3\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 & 3\cdot 1 \\ 3\cdot(-1) & 3\cdot 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}$$

$\mathbb{B} = c\mathbb{A}$

$b_{ij} = c a_{ij}$

Notation write ij th entry of $\underset{}{\overset{m\times n}{\mathbb{A}}}$ as a_{ij}

row index $1 \leq i \leq m$

column index $1 \leq j \leq n$

$\mathbb{B}$ $\qquad b_{ij}$

Sometimes: $\mathbb{A} = [a_{ij}]$

② Addition:

$\mathbb{A} + \mathbb{B}$ is only possible if $\mathbb{A}$ & $\mathbb{B}$ are the same shape

ex.

$$\underset{3\times 2}{\overset{\mathbb{A}}{\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}}} + \underset{3\times 2}{\overset{\mathbb{B}}{\begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}}} = \underset{3\times 2}{\overset{\mathbb{C}}{\begin{bmatrix} 3 & 2 \\ 3 & -1 \\ -1 & 2 \end{bmatrix}}}$$

entrywise addition

$$c_{ij} = a_{ij} + b_{ij}$$

③ Multiplication:

$\underset{m\times k}{\mathbb{A}}\ \underset{k\times n}{\mathbb{B}}$ is only possible if inner dimensions match

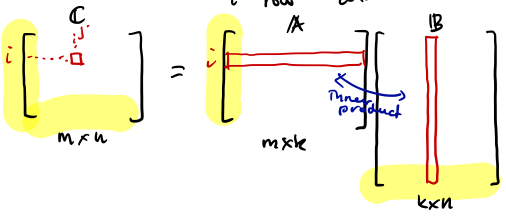
E4ap1

$$\underset{m\times n}{\mathbb{C}} = \underset{m\times k}{\mathbb{A}}\ \underset{k\times n}{\mathbb{B}}$$

Defn:

* C_{ij}, the entry for $\mathbb{C}$ in the i^{th} row and j^{th} column is the dot (inner) product of the i^{th} row of $\mathbb{A}$ and the j^{th} row of $\mathbb{B}$

* $$C_{ij} = \sum_{l=1}^{k} \underbrace{a_{il}}_{i^{th}\text{ row}} \underbrace{b_{lj}}_{j^{th}\text{ column}}$$



Rules matrix operations are pretty normal ...

$$\mathbb{A} + \mathbb{B} = \mathbb{B} + \mathbb{A}$$

commutative law for addition

$$\mathbb{A}\mathbb{B}\mathbb{C} = (\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$$

One bananapants exception:

$\mathbb{A}\mathbb{B}$ most often does not equal $\mathbb{B}\mathbb{A}$!!!!

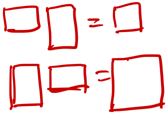
~~Two~~ Three problems

(1) $\underset{m\times k}{\mathbb{A}}\ \underset{k\times n}{\mathbb{B}}$ ✓ $\qquad \underset{k\times n}{\mathbb{B}}\ \underset{m\times k}{\mathbb{A}}$

if $n \neq m$, $\mathbb{B}\mathbb{A}$ does not make sense

(2) If $n = m$, products are both ok.

$$\underbrace{\underset{m\times k}{\mathbb{A}}\ \underset{k\times m}{\mathbb{B}}}_{m\times m} \qquad \underbrace{\underset{k\times m}{\mathbb{B}}\ \underset{m\times k}{\mathbb{A}}}_{k\times k}$$



if $k \neq m$, no good either

(3) So $m = n = k$ is required for us to even have a chance that $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$

Observe: Only possible for $n\times n$ square matrices

E4ap2

Even then, $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$ often

ex/

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 6 & 5 \end{bmatrix} \neq$$

$$\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 10 \end{bmatrix}$$

If $\underset{n\times n}{\mathbb{A}}\ \underset{n\times n}{\mathbb{B}} = \underset{n\times n}{\mathbb{B}}\ \underset{n\times n}{\mathbb{A}}$, we get very excited and say $\mathbb{A}$ & $\mathbb{B}$ commute

↑ special. spesh

Warning: Never slide matrices around in products and always be careful with order. //

Menu:
- Wizard-level matrix multiplication skills
- Inner and outer products

dot

- $\mathbb{A}\vec{x}$, $\vec{y}^T\mathbb{B}$, $\mathbb{A}\mathbb{B}$
- Block multiplication in general

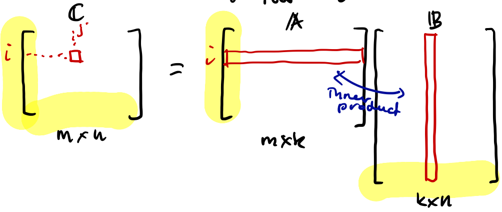
from before:

$$\underset{m\times n}{\mathbb{C}} = \underset{m\times k}{\mathbb{A}}\;\underset{k\times n}{\mathbb{B}}$$

Defn:

* C_{ij}, the entry for $\mathbb{C}$ in the i^{th} row and j^{th} column is the dot (inner) product of the i^{th} row of $\mathbb{A}$ and the j^{th} row of $\mathbb{B}$

* $$C_{ij} = \sum_{l=1}^{k} \underbrace{a_{il}}_{i^{th}\text{ row}}\underbrace{b_{lj}}_{j^{th}\text{ column}}$$



ex 1

E4b p1

$$\overset{\mathbb{A}}{\underset{2\times 3}{\begin{bmatrix}3&0&2\\1&-2&2\end{bmatrix}}}\;\overset{\mathbb{B}}{\underset{3\times 2}{\begin{bmatrix}-1&0\\2&1\\0&2\end{bmatrix}}} = \underset{2\times 2}{\begin{bmatrix}-3&4\\-5&2\end{bmatrix}}$$

C_{11} C_{12} C_{21} C_{22}

$$C_{11} = \begin{bmatrix}3&0&2\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix}$$

1st row of $\mathbb{A}$

1st col of $\mathbb{B}$

$$C_{12} = \begin{bmatrix}3&0&2\end{bmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}$$

$$C_{21} = \begin{bmatrix}1&-2&2\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix} = -5$$

$$C_{22} = \begin{pmatrix}1&-2&2\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix} = 2$$

ex2

$$\underset{1\times 3\ \checkmark}{\begin{bmatrix}1 & 2 & -1\end{bmatrix}}\begin{bmatrix}0\\1\\-2\end{bmatrix} = \underset{1\times 1}{4}$$

inner product

ex3

$$\overset{3\times \text{①}\ \checkmark}{\begin{bmatrix}0\\1\\-2\end{bmatrix}}\ \overset{3\times 1}{\underset{\text{①}\times 3\ \checkmark}{\begin{bmatrix}1 & 2 & -1\end{bmatrix}}} = \underset{3\times 3}{\begin{bmatrix}0 & 0 & 0\\1 & 2 & -1\\-2 & -4 & 2\end{bmatrix}}$$

amazingly important construction

outer product

later: see this is a rank r = 1 matrix.

See

$$= \begin{bmatrix}0\cdot\begin{bmatrix}1 & 2 & -1\end{bmatrix}\\1\cdot\begin{bmatrix}1 & 2 & -1\end{bmatrix}\\-2\cdot\begin{bmatrix}1 & 2 & -1\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}1\cdot\begin{bmatrix}0\\1\\-2\end{bmatrix} & 2\begin{bmatrix}0\\1\\-2\end{bmatrix} & -1\begin{bmatrix}0\\1\\2\end{bmatrix}\end{bmatrix}$$

ex4 block multiplication: E4bp2

$$\left[\begin{array}{c|c|c}3 & 0 & 2\\1 & -2 & 2\end{array}\right]\left[\begin{array}{cc}-1 & 0\\\hline 2 & 1\\\hline 0 & 2\end{array}\right]$$

row of 2×1's

column of 1×2's

$$= \begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}-1 & 0\end{bmatrix} + \begin{bmatrix}0\\-2\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 0\\-1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\-4 & -2\end{bmatrix} + \begin{bmatrix}0 & 4\\0 & 4\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 4\\-5 & 2\end{bmatrix}$$

ex5

$$\begin{bmatrix}\begin{bmatrix}3 & 0\\1 & -2\end{bmatrix} & \begin{bmatrix}2\\2\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}\\\begin{bmatrix}0 & 2\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}3 & 0\\1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix} + \underbrace{\underset{2\times 1}{\begin{bmatrix}2\\2\end{bmatrix}}\underset{1\times 2}{\begin{bmatrix}0 & 2\end{bmatrix}}}_{2\times 2} \quad \checkmark$$

E4bp3

$\mathbb{A}\vec{x}$:
$\mathbb{A}$ is $m \times n$, $\vec{x}$ is $n \times 1$

$$\underbrace{\overset{\mathbb{A}}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}}_{n \text{ columns}} \overset{\vec{x}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}$$

m rows; } n rows.

See this as the columns of $\mathbb{A}$ being combined with weights in vector $\vec{x}$;

$$\mathbb{A}\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{a}_{*1} & \vec{a}_{*2} & \cdots & \vec{a}_{*n} \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

column vectors inside $\mathbb{A}$

$* =$ run over all indices

$\vec{a}_{*1}$ first column vector

$$= x_1 \begin{bmatrix} | \\ \vec{a}_{*1} \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \vec{a}_{*2} \\ | \end{bmatrix} + \ldots + x_n \begin{bmatrix} | \\ \vec{a}_{*n} \\ | \end{bmatrix}$$

eg

$$\underset{2\times 3}{\overset{\mathbb{A}}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}} \underset{3\times 1}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} = (-1)\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ -2 \end{bmatrix} + (0)\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

scalings

$$= \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

$$\| \quad \underset{2\times 1}{\begin{bmatrix} -3 \\ -5 \end{bmatrix}} =$$

$\vec{y}^T \mathbb{A}$: "transpose"
$\vec{y}^T$ is $1 \times m$, $\mathbb{A}$ is $m \times n$

$$\underset{1 \times m}{\overset{\vec{y}^T \; 1\times m}{\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}}} \underset{m \times n}{\overset{\mathbb{A} \; m\times n}{\begin{bmatrix} - & \vec{a}_{1*} & - \\ - & \vec{a}_{2*} & - \\ & \vdots & \\ - & \vec{a}_{m*} & - \end{bmatrix}}}$$

n.b. $\vec{y}$ is $m \times 1$
column vector

row vectors inside $\mathbb{A}$

$\vec{a}_{1*}$ first row vector in $\vec{a}$

$$\vec{y}^T \mathbb{A} = y_1[- \; \vec{a}_{1*} \; -] + y_2[- \; \vec{a}_{2*} \; -] \; \vdots + y_m[- \; \vec{a}_{m*} \; -]$$

See this as the rows of $\mathbb{A}$ being combined with weights in vector $\vec{y}^T$.

ex

$$\underset{1\times 3}{\begin{bmatrix} 3 & 0 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}} = \underset{1\times 2}{\begin{bmatrix} -3 & 4 \end{bmatrix}}$$

$$\| \qquad\qquad \|$$

$$(3)\begin{bmatrix} -1 & 0 \end{bmatrix} + (0)\begin{bmatrix} 2 & 1 \end{bmatrix} + (2)\begin{bmatrix} 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \end{bmatrix}$$

E4bp4

$$\underset{m\times k}{\mathbb{C}} = \underset{m\times n}{\mathbb{A}}\,\underset{n\times k}{\mathbb{B}} \xrightarrow[\text{break IB into columns}]{} \mathbb{A}\begin{bmatrix} | & | & & | \\ \vec{b}_{*1} & \vec{b}_{*2} & \cdots & \vec{b}_{*k} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \overset{\text{column vector}}{(\underset{m\times n}{\mathbb{A}}\,\underset{n\times 1}{\vec{b}_{*1}})} & (\mathbb{A}\vec{b}_{*2}) & \cdots & (\mathbb{A}\vec{b}_{*k}) \\ | & | & & | \end{bmatrix}$$

C's columns are made up of A's columns

break A into rows

2 views →

↓

$$\begin{bmatrix} — \vec{a}_{1*} — \\ — \vec{a}_{2*} — \\ \vdots \\ — \vec{a}_{m*} — \end{bmatrix}\mathbb{B} = \begin{bmatrix} —(\vec{a}_{1*}\mathbb{B})— \\ —(\vec{a}_{2*}\mathbb{B})— \\ \vdots \\ —(\underbrace{\underset{1\times n}{\vec{a}_{m*}}\,\underset{n\times k}{\mathbb{B}}}_{1\times k})— \end{bmatrix}$$

C's rows are made up of IB's rows

ex

$$\overset{\mathbb{A}}{\underset{2\times 3}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}}\ \overset{\mathbb{B}}{\underset{3\times 2}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}}} = \begin{bmatrix} \overset{\mathbb{A}}{\underset{2\times3}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}}\overset{\vec{b}_{*1}}{\underset{3\times1}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}} & \overset{\mathbb{A}}{\begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}}\overset{\vec{b}_{*2}}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{2\times1}{\begin{bmatrix} -3 \\ -5 \end{bmatrix}} & \begin{bmatrix} 4 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \overset{\vec{a}_{1*}}{\underset{1\times3}{\begin{bmatrix} 3 & 0 & 2 \end{bmatrix}}}\overset{\mathbb{B}\ 3\times2}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}} \\ \overset{\vec{a}_{2*}}{\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}}\underset{\mathbb{B}}{\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} \overset{1\times2}{\begin{bmatrix} -3 & 4 \end{bmatrix}} \\ \underset{1\times2}{\begin{bmatrix} -5 & 2 \end{bmatrix}} \end{bmatrix} = \boxed{\begin{bmatrix} -3 & 4 \\ -5 & 2 \end{bmatrix}}\ ✓$$

Matrix Inverses ← delicious

Square Matrices Only

Menu:

- General goodness of inverses
- Identity matrix
- Solving $\mathbb{A}\vec{x}=\vec{b}$ • Advanced goodness
- How to w. example

Ex

$$\frac{1}{8}\overbrace{\begin{bmatrix}4&2\\2&3\end{bmatrix}}^{\mathbb{A}^{-1}}_{2\times2}\overbrace{\begin{bmatrix}3&-2\\-2&4\end{bmatrix}}^{\mathbb{A}}_{2\times2}=\frac{1}{8}\begin{bmatrix}\overset{8}{4\times3+2\times-2}&\overset{0}{4\times-2+2\times4}\\\overset{0}{2\times3+3\times-2}&\overset{8}{2\times-2+3\times4}\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}=\mathbb{I}$$

"1" for 2×2 matrices

Identity matrix $\mathbb{I}_n$

$n\times n$

leaves matrices unchanged under multiplication

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}2&3&-1\\4&2&1\end{bmatrix}=\begin{bmatrix}2&3&-1\\4&2&1\end{bmatrix}$$

$R_1'=R_1$
$R_2'=R_2$

$$\mathbb{I}_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad\mathbb{I}_4=\begin{bmatrix}1&&&\\0&1&&\\0&0&1&\\0&0&0&1\end{bmatrix}$$

lots of zeros

E5ap1

Given $\overset{n\times n}{\mathbb{A}}$, if $\mathbb{A}^{-1}$ exists then

$$\mathbb{A}^{-1}\mathbb{A}=\mathbb{A}\mathbb{A}^{-1}=\mathbb{I}$$

← order doesn't matter

Selection of Big Deals for $\mathbb{A}^{-1}$:

- Even knowing if $\mathbb{A}^{-1}$ exists is valuable ← may not have actual form.
- If $\mathbb{A}^{-1}$ exists then: ← premultiply both sides

$$\underbrace{\mathbb{A}^{-1}\underset{n\times n}{\mathbb{A}}}_{m=n}\underset{n\times 1}{\vec{x}}=\mathbb{A}^{-1}\underset{n}{\vec{b}}$$

$$\Rightarrow\quad\underset{n\times n}{\mathbb{I}}\,\underset{n\times1}{\vec{x}}=\mathbb{A}^{-1}\vec{b}$$

$$\Rightarrow\quad\vec{x}=\mathbb{A}^{-1}\vec{b}\qquad\text{Done!!}$$

But wait:

- Many systems have rectangular $\mathbb{A}$'s $m\neq n$
- $\mathbb{A}^{-1}$ may not exist
- Even if $\mathbb{A}^{-1}$ exists, computing $\mathbb{A}^{-1}$ is very hard work computationally (grows badly with n)

E5ap2

- if A^{-1} exists, then $A\vec{x}=\vec{b}$ has only one solution, always. (for all $\vec{b}$)

Simply: $\vec{x} = \underbrace{A^{-1}\vec{b}}$

- if A^{-1} does not exist then we may have 0 or ∞ly many solutions

↖ more later

- If $\exists\, \vec{x}\neq\vec{0}$ (there exists an $\vec{x}\neq\vec{0}$) such that $A\vec{x}=\vec{0}$ "$\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$"

"A maps $\vec{x}$ to $\vec{0}$"
"A crushes $\vec{x}$."

then A^{-1} does not exist

Proof $\underbrace{A\vec{x}=\vec{0}}_{\text{true}} \Rightarrow \underbrace{\underset{n\times n}{A^{-1}}\,\underset{n\times n}{A}}\,\underset{n\times 1}{\vec{x}} = \underset{n\times n}{A^{-1}}\,\underset{n\times 1}{\vec{0}}$

$\Rightarrow I\,\vec{x} = \vec{0}$

$\vec{x} = \vec{0}$

contradiction!

$\Rightarrow A^{-1}$ cannot exist

- Foreshadowing: if $A\vec{x}=\vec{0}$ we say $\vec{x}\in N(A)$

↑ null space of A

- $(\underset{n\times n}{A}\,\underset{n\times n}{B})^{-1} = B^{-1}A^{-1}$

See $B^{-1}\underbrace{A^{-1}(A}B) = B^{-1}\overset{B}{(I\,B)} = B^{-1}B$

$(A\underbrace{B)\,B^{-1}}A^{-1} = \underset{A}{(A\,I)}A^{-1} = A\,A^{-1}$

$= I$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

- If we have $\underset{n\times n}{A}, \underset{n\times n}{Z_L}, \underset{n\times n}{Z_R}$ such that

$A Z_R = I$ ↙ right inverse & $Z_L A = I$ ↙ left inverse

then $A^{-1} = Z_R = Z_L$

Reason

$\overset{I}{Z_L(A}Z_R) = \underbrace{Z_L(I)}$

premultiply

$\cancel{I}\,Z_R = Z_L$

Using Gauss-Jordan Elimination to find A^{-1}

- general story (it's $A\vec{x}=\vec{b}$ again!)
- example

Game: given $\underset{n\times n}{A}$, find $\overset{n\times n}{A^{-1}}$

$$\downarrow$$

$$A^{-1}A = \underbrace{AA^{-1} = I}_{\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}\text{ ish}}$$

Consider: $\overset{n\times n}{A}\,\overset{n\times n}{Z} = \overset{n\times n}{I}$ ✓ ? ✓

2×2 general ex

$$\underset{2\times 2}{A}\begin{bmatrix} | & | \\ \vec{z}_1 & \vec{z}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2×1

⇓ wrangling

$$\begin{bmatrix} | & | \\ \underset{2\times2}{A}\,\underset{2\times1}{\vec{z}_1} & \underset{2\times2}{A}\,\underset{2\times1}{\vec{z}_2} \\ | & | \\ 2\times1 & 2\times1 \end{bmatrix} = \begin{bmatrix} \overset{\hat{i}}{\begin{bmatrix}1\\0\end{bmatrix}} & \overset{\hat{j}}{\begin{bmatrix}0\\1\end{bmatrix}} \end{bmatrix}$$

$\Rightarrow$ Solve $A\vec{z}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ & $A\vec{z}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

↖ $A\vec{x}=\vec{b}$!!

E5bp1

Note: we would make A become U w. row reduction for both equation.

Do all at once with a super augmented matrix:

$$\underbrace{\left[\begin{array}{c|c} \underset{n\times n}{A} & \underset{n\times n}{I} \end{array}\right]}_{n\times 2n}$$

#awesome

Use row ops to turn A into U then I

I will change into A^{-1}

actually:

- finding right inverse of A; later we show it's the true inverse
- only works if A has n pivots

Example:

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{cc|cc} \overset{D_1}{\boxed{3}} & -2 & 1 & 0 \\ \boxed{-2} & 4 & 0 & 1 \end{array}\right]$$

* * *

$$R_2' = R_2 - \left(\frac{-2}{\underset{D_1}{3}}\right) R_1$$

$$\ell_{21} = -\frac{2}{3}$$

$$\left[\begin{array}{cc|cc} \overset{D_1}{3} & \overset{U}{-2} & 1 & 0 \\ 0 & \underset{D_2}{8/3} & 2/3 & 1 \end{array}\right]$$

$$8/3 = 4 - \frac{4}{3}, \quad 4 = 12/3$$

* * *

$$R_1' = R_1 - \left(\frac{-2}{\underbrace{8/3}_{+\frac{3}{4}}}\right) R_2 \quad D_2$$

$$\left[\begin{array}{cc|cc} 3 & 0 & 3/2 & 3/4 \\ 0 & 8/3 & 2/3 & 1 \end{array}\right]$$

$$3/2 = 1 + \frac{2}{3} \cdot \frac{3}{4}$$

divide by pivots

* * *

$$R_1' = \frac{1}{\underset{D_1}{\boxed{3}}} R_1$$

$$R_2' = \frac{1}{\underset{D_2}{\boxed{8/3}}} R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/4 & 3/8 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \qquad \underbrace{\begin{matrix} 1/2 & 1/4 \\ 1/4 & 3/8 \end{matrix}}_{A^{-1}}$$

see E5ap1.

tidying up

$$A = \frac{1}{\boxed{8}} \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

turns out this is a very special number for A

notation

- Determinant of A
- Det(A)
- $|A|$

More later!!

E5bp3

3x3 plan

order of elimination

$$\left[\begin{array}{ccc|ccc} 2 & \overset{⑥}{3} & \overset{⑤}{4} & 1 & 0 & 0 \\ \overset{①}{1} & -1 & \overset{④}{7} & 0 & 1 & 0 \\ \overset{②}{13} & \overset{③}{2} & 17 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{}_{A} \quad \underbrace{}_{I}$$

Hidden Secrets of Inverses:

- $\mathbb{A}^{-1}$ and elimination matrices
- Inverses of elimination matrices
- Missing pivots → $\mathbb{A}^{-1}$ does not exist

Pratchett upon learning more about inverses →

- Curious things about columns...

$$\mathbb{A} = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$$

aside $\mathbb{A} = \mathbb{A}^T$ transpose

$$\mathbb{A}^{-1} = \frac{1}{8}\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

$(\mathbb{A}^{-1})^T = \mathbb{A}^{-1}$

Row reduction

⇒ Elimination matrices

for Solving $[\mathbb{A} \mid \mathbb{I}] \Leftrightarrow \mathbb{A}\mathbb{Z} = \mathbb{I}$

row op 1

$$\mathbb{E}_{21} = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}$$

row 2 row 1 $-\ell_{21}$

row op 2

multiplier

$$\mathbb{E}_{12} = \begin{bmatrix} 1 & 3/4 \\ 0 & 1 \end{bmatrix}$$

ESc p1

pivot matrix

$$\mathbb{D} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 8/3 \end{bmatrix}$$

$$\mathbb{D}^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 3/8 \end{bmatrix}$$

undo each other

$$\left[\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ \cancel{-4} & 4 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/4 & 3/8 \end{array}\right]$$

-2

[note: transcribed incorrectly in video]

$$\underbrace{\mathbb{D}^{-1}\mathbb{E}_{12}\mathbb{E}_{21}}_{\mathbb{A}^{-1}}\mathbb{A}\mathbb{Z} = \underbrace{\mathbb{D}^{-1}\mathbb{E}_{12}\mathbb{E}_{21}}_{\mathbb{A}^{-1}}\mathbb{I}$$

$$\mathbb{I}\mathbb{Z}_1 = \mathbb{A}^{-1}$$

found by row operations

made by $\mathbb{E}_{ij}$ & D matrices.

Big Deal:

See $\mathbb{A}^{-1}$ is a product of $\mathbb{E}_{ij}$'s, $\mathbb{D}^{-1}$, $\mathbb{P}$

pivots

permutation for row swaps

⇑ Huge: Demonstrates that $\mathbb{A}^{-1}$ is a left and right inverse

Next: Elimination matrices have simple inverses.

$$\mathbb{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \longleftrightarrow R_3' = R_3 - 2R_1$$

$-\ell_{31}$

undo with

$R_3' = R_3 + 2R_1$

$$\Rightarrow \mathbb{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +2 & 0 & 1 \end{bmatrix}$$

$+\ell_{31}$ swap sign

check

$\mathbb{E}_{31}\mathbb{E}_{31}^{-1} = \mathbb{E}_{31}^{-1}\mathbb{E}_{31}$

$= \mathbb{I}.$

In general flip sign of lone off diagonal element to turn $\mathbb{E}_{ij}$ into $\mathbb{E}_{ij}^{-1}$

Monks made us do this...
Sneaky plan.

Permutation matrices:

$$\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \leftarrow \begin{aligned} R_1' &= R_1 \\ R_2' &= R_3 \\ R_3' &= R_2 \end{aligned}$$

here: $\mathbb{P}^{-1} = \mathbb{P}$

But in general $\mathbb{P}^{-1} = \mathbb{P}^T$

Missing pivots

E5cp2

What if $[\mathbb{A} \mid \mathbb{I}] \rightarrow$ one or more rows of zeros on left? i.e., missing pivots?

from before: if $\vec{x} \neq \vec{0}$ solves $\mathbb{A}\vec{x} = \vec{0}$ then $\mathbb{A}^{-1}$ cannot exist

so

$$[\,\mathbb{A} \mid \underset{\vec{b}}{\vec{0}}\,] \underset{\text{Row ops}}{\sim} [\,\underset{\text{upper triangular}}{\mathbb{U}} \mid \underset{\vec{c}}{\vec{0}}\,]$$

Row of 0's in $\mathbb{U}$ → ∞ly many solns
→ $\mathbb{A}\vec{x} = \vec{0}$ is solved by $\vec{x} \neq \vec{0}$
→ $\mathbb{A}^{-1}$ does not exist

ex

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & 4 & 0 \end{array}\right] \underset{R_2' = R_2 - \left(\frac{6}{3}\right)R_1}{\sim} \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

missing pivot

Upshot:

$\underset{n\times n}{\mathbb{A}}^{-1}$ exists

$\Longleftrightarrow$ $\mathbb{A}$ has n pivots

$\Longleftrightarrow$ $\mathbb{A}\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution

$\Longleftrightarrow$ $\det(\mathbb{A}) \neq 0$

later

parallelograms will be involved

If $\mathbb{A}_{2\times3}$ has column 1 + column 2 = column 3

show $\mathbb{A}^{-1}$ does not exist.. (weird)

(a) See $\mathbb{A}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ⇐ non-zero solut to $\underline{\underline{\mathbb{A}\vec{x}=0}}$

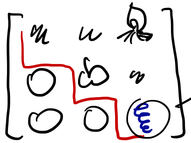
column picture $\quad \underbrace{1\vec{a}_1}_{3\times1} + 1\cdot\vec{a}_2 - 1\cdot\vec{a}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

↓

$\mathbb{A}^{-1}$ does not exist ⇐ pivot missing.

(b) Another aspect:
Row operations destroy rows But
Column relationships are unchanged.

row reduct →



must be 0 → 3rd pivot is missing ✓

$c_1+c_2=c_3 \Rightarrow \xi = 0+0=0$ ✓

LE5cp3

$$\begin{array}{c}\begin{bmatrix}1&2&3\\1&3&4\\1&4&5\end{bmatrix}\\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 = \vec{a}_1+\vec{a}_2\end{array} \sim \cdots \sim \begin{bmatrix}1&2&3\\0&1&1\\0&2&2\end{bmatrix}$$

↑ col 1+col 2 = col 3

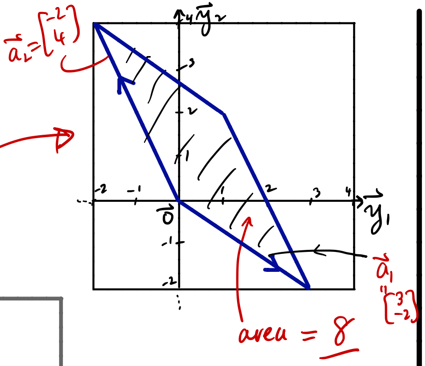
$$\sim \cdots \sim \begin{bmatrix}1&2&3\\0&1&1\\0&0&0\end{bmatrix}$$

columns are "linearly dependent"

⇒ connects to $\mathbb{A}^{-1}$ not existing

Foreshadowing:

Determinant
matrix.



from p5bp2

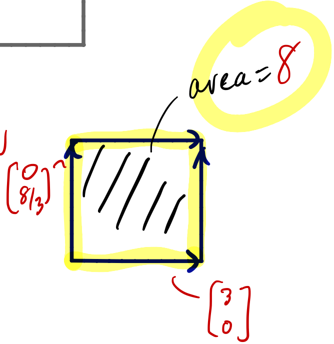
$$[\mathbb{A} \mid \mathbb{I}] = \left[\begin{array}{cc|cc} \overset{D_1}{3} & \overset{\downarrow}{-2} & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right]$$

$$\underbrace{R_2' = R_2 - \left(\frac{-2}{\underset{D_1}{3}}\right) R_1}_{\ell_{21} = -\frac{2}{3}} \qquad \left[\begin{array}{cc|cc} \overset{D_1}{3} & \overset{U}{-2} & 1 & 0 \\ 0 & \underset{D_2}{8/3} & 2/3 & 1 \end{array}\right]$$

$$4 - \frac{4}{3} \qquad \frac{12}{3}$$

$$R_1' = R_1 - \underbrace{\left(\frac{-2}{\underset{D_2}{8/3}}\right)}_{+\frac{3}{4}} R_2 \qquad \left[\begin{array}{cc|cc} 3 & 0 & \overset{1 + \frac{2}{3} \cdot \frac{3}{4}}{3/2} & 3/4 \\ 0 & 8/3 & 2/3 & 1 \end{array}\right]$$

$$|\mathbb{A}| = 3 \times 4 - (-2)(-2)$$
$$= 12 - 4 = 8$$



Triangle × Triangle = Rectangle

Menu:

- Our first factorization: $\mathbb{L}\mathbb{U}$
- Method first
- The l_{ij}'s serve us well (as promised by mysterious monks)

← t-shirt for each factorization

Ex

$$x_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + x_2 \begin{bmatrix}1\\2\\3\end{bmatrix} + x_3 \begin{bmatrix}1\\3\\6\end{bmatrix} = \begin{bmatrix}5\\7\\11\end{bmatrix}$$

$\vec{a}_1$ $\vec{a}_2$ $\vec{a}_3$ $\vec{b}$

$$\underset{\substack{3\times3\\m\times n}}{\mathbb{A}}\ \underset{3\times1}{\vec{x}} = \underset{3\times1}{\vec{b}}$$

Normal plan:

$$\text{set up } [\mathbb{A} \mid \vec{b}] \xrightarrow{\text{row ops}} [\mathbb{U} \mid \vec{c}] \xrightarrow{\text{back sub}} \vec{x}$$

Now: focus on reducing $\mathbb{A}$ by itself

↖ very good if $\vec{b}$ is changed.

$$\mathbb{A} = \begin{bmatrix}1&1&1\\1&2&3\\1&3&6\end{bmatrix} \quad \underbrace{\begin{aligned}R_2' &= R_2 - \left(\tfrac{1}{1}\right)R_1\\ R_3' &= R_2 - \left(\tfrac{1}{1}\right)R_1\end{aligned}} \quad \begin{bmatrix}1&1&1\\0&1&2\\0&2&5\end{bmatrix}$$

$l_{21} = 1$

$l_{31} = 1$

$$\underbrace{R_3' = R_3 - \left(\tfrac{2}{1}\right)R_2} \quad \begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix} \leftarrow \mathbb{U}$$

$l_{32} = 2$

$D_1 = 1$
$D_2 = 1$
$D_3 = 1$

← not exciting

Elimination matrix story

E6ap1

$$\mathbb{A} \rightarrow \mathbb{U} = \mathbb{E}_{32}\mathbb{E}_{31}\underbrace{\mathbb{E}_{21}\mathbb{A}}$$

↑ ↑ ↑ powerful encoding of our row operations

Monks whisper: "invert $\mathbb{E}_{ij}$'s"

RHS ↕ ↕ LHS

$$\underbrace{\mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\underbrace{\mathbb{E}_{32}^{-1}\mathbb{E}_{32}}_{\mathbb{I}}\mathbb{E}_{31}\mathbb{E}_{21}}_{\mathbb{I}}\mathbb{A} = \mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\mathbb{E}_{32}^{-1}\mathbb{U}$$

$$\Rightarrow \mathbb{A} = \underbrace{\mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\mathbb{E}_{32}^{-1}}_{\text{row operations in reverse}}\mathbb{U} \leftarrow \text{upper triangular}$$

⇙

Tells us how to combine $\underbrace{\text{rows of } \mathbb{U}}_{\text{simple-ish}}$ to make rows of $\mathbb{A}$

Amazingly: we'll see

$$\overset{m\times n}{\mathbb{A}} = \overset{\substack{\text{square}\\m\times m}}{\mathbb{L}}\ \overset{m\times n}{\mathbb{U}}$$

upper triangular

$$\text{lower triangular} \underset{m\times m}{\begin{bmatrix}1&0&0\\+l_{21}&1&0\\+l_{31}&+l_{32}&1\end{bmatrix}} \qquad \begin{bmatrix}D_1&*&*\\0&D_2&*\\0&0&D_3\end{bmatrix}$$

▨ = ▨ ▨

LE6ap2

Now

$$\mathbb{A} = \overbrace{\underbrace{\mathbb{E}_{21}^{-1}\,\mathbb{E}_{31}^{-1}\,\mathbb{E}_{32}^{-1}}_{\text{know these are simple}}}^{\mathbb{L}}\,\mathbb{U}$$

ex recall

$$\mathbb{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ +l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2' = R_2 - l_{21} R_1$ $\qquad$ $R_2' = R_2 + l_{21} R_1$

Big Deals:

(1) $\mathbb{E}_{ij}^{-1}$ is $\mathbb{E}_{ij}$ with single off diagonal element flipped in sign

(2) $\mathbb{E}_{ij}$'s & $\mathbb{E}_{ij}^{-1}$'s are all lower triangular

(3) $\mathbb{E}_{ij}$ is $\mathbb{I}$ with $-l_{ij}$ replacing 0 in ij position

(4) Remarkably:

$$\mathbb{E}_{21}^{-1}\,\mathbb{E}_{31}^{-1}\,\mathbb{E}_{32}^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ +l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +l_{31} & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +l_{32} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \mathbb{L}$$

$\mathbb{L}$ always has 1's on the diagonal.

Back to example:

$$\mathbb{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

So:

$$\mathbb{A} = \begin{bmatrix}1&1&1\\1&2&3\\1&3&6\end{bmatrix} = \mathbb{L}\mathbb{U}$$

$$= \begin{bmatrix}1&0&0\\1&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}$$

D_1 D_2 D_3.

ℓ_{21} ℓ_{31} ℓ_{31}

//

Now Solve $\mathbb{A}\vec{x} = \vec{b}$ if $\vec{b} = \begin{bmatrix}5\\7\\11\end{bmatrix}$ by solving two (easy) triangular systems

$$\mathbb{L}\,\underbrace{\mathbb{U}\vec{x}}_{C} = \begin{bmatrix}5\\7\\11\end{bmatrix} \leftarrow \vec{b}.$$

From before $\mathbb{U}\vec{x}=\vec{c}$

→ $\underset{\checkmark}{\mathbb{L}}\ \underset{?}{\vec{c}} = \underset{\checkmark}{\begin{bmatrix}5\\7\\11\end{bmatrix}}$

$$\begin{bmatrix}1&0&0\\1&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}5\\7\\11\end{bmatrix}$$ ← Solve by forward substitution

R1: $c_1 = 5$

R2: $\underset{5}{\cancel{c_1}} + c_2 = 7 \Rightarrow c_2 = 2$

R3: $\underset{5}{\cancel{c_1}} + 2\underset{2}{\cancel{c_2}} + c_3 = 11 \Rightarrow c_3 = 2$

$$\vec{c} = \begin{bmatrix}5\\2\\2\end{bmatrix}$$

Now solve $\underset{\checkmark}{\mathbb{U}}\underset{?}{\vec{x}} = \underset{\checkmark}{\vec{c}}$ with back substitution

E6ap3

$$\uparrow \begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}5\\2\\2\end{bmatrix}$$

R_3: $x_3 = 2$

R_2: $x_2 + 2\overset{2}{\cancel{x_3}} = 2 \rightarrow x_2 = -2$

$x_1 + \underset{-2}{\cancel{x_2}} + \underset{2}{\cancel{x_3}} = 5 \rightarrow x_1 = 5$

$$\vec{x} = \begin{bmatrix}5\\-2\\2\end{bmatrix}$$

done

Big deal: Swap $\vec{b}$, easy to solve, Row reduction is done once and is encoded in $\mathbb{L}$ & $\mathbb{U}$.

// Extra pieces:

Our $\mathbb{A}$ was special b/c $\mathbb{A} = \mathbb{A}^T$
$\Rightarrow$ $\mathbb{L}$ and $\mathbb{U}$ are transposes of each other

But only b/c $D_1 = D_2 = D_3 = 1$

Also very useful:
Separate out pivots

$$\underset{\mathbb{A}}{\underset{m\times n}{\begin{bmatrix}2&1&3\\2&0&0\\4&3&13\end{bmatrix}}} = \overset{\mathbb{L}}{\underset{\substack{m\times m\\ \text{square}}}{\begin{bmatrix}1&0&0\\1&1&0\\2&-1&1\end{bmatrix}}}\overset{\mathbb{U}}{\begin{bmatrix}2&1&3\\0&-1&-3\\0&0&4\end{bmatrix}}$$

always 1s

$\ell_{21} = 1$
$\ell_{31} = 2$
$\ell_{32} = -1$

$D_1 = 2,\ D_2 = -1,\ D_3 = 4$

Alternate factorization ($\mathbb{U}$ is different!)

$$\mathbb{A} = \underset{m\times m}{\begin{bmatrix}1&0&0\\1&1&0\\2&-1&1\end{bmatrix}}\underset{m\times m}{\begin{bmatrix}2&0&0\\0&-1&0\\0&0&4\end{bmatrix}}\begin{bmatrix}1&1/2&3/2\\0&1&3\\0&0&1\end{bmatrix}$$

$$= \mathbb{L}\,\mathbb{D}\,\mathbb{U}$$

* If $\underset{\substack{\uparrow\\ \text{next}}}{\mathbb{A} = \mathbb{A}^T}$ then $\mathbb{A} = \mathbb{L}\mathbb{D}\underbrace{\mathbb{L}^T}_{\mathbb{U}}$

LE6ap4

Bad notation:
$\mathbb{U}$ in $\mathbb{L}\mathbb{U} \neq \mathbb{U}$ in $\mathbb{L}\mathbb{D}\mathbb{U}$

Must state which form we're using from the start.

$=$

Last thing: Row Swaps

do at the start

$$\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U}$$

or

$$\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{D}\mathbb{U}$$

possible for every matrix $\mathbb{A}$

Amazing!!

Why $\mathbb{L}\mathbb{U}$ works:

mad!

Claim: $\mathbb{E}_{ij}$ matrices always combine to produce a lower triangular matrix with l_{ij}'s in the right spots & 1's along the main diagonal

$$\mathbb{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Why does $\mathbb{E}_{21}^{-1}\mathbb{E}_{31}^{-1}\mathbb{E}_{32}^{-1}$ work so simply?

Reason:

As we uncover $\mathbb{U}$ with row operations, we only use rows of $\mathbb{U}$ to modify lower rows of $\mathbb{A}$.

$\Rightarrow$ $\mathbb{L}$ will be lower triangular

↑

"combining" matrix tells us how to combine $\mathbb{U}$'s rows to produce $\mathbb{A}$.

3x3 example (ignoring row swaps):

Row 1 of $\mathbb{U}$ = Row 1 of $\mathbb{A}$

Row 2 of $\mathbb{U}$ = Row 2 of $\mathbb{A}$ $-\, l_{21} \times \underbrace{\text{Row 1 of } \mathbb{U}}_{\text{Row 1 of } \mathbb{A}}$

Row 3 of $\mathbb{U}$ = Row 3 of $\mathbb{A}$
$-\, l_{31} \times$ Row 1 of $\mathbb{U}$
$-\, l_{32} \times$ Row 2 of $\mathbb{U}$

Invert

Row 1 of $\mathbb{A}$ = $\overset{1\times}{\text{Row 1 of }}$ $\mathbb{U}$

Row 2 of $\mathbb{A}$ = $\overset{1\times}{\text{Row 2 of }}$ $\mathbb{U}$
$+\, l_{21} \times$ Row 1 of $\mathbb{U}$

Row 3 of $\mathbb{A}$ = $\overset{1\times}{\text{Row 3 of }}$ $\mathbb{U}$
$+\, l_{31} \times$ Row 1 of $\mathbb{U}$
$+\, l_{32} \times$ Row 2 of $\mathbb{U}$

RHS is simple

Transposes and Symmetric Matrices ↖ ♡

Menu:

- Transposes
- Symmetric matrices
- Properties of peculiar nature

//

Defn: $\underset{n\times m}{A^T}$ = the transpose of $\underset{m\times n}{A}$

= A flipped about the main diagonal

[A^T's columns are the rows of A
& " rows " " columns of A

ex *
$$\underset{\substack{2\times 3\\ m\times n}}{\begin{bmatrix} 2 & 7 & 3 \\ -1 & 2 & 4 \end{bmatrix}^T} = \underset{3\times 2}{\begin{bmatrix} 2 & -1 \\ 7 & 2 \\ 3 & 4 \end{bmatrix}}$$

ex.
$$\begin{bmatrix} 3 & 9 \\ 17 & 23 \end{bmatrix}^T = \begin{bmatrix} 3 & 17 \\ 9 & 23 \end{bmatrix}$$

Defn again: $(A^T)_{ij} = (A)_{ji}$

* $(A)_{21} = (A^T)_{12}$
$= -1$

LE 7a p1

Big Deal:
The transpose of A will matter ridiculously everywhere and especially for solving $A\vec{x} = \vec{b}$
↖our pal.

//

Defn: if $A = A^T$ (means A must be square)
then we say A is symmetric and we are very happy.
↙
super, super special matrices

//

Properties:

$$(A+B)^T = A^T + B^T \quad ✓$$

$$\underbrace{\left(\underbrace{\underbrace{A}_{m\times q}\underbrace{B}_{q\times n}}_{m\times n}\right)^T}_{n\times m} \overset{?}{=} \cancel{\underbrace{A^T}_{q\times m}\underbrace{B^T}_{n\times q}}$$

can't be general...

$$= \underbrace{\underbrace{B^T}_{n\times q}\underbrace{A^T}_{q\times m}}_{n\times m}$$ ← always true

← right dimensions.

$$\boxed{(AB)^T = B^T A^T}$$ ← proof later

What about $(A^{-1})^T$?

know

$$A^{-1}A = I = AA^{-1}$$

take transposes:

$$(A^{-1}A)^T = I^T = (AA^{-1})^T$$

$$(A^T)(A^{-1})^T = I = (A^{-1})^T(A)^T$$

$$\Rightarrow (A^T)^{-1} = (A^{-1})^T$$

If $A = A^T$, $\quad (A^T)^{-1} = (A^{-1})^T$

$$\| $$

$$(A)^{-1} \qquad (A^{-1})^T = A^{-1}$$

So if A is symmetric, so its inverse

E7ap2

Crazily important objects:

Square matrices: A does not have to be square

$$\underbrace{\underbrace{A^T}_{n\times m}\underbrace{A}_{m\times n}}_{n\times n} \quad \& \quad \underbrace{\underbrace{A}_{m\times n}\underbrace{A^T}_{n\times m}}_{m\times m}$$

$$(A^T A)^T = (A^T)\,\overset{\text{undo}}{(A^T)^T} = A^T A$$

So A^TA is always symmetric

Check: true for AA^T as well.

ex

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

3×2

$$\underset{2\times 2}{A^T A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

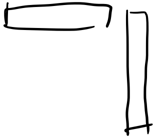
$$\underset{3\times 3}{A A^T} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

#awesome

E7ap3

Inner product:

row vector → $\vec{a}^T \vec{b}$

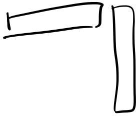


what happens with?:

$$(\vec{a}^T\vec{b})^T$$

$$= \vec{b}^T(\vec{a}^T)^T$$

$$= \underbrace{\underbrace{\vec{b}^T}_{1\times n}\ \underbrace{\vec{a}}_{n\times 1}}_{1\times 1}$$



More advanced inner producting:

$$\underbrace{A\vec{x}}_{\text{transformation of } \vec{x}} \ \& \ \vec{y}$$

ex: $(A\vec{x})^T\vec{y} = \vec{x}^T A^T \vec{y}$

$= \vec{x}^T(A\vec{y})$

Inner product of $\vec{x}$ & $A^T\vec{y}$

Transform $\vec{y}$ with A^T first

More on the Transpose

menu:

- example of $(AB)^T = B^T A^T$
- three different proofs

ex
$$\left(\underset{A}{\begin{bmatrix}2&3\\1&-1\end{bmatrix}}\underset{B}{\begin{bmatrix}-1&4\\3&1\end{bmatrix}}\right)^T = \begin{bmatrix}7&11\\-4&3\end{bmatrix}^T = \begin{bmatrix}7&-4\\11&3\end{bmatrix}$$

$$\overset{B^T}{\begin{bmatrix}-1&4\\3&1\end{bmatrix}^T}\overset{A^T}{\begin{bmatrix}2&3\\1&-1\end{bmatrix}^T}$$

$$=\begin{bmatrix}-1&3\\4&1\end{bmatrix}\begin{bmatrix}2&1\\3&-1\end{bmatrix}=\begin{bmatrix}7&-4\\11&3\end{bmatrix}$$

E7bp1

$$(AB)^T_{ij}$$

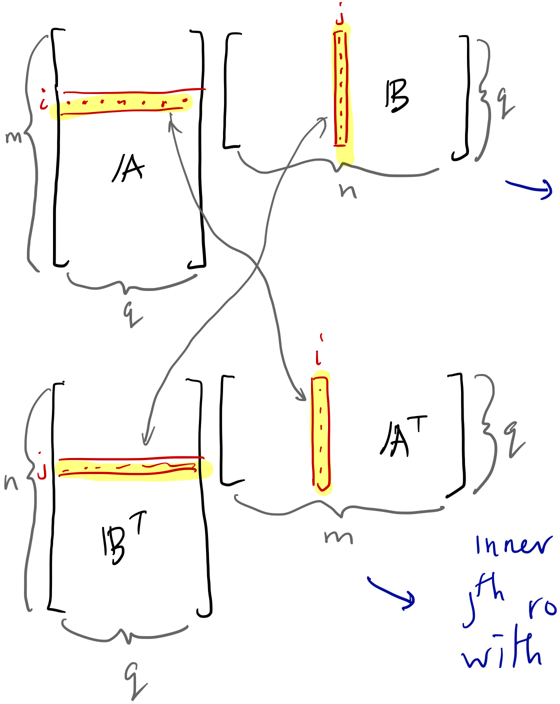
$$= (\underset{m\times q}{A}\;\underset{q\times n}{B})_{ji} = \sum_{k=1}^{q} a_{jk} b_{ki}$$

inner product of jth row of A & ith column of B

$$= \sum_{k=1}^{q} (A^T)_{kj} (B^T)_{ik}$$

$$= \sum_{k=1}^{q} (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}$$



→ Inner product of i^{th} row of $\mathbb{A}$ with j^{th} column of $\mathbb{B}$

$$= (\mathbb{A}\mathbb{B})_{ij}$$

$$= ((\mathbb{A}\mathbb{B})^T)_{ji}$$

→ Inner product of j^{th} row of $\mathbb{B}^T$ with i^{th} column of $\mathbb{A}^T$

$$= (\mathbb{B}^T\mathbb{A}^T)_{ji}$$

$$(\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T$$

Yet another way:

column picture

$$\underset{m\times n}{(A}\underset{n\times 1}{\vec{x})}^T = \left(x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n\right)^T$$

$$= x_1\vec{a}_1^T + \dots + x_n\underbrace{\vec{a}_n^T}_{\text{row vector}}$$

$$= [x_1 \dots x_n]\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_n^T & - \end{bmatrix}$$

$$= \boxed{\vec{x}^T A^T}$$

use here

$$= (\underset{m\times q}{A}\underset{q\times n}{B})^T$$

$$= \left(A\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \\ | & | & & | \end{bmatrix}\right)^T$$

$$= \left(\begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \dots & A\vec{b}_n \\ | & | & & | \end{bmatrix}\right)^T$$

E7bp3

$$= \begin{bmatrix} - & (A\vec{b}_1)^T & - \\ - & (A\vec{b}_2)^T & - \\ & \vdots & \\ - & (A\vec{b}_n)^T & - \end{bmatrix}$$

$$= \begin{bmatrix} - & \vec{b}_1^T A^T & - \\ - & \vec{b}_2^T A^T & - \\ & \vdots & \\ - & \vec{b}_n^T A^T & - \end{bmatrix}$$

$$= \begin{bmatrix} - & \vec{b}_1^T & - \\ & \vdots & \\ - & \vec{b}_n^T & - \end{bmatrix} A^T$$

$$= B^T A^T$$

Yes!!

"Paging Dr. Vector Spaceman"

Menu:

- Our new plan for $A\vec{x}=\vec{b}$
- Vector spaces, introduction to

The Column picture for $A\vec{x}=\vec{b}$

$$A = \begin{array}{c} \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{matrix} \end{array} \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

We solve $x_1\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Game:

Find out how we may combine column vectors of A to create/generate/reach $\vec{b}$

New idea:

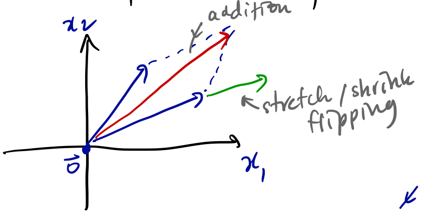
Understand the places ("spaces") where $\vec{a}$'s, $\vec{x}$'s, and $\vec{b}$'s live

E8op1

Big things coming:

- Null space of A
- Column Space of A
- Row Space of A
- Left Null Space of A
- Beautiful connection to A^T, A's transpose

Example Vector Space: Idealized plane



$$V = R^{\textcircled{2}} = \underset{\substack{\uparrow \\ \text{curly} \\ \text{braces}}}{\Big\{} \overset{\substack{\text{basic} \\ \text{structure} \\ \swarrow}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} ; \overset{\substack{\text{restrictions} \\ \downarrow}}{x_1, x_2} \in \underset{\substack{\uparrow \\ \text{real} \\ \text{numbers} \\ R=R^1}}{R} \Big\}$$

Two (pretty obvious) features of vector spaces:

They are closed under addition and scalar multiplication.

(1) If we add any two vectors in V, we get another vector that's still in V

(2) If we multiply a vector in V by a Scalar (for us: a real number), the result is still in V.

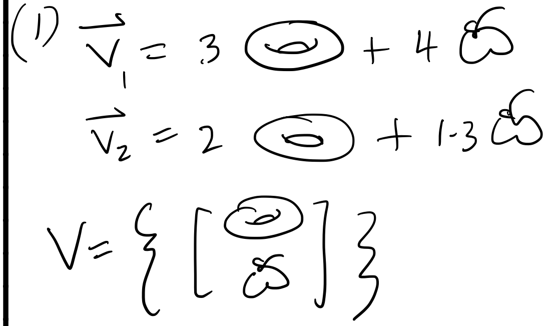
$$\underset{\in R^2}{\begin{bmatrix}3\\4\end{bmatrix}} + \underset{\in R^2}{\begin{bmatrix}773\\6\end{bmatrix}} = \underset{\in R^2}{\begin{bmatrix}776\\10\end{bmatrix}}$$

"is an element of"

$$\underbrace{7}_{\in R \text{ (real numbers)}} \underbrace{\begin{bmatrix}3\\4\end{bmatrix}}_{\in R^2} = \underbrace{\begin{bmatrix}21\\28\end{bmatrix}}_{\in R^2}$$

E8ap2

Examples of things that are and are not Vector Spaces:



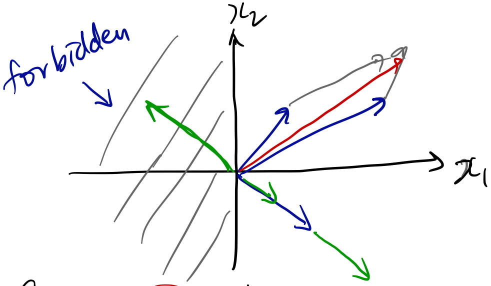
$$(2)\quad V = \left\{ f(x) = \overset{x}{c_2}x^2 + \overset{y}{c_1}x + \overset{z}{c_0} \ \overset{\text{such that}}{\Big|}\ c_2, c_1, c_0 \in R \right\}.$$

$$f_1(x) = 2x^2+3$$

$$f_2(x) = -7x^2+3x+4$$

$$f_1(x)+f_2(x) = -5x^2+3x+7$$

(3)



$$V = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \underbrace{x_2 \in R}_{-\infty < x_2 < \infty},\ x_1 \geq 0 \right\}$$

Not a vector space

Observe:

Addition works

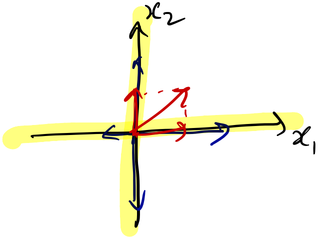
if $\vec{v}_1, \vec{v}_2 \in V$ then $\vec{v}_1 + \vec{v}_2 \in V$

Scalar multiplization fails!

$$-3 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\in V} = \underset{\notin V}{\begin{bmatrix} -3 \\ -3 \end{bmatrix}}$$

Note: we're starting to talk about subspaces

(4) All points on axes of R^2 E8ap3



Now see

Scalar multiplication works but addition fails

$$\underset{\in V}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \underset{\in V}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

not in V

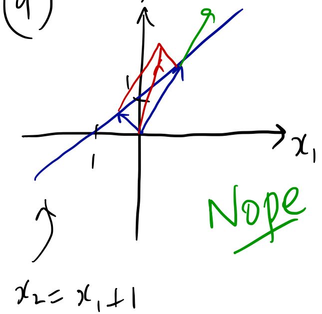
(5) All m×n matrices form a vector space $\equiv R^{mn}$

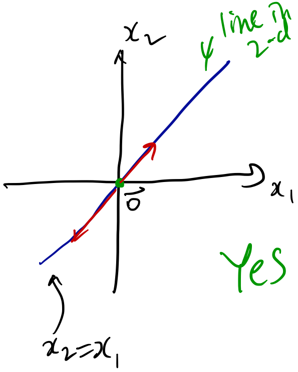
(6) What about the integers ✗

(7) " " rational numbers ✗

(8) " " real numbers ✓

(a)





Vector Spaces Inside Vector Spaces

Menu:
- vector space requirements
- subspace requirements

General Requirements of a Vector Space:

VSP1 if $\vec{x}_1, \vec{x}_2 \in V$ then $\vec{x}_1 + \vec{x}_2 \in V$

VSP2 if $\vec{x} \in V$ then $c\vec{x} \in V$ for all $c \in R$

VSP3 $\vec{0} \in V$ and $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x}$ in V

vector space property

+ a series of increasingly boring conditions such as $c(\vec{x}_1 + \vec{x}_2) = cx_1 + cx_2$

E8bp1

Our focus: R^n, $n = 0, 1, 2, \ldots$

Super big deal:

Vector Spaces have vector spaces inside them and we call these subspaces

Need three properties for a subset S of V to be a subspace:

SSP1 if $\vec{x}_1 \& \vec{x}_2 \in S$ then $\vec{x}_1 + \vec{x}_2 \in S$

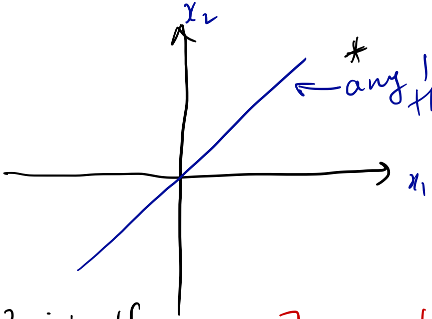
SSP2 if $\vec{x} \in S$ then $c\vec{x} \in S$ for all $c \in R$

SSP3 $\vec{0} \in S$

E86p2

Examples of subspaces:

R^2



∞ly many

* any line through $\vec{0}$

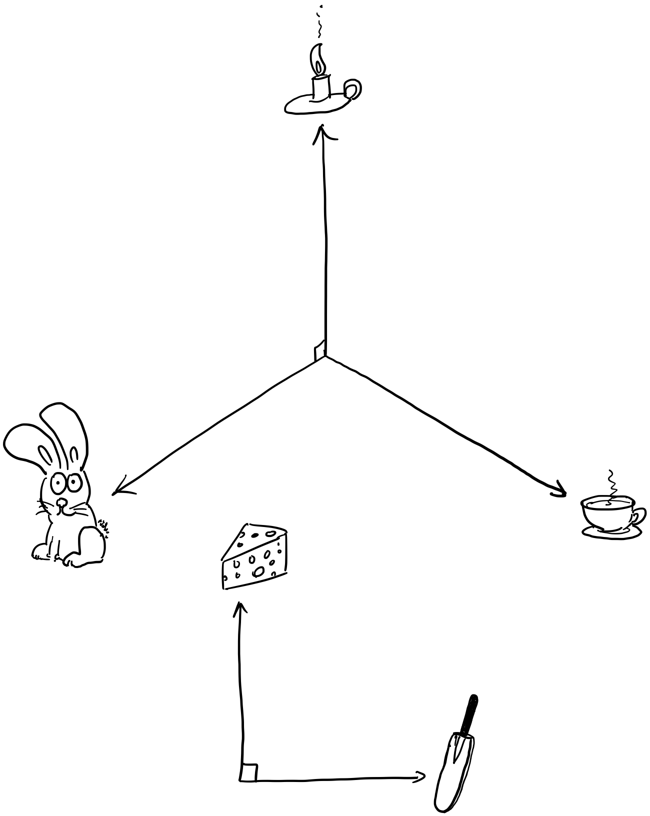
- * R^2 itself
- * $\{\vec{0}\}$ works too

important!

R^3: Subspaces

- * R^3 itself
- * $\{\vec{0}\}$
- * any line through $\vec{0}$
- * any plane " $\vec{0}$

Bonus very silly Spaces:



"Danger Will Robinson! We are entering column space!!"

Menu: • Column space for $A\vec{x}=\vec{b}$
→ the first of four awesome subspaces

Our [beloved / belated] problem $\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$

delete as applicable

rows ↙ ↘ columns

The column picture:

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$$

↑ each column has m entries

Observation / Big deal:

Columns of A and $\vec{b}$ live in R^m (not R^n)

↑ $\vec{x}$ lives here

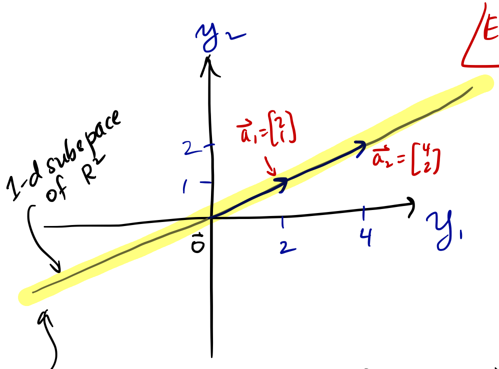
ex

$$A=\begin{bmatrix}2&4\\1&2\end{bmatrix}$$

$$\vec{a}_1=\begin{bmatrix}2\\1\end{bmatrix},\quad \vec{a}_2=\begin{bmatrix}4\\2\end{bmatrix}$$

↑ live in y-space

E8cp1



all linear combinations of $\vec{a}_1$ & $\vec{a}_2$, i.e. $x_1\vec{a}_1+x_2\vec{a}_2$, live on this line which is a subspace of R^2

Huge:

↙ notation

$$C(\underset{m\times n}{A}) = \text{Column Space of } A$$
$$= \text{Subspace of } R^m$$

Here:

$$C(A) = \left\{ \vec{y}\in R^2 \;\middle|\; \vec{y} = c\begin{bmatrix}2\\1\end{bmatrix},\ c\in R \right\}$$

↑ by eye

↑ "such that"

Big deal:

$A\vec{x} = \vec{b}$ has a solution (1 or ∞ly many) only if $\vec{b} \in C(A)$

" $\vec{b}$ lives in the column of A ".

$\Rightarrow$ If $\vec{b} \notin A$, $A\vec{x} = \vec{b}$ has no solution.

For $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 38 \\ 19 \end{bmatrix} \in C(A)$ ← we'll see means there are ∞ly many solut.

$\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin C(A)$

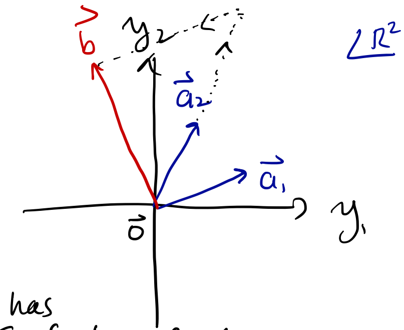
↑ so no solution to $A\vec{x}=\vec{b}$

∠E8cp2

ex

$A = \Big[\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\vec{a}_1} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{a}_2} \Big]$

∠R^2



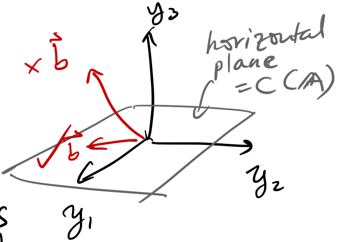
See that $A\vec{x} = \vec{b}$ always has a solution. In fact, only 1.

$\Rightarrow C(A) = R^2$

ex $\underset{3\times2}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

tall ↗

$\hat{i}$ ↖ $\hat{j}$ 2 vectors in 3-d.



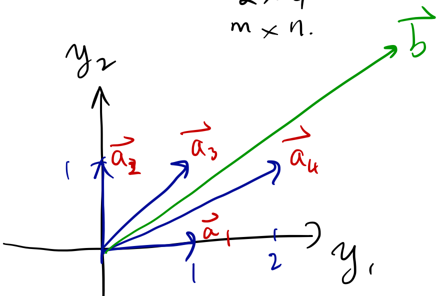
↓

$C(A) \neq R^3$ ← $m=3$

ex $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

wide A →

2×4
$m \times n$.



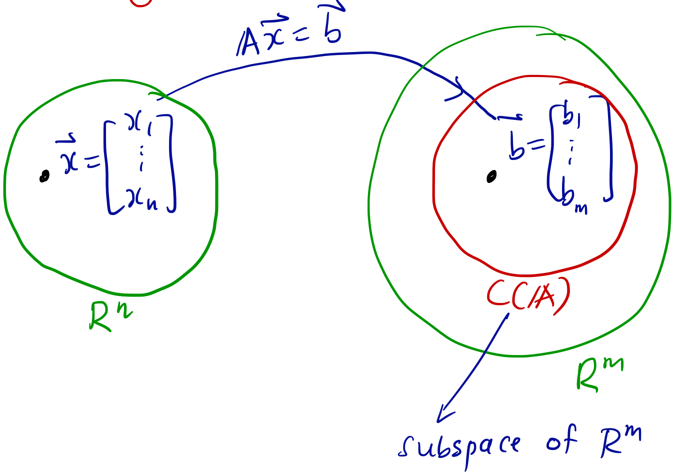
See: any two column vectors of A will work.

$\Rightarrow$ ∞ly many solutions

Again $C(A) = R^2 \leftarrow m=2$

Emerging Picture

LE8cp3



A new realm opens up: Null Space

Menu:
- Definition of the Null space of $\mathbb{A}$, $N(\mathbb{A})$
- what $C(\mathbb{A})$ & $N(\mathbb{A})$ mean for $\mathbb{A}\vec{x}=\vec{b}$

Consider $\mathbb{A}\vec{x}=\vec{0}$ ← very special $\vec{b}$

called Nullspace Equation
or Homogeneous Equation

how can we combine columns of $\mathbb{A}$ to produce nothing?

Immediate Observation: $\mathbb{A}\vec{0}=\vec{0}$
So: Always a solution → $\vec{0}\in C(\mathbb{A})$ always
→ May be 1 or ∞ly many

Example: Solve $\mathbb{A}\vec{x}=\boxed{\vec{0}}$ for:

$$\mathbb{A}=\underset{m\,3\times 3\,n}{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}} \overset{\text{row ops}}{\dashrightarrow}_{\text{pain \& suffering}} \quad \vec{x}=c\begin{bmatrix}1\\1\\-1\end{bmatrix} \text{ where } c\in\mathbb{R}$$

✓

$$N(\mathbb{A})=\left\{\vec{x}\in\mathbb{R}^{③}\ \middle|\ \vec{x}=c\begin{bmatrix}1\\1\\-1\end{bmatrix},\ c\in\mathbb{R}\right\}$$

↑ Null space of $\mathbb{A}$ (n)

↖ subspace of $\mathbb{R}^3$.

E9ap1

Notation:
$\vec{x}_n$ for a null space vector
Also $\vec{x}_h$ ← homogeneous

Now solve:

$$\mathbb{A}\vec{x}=\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\-1\\1\end{bmatrix}$$

$\vec{b}$ ↑ non-zero interesting $\vec{b}$

more pain and suffering (row operations)

find $\vec{x}_r=\begin{bmatrix}1\\0\\1\end{bmatrix}$ ← could be called $\vec{x}_p$ where p is "particular"

we'll write $\vec{x}_r$ because some dying monk said we should

So $\mathbb{A}\vec{x}_n=0$ & $\mathbb{A}\vec{x}_r=\begin{bmatrix}4\\-1\\1\end{bmatrix}$ ← $\vec{b}$

⇒ See $\vec{x}_r$ is not a unique solution b/c $\underbrace{\mathbb{A}(\vec{x}_r+\vec{x}_n)}_{=\vec{b}}$

More:

$$\underset{A}{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}}\left(\underset{\vec{x}_r}{\begin{bmatrix}1\\0\\1\end{bmatrix}}+c\underbrace{\begin{bmatrix}1\\1\\-1\end{bmatrix}}_{\vec{x}_n}\right)$$

$$=\underbrace{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}}+c\underbrace{\begin{bmatrix}1&2&3\\1&-3&-2\\0&1&1\end{bmatrix}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$=\begin{bmatrix}4\\-1\\1\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$=\begin{bmatrix}4\\-1\\1\end{bmatrix}=\vec{b}$$

$\Rightarrow$ There are infinitely many solutions b/c $N(A)\neq\{\vec{0}\}$

In general:

$$A(\vec{x}_r+\vec{x}_n)=\underset{\vec{b}}{A\vec{x}_r}+\overset{\vec{0}}{\cancel{A\vec{x}_n}}$$

E9ap2

The Big Deals:

(1) All vectors $\vec{x}\in R^n$ for which $A\vec{x}=\vec{0}$ form a subspace of R^n

SSP1 $\quad A(\vec{x}_1+\vec{x}_2)=\overset{\vec{0}}{\cancel{A x_1}}+\overset{\vec{0}}{\cancel{A x_2}}=\vec{0}$

SSP2 $\quad A(c\vec{x})=c(A\vec{x})=c\vec{0}=\vec{0}$

SSP3 $\quad \vec{0}\in N(A)$ b/c $A\vec{0}=\vec{0}$

(2)
- If $N(A)=\{\vec{0}\}$ and $\vec{b}\in C(A)$ then $A\vec{x}=\vec{b}$ has one, unique solution
- If $N(A)\neq\{\vec{0}\}$ and $\vec{b}\in C(A)$ then $A\vec{x}=\vec{b}$ has infinitely many solutions
- if $\vec{b}\notin C(A)$ then $A\vec{x}=\vec{b}$ has no solutions

Row Reduction, as you wish.

menu:

- Turning $\mathbb{A}\vec{x}=\vec{b}$ into $\mathbb{R}_{\mathbb{A}}\vec{x}=\vec{d}$
- Reduced Row Echelon Forms (RREFs)
- Pivot and free variables
- The rank r of a matrix
- Fezzik

so much winning

$$\mathbb{A}=\begin{bmatrix}2&4&3&4\\2&4&6&10\\6&12&12&18\end{bmatrix}\qquad \vec{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Fezzik

3×4

$\vec{x}\in\mathbb{R}^4$

arbitrary

Monks tell us:

Solve $\mathbb{A}\vec{x}=\vec{b}$ for general $\vec{b}$

$$[\mathbb{A}\,|\,\vec{b}]=\begin{bmatrix}\overset{D_1}{2}&4&3&4&b_1\\\boxed{2}&4&6&10&b_2\\\boxed{6}&12&12&18&b_3\end{bmatrix}$$

$$R_2'=1\cdot R_2-\left(\frac{2}{\underset{D_1}{2}}\right)R_1$$

$$R_3'=1\cdot R_3-\left(\frac{6}{\underset{D_1}{2}}\right)R_1$$

$$\left[\begin{array}{cccc|c}\overset{D_1}{2}&4&3&4&b_1\\0&0&\overset{D_2}{3}&6&b_2-b_1\\0&0&3&6&b_3-3b_1\end{array}\right]$$

$$R_3'=1\cdot R_3-\left(\frac{3}{3}\right)$$

E9bp1

$$\left[\begin{array}{cccc|c}\text{pivot column}&\text{free column}&\text{pivot column}&\text{free column}&\\2&4&3&4&b_1\\0&0&3&6&b_2-b_1\\0&0&0&0&b_3-b_2-2b_1\end{array}\right]$$

$U\vec{x}=\vec{c}$.

$(b_3-3b_1)-(b_2-b_1)$

Keep Going!! (as with inverses).

$$R_1'=1\cdot R_1-\left(\frac{3}{3}\right)R_2\qquad\left[\begin{array}{cccc|c}2&4&0&-2&2b_1-b_2\\0&0&3&6&b_2-b_1\\0&0&0&0&b_3-b_2-2b_1\end{array}\right]$$

last step: divide through by pivots

$R_1'=\frac{1}{2}R_1$

$R_2'=\frac{1}{3}R_2$

1st 2 columns of Identity matrix

$$\left[\begin{array}{cccc|c}1&2&0&-1&\frac{1}{2}(2b_1-b_2)\\0&0&1&2&\frac{1}{3}(b_2-b_1)\\0&0&0&0&b_3-b_2-2b_1\end{array}\right]$$

$$=\left[\,\mathbb{R}_{\mathbb{A}}\,\middle|\,\vec{d}\,\right]$$

Reduced Row Echelon Form of $\mathbb{A}$

Big Deal Things:

- We can't reduce any further
- $\mathbb{R}_{\mathbb{A}}$ is unique for any $\mathbb{A}$
- Row swaps are still part of the game
- New — pivots may appear irregularly

- Pivot columns match Identity Matrix Columns
- We call x_j that match up with pivot columns, the pivot variables.
- Similarly: free columns ⇔ free variables

For Fezzik x_1 & x_3 are pivot variables
x_2 & x_4 are free variables

Very, very big deal:

Definition:

$$\boxed{\begin{array}{l}\#\text{ pivot columns in } \mathbb{R}_{\mathbb{A}} \\ = \text{rank of } \mathbb{A}\end{array}}$$

Notation: rank of $\mathbb{A} = r$

For Fezzik: $r=2$

→ $m=3, n=4, r=2$

Huge idea: Inside every matrix $\underset{m\times n}{\mathbb{A}}$ there is an invertible $r \times r$ square matrix

The Search for Column Space ...

← C(A)

Our Story:

C(A) = all $\vec{b}$'s for which $A\vec{x} = \vec{b}$ has a solution.

↓ subspace of R^m

Method 1 of 3 for finding C(A):

reduce $[A \mid \vec{b}]$ ← general $\vec{b}$ to $[R_A \mid \vec{d}]$

and for all rows of 0 in R_A, set matching entries in $\vec{d}$ to 0.

Our friend Fezzik:

$$[A \mid \vec{b}] = \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 10 & b_2 \\ 6 & 12 & 12 & 18 & b_3 \end{array}\right]$$

previous row reduction suffering ↓

↙ $\vec{d}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1 - b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2 - b_1) \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right] = [R_A \mid \vec{d}]$$

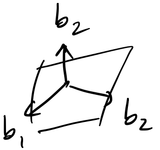
E9cp1

$$\underbrace{0x_1 + 0x_2 + 0x_3 + 0x_4}_{0} = \underbrace{b_3 - b_2 - 2b_1}_{\text{must be } 0}$$

→

$$\Rightarrow \quad b_3 - b_2 - 2b_1 = 0$$

↖ eq of a plane in



↗ true but not useful.

Better (but not only way):

Set $b_3 = b_2 + 2b_1$, where $b_1, b_2 \in R$

↖ b_3 depends on ↗

fixed vectors

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2 + 2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

always do this.

where $b_1, b_2 \in R$.

Formal result:

$$C(A) = \left\{ \vec{b} \in R^3 \;\middle|\; \vec{b} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ b_1, b_2 \in R \right\}$$

LE9cp2

Big deal:

rank of A, r

See $C(A)$ is a $\boxed{2}$-d subspace (plane) of R^3 $(m=3)$

↗ #awesome

Notes

* Because $C(A)$ does not fill up R^3 then $A\vec{x}=\vec{b}$ may or may not have solutions

* Nothing super special about $\begin{bmatrix}1\\0\\2\end{bmatrix}$ & $\begin{bmatrix}0\\1\\1\end{bmatrix}$

×2   ×−3

$\begin{bmatrix}2\\0\\4\end{bmatrix}$, $\begin{bmatrix}0\\-3\\-3\end{bmatrix}$ would work

*we had $b_3-b_2-2b_1=0$

$$b_2=b_3-2b_1$$

↑ dependent var

very nutritious ↓

* $b_3-b_2-2b_1=0$

$$\underset{1\times 3}{\begin{bmatrix}-2 & -1 & 1\end{bmatrix}}\underset{3\times 1}{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}=\underset{1\times 1}{\begin{bmatrix}0\end{bmatrix}}$$

$$A \qquad \vec{x} \quad = \quad \vec{0}$$

↗

Solve a nullspace problem to find $C(A)$

The Search for Null Space, $N(A)$:

Quest: find all $\vec{x}$ such that $\underbrace{A\vec{x}}=\vec{0}$

$\vec{x}$ is on right of A
⇒ Right null space

Again, with Fezzik:

$[A\,|\,\cancel{\vec{b}}^{\,\vec{0}}] =$

$$\left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 10 & b_2 \\ 6 & 12 & 12 & 18 & b_3 \end{array}\right]$$

$\vec{0}$

previous row reduction suffering

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1-b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2-b_1) \\ 0 & 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right] = [R_A\,|\,\vec{d}\,]$$

$\vec{d}$ $\vec{0}$ $\vec{0}$

For $N(A)$, set $\vec{b}=\vec{0}$ (or start with $\vec{b}=\vec{0}$)

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- hats not in video
- the shame

E9dp1

Usual story:

$$\begin{aligned} x_1 + 2x_2 \qquad\quad -x_4 &= 0 \\ x_3 \quad +2x_4 &= 0 \end{aligned}$$

free variables

pivot variables

Always do the following well defined procedure:

Express pivot variables (dependent) in terms of the free variables (independent)

$$\Rightarrow \begin{aligned} x_1 &= -2x_2 + x_4 \\ x_3 &= \quad 0 \quad -2x_4 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2+x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

where $x_2, x_4 \in R$

always do this!

replace pivot vars with free vars

$A\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

$A\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \vec{0}$

LE9d p2

Formally:

$$N(A) = \left\{ \vec{x} \in \mathbb{R}^{\overset{n}{\boxed{4}}} \;\middle|\; \vec{x} = x_2 \begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4 \begin{bmatrix}1\\0\\-2\\1\end{bmatrix} \right\}$$

$x_2, x_4 \in \mathbb{R}$

For Fezzik, this is a plane in 4 dimensions

Notes:

- b/c $N(A) \neq \{\vec{0}\}$, if $A\vec{x} = \vec{b}$ has a solution (i.e, $\vec{b} \in C(A)$) then there a ∞ly many solutions

- Soon we we'll see that the dimension of $N(A)$ is

$$\dim N(A) = \underset{\text{\# columns}}{n} - \underset{\text{rank of } A}{r}$$

Fezzik: $4 - 2 = 2$ ✓

Solving $A\vec{x}=\vec{b}$ the Subspace Way:

- Fezzik with $\vec{b}$ in $C(A)$ & $\vec{b} \neq \vec{0}$

From before:

$$[A\,|\,\vec{b}] = \left[\begin{array}{cccc|c} 2 & 4 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 10 & b_2 \\ 6 & 12 & 12 & 18 & b_3 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} p & F & p & F & \\ 1 & 2 & 0 & -1 & \frac{1}{2}(2b_1-b_2) \\ 0 & 0 & 1 & 2 & \frac{1}{3}(b_2-b_1) \\ 0 & 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right] = [R_A\,|\,\vec{d}]$$

example: $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$ b_1 b_2 b_3 ← see this is the first column!

$$[R_A\,|\,\vec{d}] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \checkmark$$

$\vec{b} \in C(A)$

Plan: Use same steps as for finding $N(A)$

E9ep1

$$\begin{array}{ccccc} p & F & p & F & \\ x_1 & +2x_2 & & -x_4 & =1 \\ & & x_3 & +2x_4 & =0 \end{array}$$

put free variables on RHS

each pivot variable appears only once in all equations

$$x_1 = 1 - 2x_2 + x_4$$
$$x_3 = 0 + 0x_2 - 2x_4$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-2x_2+x_4 \\ 0+x_2+0x_4 \\ 0-2x_4+0x_2 \\ 0+x_4+0x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{\vec{x}_p} + \underbrace{x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\vec{x}_h}$$

where $x_2, x_4 \in \mathbb{R}$

$\vec{x}_p$ p = particular

$\vec{x}_h$ h = homogeneous

replace pivot vars w. free var expression

General Story:

$$A(\overbrace{\vec{x}_p + \vec{x}_h}^{\vec{x}}) = \underbrace{A\vec{x}_p}_{\vec{b}} + \underbrace{\cancel{A\vec{x}_h}}_{\vec{0}} = \vec{b}$$

not unique

not necessarily the same

Later:

$$A(\overbrace{\vec{x}_r + \vec{x}_n}^{\vec{x}}) = \underbrace{A\vec{x}_r}_{\vec{b}} + \underbrace{\cancel{A\vec{x}_n}}_{\vec{0}}$$

think of as most excellent $\vec{x}_p$

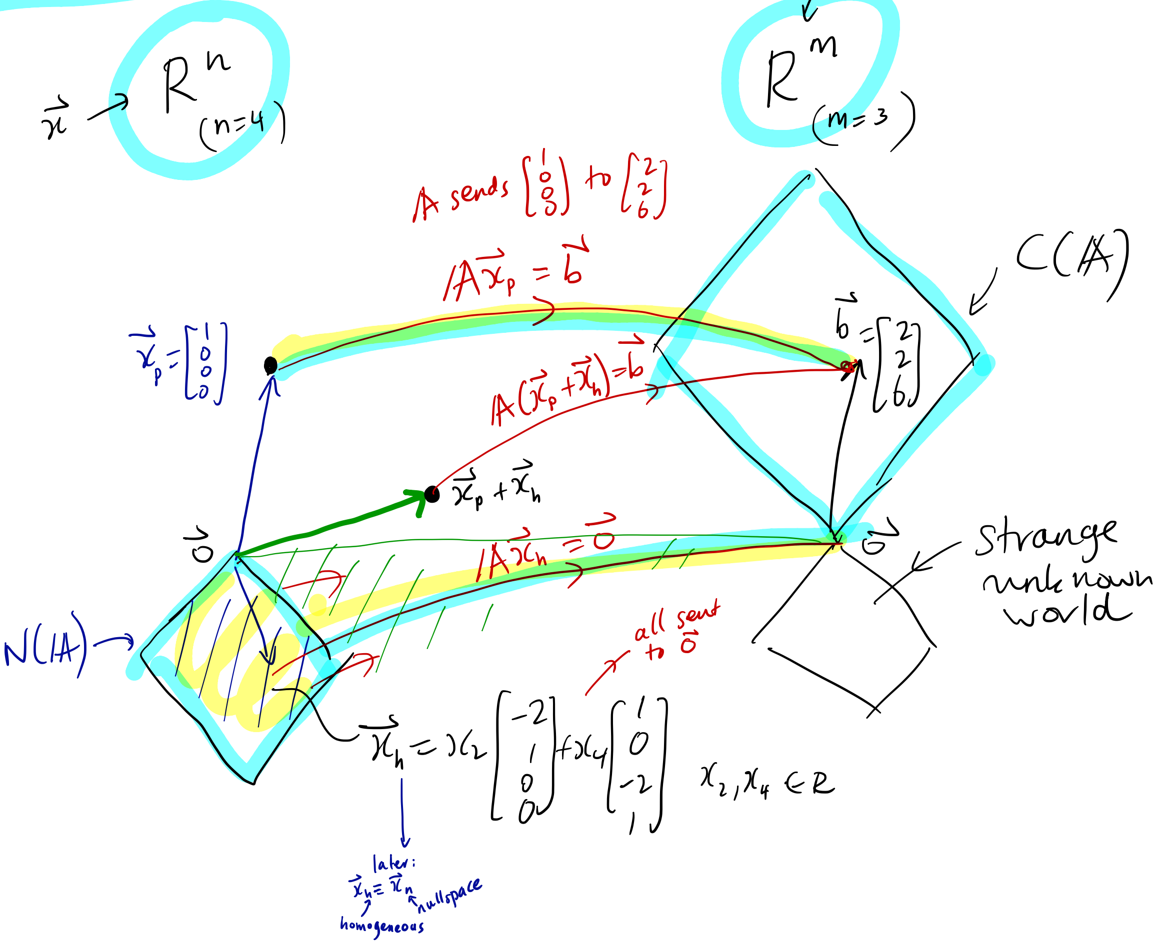
unique

row

null

Big picture emerging for $A\vec{x}=\vec{b}$.

LE9ep2



E9fp1

"I see null vectors"

- Jumping to the form of $N(\mathbb{A})$ from $\mathbb{R}_{\mathbb{A}}$

Fezzik's $\mathbb{R}_{\mathbb{A}}$:

$$\begin{array}{cccc} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ \hline P & F & P & F \end{array} \longrightarrow \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \mathbb{F}$$

pivot columns: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}_2$

$\uparrow$ $r=2$

Our two null vectors from our earlier solution for $N(\mathbb{A})$:

make a matrix

$$\mathbb{N} = \begin{array}{c|cc} P & -2 & 1 \\ F & 1 & 0 \\ P & 0 & -2 \\ F & 0 & 1 \end{array}$$

$-\mathbb{F} = -\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

See $\underbrace{\mathbb{R}_{\mathbb{A}}}_{m\times n}\ \underbrace{\mathbb{N}}_{n\times(n-r)} = \underbrace{\mathbb{0}}_{m\times(n-r)}$

$\uparrow$ secret

$\uparrow$ all zeros = kevin the destroyer of words

also $\mathbb{A}\,\mathbb{N} = \mathbb{0}$

↳ $\mathbb{A}$ & $\mathbb{R}_{\mathbb{A}}$ have the same Nullspace.

General Story

permutation of x_i's

$$\underbrace{\mathbb{R}_{\mathbb{A}}}_{m\times n} = \left[\begin{array}{c|c} \overset{r\times r}{\mathbb{I}} & \overset{r\times(n-r)}{\mathbb{F}} \\ \hline 0\ 0 \cdots & \cdots 0 \end{array}\right] \leftarrow (m-r)\times n$$

m−r rows of 0s

$\mathbb{0}$ ← all zeros

$$\underbrace{\mathbb{N}}_{n\times(n-r)} = \begin{bmatrix} -\mathbb{F} \\ \hline \mathbb{I} \end{bmatrix}$$

$n-r = \dim N(\mathbb{A})$

$$\underbrace{\mathbb{R}_{\mathbb{A}}}_{m\times n}\ \underbrace{\mathbb{N}}_{n\times(n-r)} = \begin{bmatrix} -\mathbb{I}\overbrace{\mathbb{F}}^{r\times(n-r)} + \mathbb{F}\overbrace{\mathbb{I}}^{r\times r} \\ \hline \underbrace{\mathbb{0}}_{(m-r)\times(n-r)} \end{bmatrix} = \underbrace{\mathbb{0}}_{m\times(n-r)}$$

absent in videos due to cavalier attitude

E9gp1

Solving $\mathbb{A}\vec{x}=\vec{b}$ the Subspace way: simpler examples

(1)

$$\underset{m \times n}{\mathbb{A} = \begin{bmatrix}1&0\\0&1\end{bmatrix}} \leftarrow \text{the identity matrix}$$

$\mathbb{A}\vec{x}=\vec{b}$ is always solvable!

$\mathbb{I} \rightarrow \vec{x}=\vec{b}$

see $\mathbb{R}_{\mathbb{A}} = \mathbb{A} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\begin{array}{cc}\uparrow&\uparrow\\p&p\end{array}$

$\mathbb{R}_{\mathbb{A}}$ already here

* Column Space

Solve $\left[\begin{array}{cc|c}1&0&b_1\\0&1&b_2\\p&p&\end{array}\right]$ for $C(\mathbb{A})$

$\Rightarrow b_1, b_2 \in R$ (no restrictions)

$$\vec{b} = \underbrace{\begin{bmatrix}b_1\\b_2\end{bmatrix}}_{\text{always}} = b_1\begin{bmatrix}1\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\end{bmatrix}$$

$b_1, b_2 \in R$

$\Rightarrow C(\mathbb{A}) = R^2 \leftarrow m=2$

* Null Space

Solve $\left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$

$\Rightarrow \begin{array}{l}x_1=0\\x_2=0\end{array} \Rightarrow \vec{x} = \begin{bmatrix}0\\0\end{bmatrix} = \vec{0}$ is only solution

$N(\mathbb{A}) = \{\vec{0}\}$.

Delectable Upshots:

- Every $\vec{b} \in C(\mathbb{A})$ so $\mathbb{A}\vec{x}=\vec{b}$ is always solvable
- Because $N(\mathbb{A})=\{\vec{0}\}$, every solution is unique.

$$\dim C(\mathbb{A}) = 2 \;(= \overset{\text{later}}{r})$$

$$\dim N(\mathbb{A}) = 0 \;(= \underset{2-2}{n-r})$$

(2)

* see E8cp1 for first examination of this A

$A = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ with $\vec{b} = \begin{bmatrix}2\\4\end{bmatrix}$ & then $\vec{b} = \begin{bmatrix}2\\2\end{bmatrix}$

see columns are multiples of each other as are rows

Find $C(A)$:

$$[A \mid \vec{b}] = \left[\begin{array}{cc|c} \overset{D_1}{1} & 2 & b_1\\ 2 & 4 & b_2\end{array}\right]$$

$$\sim \quad R_2' = R_2 - \left(\frac{2}{\underset{D_1}{1}}\right) R_1 \qquad \left[\begin{array}{cc|c} P & F & \\ 1 & 2 & b_1\\ 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

rank $r = 1$

$\underbrace{\quad}_{R_A}$ $\underbrace{\quad}_{d}$ $=0$

$\Rightarrow$ $b_2 - 2b_1 = 0$

$b_2 = 2b_1$ where $b_1 \in R$

dependent independent.

$$\vec{b} = \begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}b_1\\ 2b_1\end{bmatrix} = b_1\begin{bmatrix}1\\2\end{bmatrix} \text{ where } b_1 \in R$$

E9gP2

Find $N(A)$:

$$[A \mid \vec{0}] \sim \left[\begin{array}{cc|c} P & F & \\ 1 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$$

$\Rightarrow$ $\underset{P}{x_1} + \underset{F}{2x_2} = 0$

$\Rightarrow$ $x_1 = -2x_2$

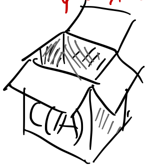
replace pivot var

$$\vec{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-2x_2\\ x_2\end{bmatrix} = x_2\begin{bmatrix}-2\\1\end{bmatrix} \text{ where } x_2 \in R$$

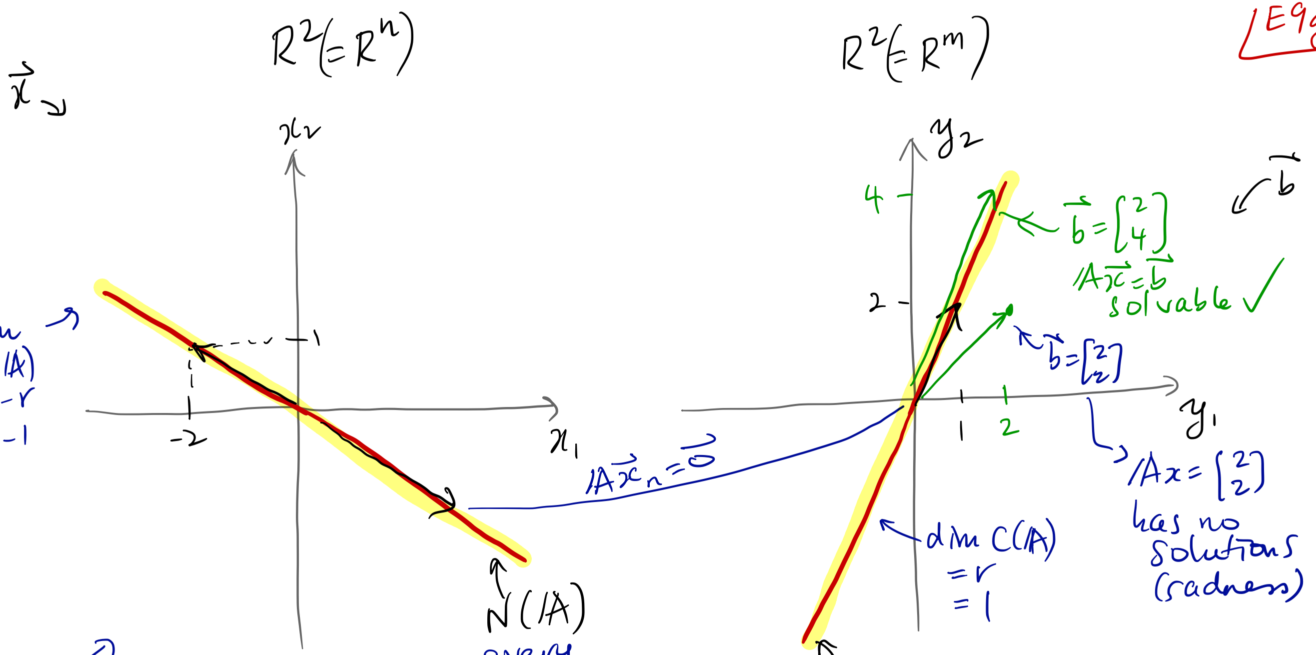
$$C(A) = \left\{ \vec{b} \in R^2 \;\middle|\; \vec{b} = b_1\begin{bmatrix}1\\2\end{bmatrix}, b_1 \in R \right\}$$

$$N(A) = \left\{ \vec{x} \in R^2 \;\middle|\; \vec{x} = x_2\begin{bmatrix}-2\\1\end{bmatrix}, x_2 \in R \right\}$$

box of vectors



E9gP3



dim N(A) = n − r = 2 − 1 = 1

$\vec{b} = \begin{bmatrix}2\\4\end{bmatrix}$

$A\vec{x} = \vec{b}$ solvable ✓

$\vec{b} = \begin{bmatrix}2\\2\end{bmatrix}$

$Ax = \begin{bmatrix}2\\2\end{bmatrix}$ has no solutions (sadness)

dim C(A) = r = 1

b/c N(A) ≠ $\{\vec{0}\}$ there are ∞ly many solution if $\vec{b} \in C(A)$

N(A) every vector on this line is sent to zero by A.

C(A) Any $\vec{b}$ in C(A) can be made by $A\vec{x}$

$\Longleftrightarrow$ $A\vec{x} = \vec{b}$ has a solution

What $\mathbb{R}_A$ tells us

Menu:

- The four basic kinds of $\mathbb{R}_A$
- How these forms for $\mathbb{R}_A$ dictate $A\vec{x}=\vec{b}$'s solution
- Later: $\mathbb{R}_{A^T}$ gives us the rest of what we need to know

---

The Story so far:

$\mathbb{R}_A$ provides us with

(1) The rank r of A (# pivot columns)

(2) Nullspace N(A) (solve $\mathbb{R}_A\vec{x}=\vec{0}$)

(3) The number of possible solutions to $A\vec{x}=\vec{b}$

(1),(2) → (3) because:

- If $r < m$, one or more rows of $\mathbb{R}_A$ are all 0's and therefore some $\vec{b}$'s will lead to no solution for $A\vec{x}=\vec{b}$
- If $N(A) \neq \{\vec{0}\} \iff \mathbb{R}_A$ has one or more free columns then $A\vec{x}=\vec{b}$ will have ∞ly many solutions if it is solvable (i.e., if $\vec{b} \in C(A)$)

Four examples:

E10ap1

(i)

$$\mathbb{R}_A = \begin{array}{cccccc} \text{pivot} & \text{free} & f & p & f & p \\ 1 & \omega_{12} & \omega_{13} & 0 & \omega_{15} & 0 \\ 0 & 0 & 0 & 1 & \omega_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

from a wide A

no row of zeros

$m=3$, $n=6$, $r=3$

See: always a solution to $A\vec{x}=\vec{b}$

$\Rightarrow C(A) = R^3 = R^m = R^r$

Also N(A) is a 3-d subspace of R^6 ($n=6$)

$\vec{x}$ lives in R^6 — to be proven

Know $N(A) \neq \{\vec{0}\}$

So: $A\vec{x}=\vec{b}$ always has $\overbrace{\text{a solut}}^{C(A)}$ and there are always $\underbrace{\infty\text{ly many}}_{N(A)}$

Note: Wide A's always have at least 1 free variable $\Rightarrow N(A) \neq \{\vec{0}\}$

(ii) $$\mathbb{R}_{/A} = \begin{array}{c} \begin{array}{ccc} p & p & p \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \mathbb{I}$$

$m = n = r = 3$

⇒ No free variables

See: $A\vec{x} = \vec{b}$ is always solvable

and $N(A) = \{\vec{0}\}$

So: $A\vec{x} = \vec{b}$ always has 1, unique solut.

For square invertible matrices $(n \times n)$

$\mathbb{R}_{/A} = \mathbb{I}$ always.

↖ 1-1 mapping from $R^n \to R^n$

(iii) $$\mathbb{R}_{/A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$m = 4$

$n = r = 3$

tall →

tall matrices must always have a row of zeros

See;

$N(A) = \{\vec{0}\}$

+ Possible: no solutions

$A\vec{x} = \vec{b}$ has 0 or 1 solutn

$\vec{b} \notin C(A)$ ↗ $\vec{b} \in C(A)$

E10ap2

(iv)

m and n > r

$$\begin{array}{c} \begin{array}{cccc} P & F & P & F \end{array} \\ \begin{bmatrix} 1 & \omega_{12} & 0 & \omega_{14} \\ 0 & 0 & 1 & \omega_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$m = 3$

$n = 4$

$r = 2$

↖ later: really a 2 by 2 matrix in a 3×4 matrix

A sends a plane in R^4 to a plane in R^3

See: $N(A) \neq \{\vec{0}\}$ (2 free variables)

$C(A)$ may or may not contain $\vec{b}$ (row of 0's in $\mathbb{R}_{/A}$)

⇒ either 0 or ∞ly many solutn.

E10ap3

Case:	example R_A	# Solutions
(i) $m=r$ $n=r$ square	$\begin{array}{ccc} p & p & p \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$	1 always $\left[\begin{matrix} C(A)=\mathbb{R}^m \\ N(A)=\{\vec{0}\} \end{matrix}\right.$
(ii) $m=r$ $n>r$ wide	$\begin{array}{ccccc} P & F & P & F & P \\ \hline 1 & -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}$	∞ always $\left[\begin{matrix} C(A)=\mathbb{R}^m \\ N(A)\neq\{\vec{0}\} \end{matrix}\right.$
(iii) $m>r$ $n=r$ tall	$\begin{array}{ccc} p & p & p \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$	0 or 1 $\left[\begin{matrix} C(A)\neq\mathbb{R}^m \\ N(A)=\{\vec{0}\} \end{matrix}\right.$
(iv) $m>r$ $n>r$ many possibilit.	$\begin{bmatrix} 1 & -2 & 0 & 12 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	0 or ∞ $\left[\begin{matrix} C(A)\neq\mathbb{R}^m \\ N(A)\neq\{\vec{0}\} \end{matrix}\right.$

Next: • find bases for $C(A)$ & $N(A)$
• $\dim C(A)=r$, $\dim N(A)=n-r$

Getting to know your Subspaces:

Menu:

- Care and feeding
- Spanning sets
- Bases
- Dimensions

New friend: Inigo $\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

Plan: explore $\mathbb{A}\vec{x} = \vec{b}$ with Inigo

First ~ find $C(\mathbb{A})$ and $N(\mathbb{A})$

Solve $\begin{bmatrix} \overset{\swarrow D_1}{1} & 2 & 1 & | & b_1 \\ 2 & 4 & 2 & | & b_2 \end{bmatrix} = [\mathbb{A} \mid \vec{b}]$

$R_2' = R_2 - \left(\frac{2}{1}\right)R_1$ (2 ← ℓ_{21}, 1 ← D_1)

$$\sim \begin{array}{cccc} P & F & F & \\ \left[\begin{matrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{matrix}\right. & & & \left|\begin{matrix} b_1 \\ b_2 - 2b_1 \end{matrix}\right] \end{array} = [\mathbb{R}_{\mathbb{A}} \mid \vec{d}]$$

$m = 2$ rows
$n = 3$ columns
$r = 1$, rank

$C(\mathbb{A})$: Must have $b_2 - 2b_1 = 0$ for solution to be possible

$\Rightarrow \overset{\swarrow \text{dependent}}{b_2} = 2\overset{\swarrow \text{independent}}{b_1}$

$\Rightarrow \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$C(\mathbb{A}) = \left\{ \vec{y} \in R^2 \;\middle|\; \vec{y} = c\begin{bmatrix} 1 \\ 2 \end{bmatrix},\; c \in R \right\}$ ← line through origin

← 1-d subspace of R^2 ← $r=1$, $m=2$

E11ap1

$N(\mathbb{A})$: Solve $\mathbb{A}\vec{x} = \vec{0}$

$\Rightarrow$ Set $\vec{b} = 0$ in previous

$$\begin{array}{cccc} P & F & F & \\ \left[\begin{matrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{matrix}\right. & & & \left|\begin{matrix} 0 \\ 0 \end{matrix}\right] \end{array} = [\mathbb{R}_{\mathbb{A}} \mid \vec{0}]$$

$\Rightarrow \underset{P}{x_1} + 2\underset{F}{x_2} + \underset{F}{x_3} = 0 \Rightarrow \underset{P}{x_1} = -2\underset{F}{x_2} - \underset{F}{x_3}$

express pivot variables in terms of the free variables

replace pivot variables

$\vec{x} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{always}} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where $x_2, x_3 \in R$

$N(\mathbb{A}) = \left\{ \vec{x} \in R^{\overset{n=3}{3}} \;\middle|\; \vec{x} = c_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};\; c_1, c_2 \in R \right\}$

↑ plane in R^3

Always true:

$C(\mathbb{A}) \subset R^m$ & $N(\mathbb{A}) \subset R^n$

"is a subspace of" ($\subset$), $\vec{b}$'s (R^m), $\vec{x}$'s (R^n)

Boring but important:

How do we know C(A) & N(A) are really subspaces and not some wheedy subsets?

N(A) for Inigo comprises $\underbrace{\text{all linear combinations}}_{\text{automatic subspacification}}$ of $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Check supspace properties:

SSP1 if $\vec{x}_1, \vec{x}_2 \in N(A)$, $\vec{x}_1+\vec{x}_2 \in N(A)$

$$\vec{x}_1 = c_{11}\begin{bmatrix}-2\\1\\0\end{bmatrix} + c_{12}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad \vec{x}_2 = c_{21}\begin{bmatrix}-2\\1\\0\end{bmatrix} + c_{22}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$\Rightarrow \vec{x}_1+\vec{x}_2 = (c_{11}+c_{21})\begin{bmatrix}-2\\1\\0\end{bmatrix} + (c_{12}+c_{22})\begin{bmatrix}-1\\0\\1\end{bmatrix} = \text{Vector in } N(A)$$

SSP2 if $\vec{x}_1 \in N(A)$, $c\vec{x}_1 \in N(A)$ for all $c \in R$

Yes: $$\underbrace{c\times c_{11}}_{\text{still a real number}}\begin{bmatrix}-2\\1\\0\end{bmatrix} + \underbrace{c\times c_{21}}_{\text{same}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

E11ap2

SSP3 $\vec{0} \in N(A)$

Yes: set $c_{11}=c_{12}=0 \Rightarrow \vec{x}_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

General Story:

Sets made up of all linear combinations of some set of vectors are automatically subspaces.

Terminology:

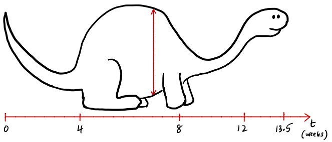
We say $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ "span" the nullspace of A

and that $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ are a "spanning set" for N(A)

"All your bases are belong to us"

Menu:
- Spanning sets for vector spaces & subspaces
- Bases for vector spaces & subspaces
- How bases are all about $\mathbb{A}\vec{x}=\vec{0}$ and the Nullspace of $\mathbb{A}$
- The dimensions of subspaces
- And this:

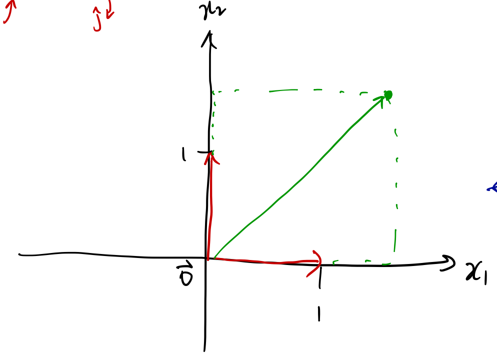
Melvin the Course Difficulty Dinosaur:



Three Examples of Spanning sets for $\mathbb{R}^2$

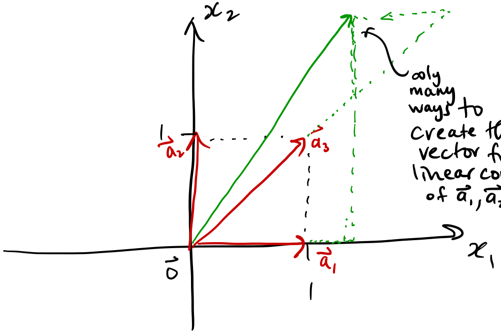
(1) $\left\{ \overset{\vec{a}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \overset{\vec{a}_2}{\begin{bmatrix}0\\1\end{bmatrix}} \right\}$ basis

$\hat{i}$ $\hat{j}$



E11bp1

(2) $\left\{ \overset{\vec{a}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \overset{\vec{a}_2}{\begin{bmatrix}0\\1\end{bmatrix}}, \overset{\vec{a}_3}{\begin{bmatrix}1\\1\end{bmatrix}} \right\}$ not a basis



only many ways to create this vector from a linear combinat of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$

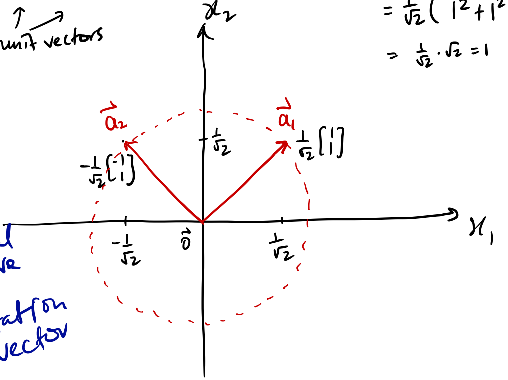
(3) $\left\{ \overset{\vec{a}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}, \overset{\vec{a}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}} \right\}$ ← basis.

unit vectors

note: $\|\vec{a}_1\|$ ← length

$= \frac{1}{\sqrt{2}}(1^2+1^2)^{1/2}$

$= \frac{1}{\sqrt{2}}\cdot\sqrt{2} = 1$



2 non-parallel vectors give unique representation of each vector

Observe:

- Examples (1) & (3) are special because we need both vectors
- For (2), we could take any one vector away, and the remaining two would still span R^2

The right words for the above:

(1) & (3) have linearly independent sets of vectors

(2) has a linearly dependent set of vectors

Big Deal time:

LE11bp2

Defn: A set of vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ in R^m is linearly independent if

ER

$$\underline{x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{0}}$$ only

Nullspace Equation $A\vec{x} = \vec{0}$

has $x_1 = x_2 = \ldots = x_n = 0$ as a solution

(x_i is a scalar)

Why? If one or more $x_i \neq 0$, then we can express one vector in terms of the others

ex (2)

$$\underset{\vec{a}_3}{\begin{bmatrix}1\\1\end{bmatrix}} = \underset{\vec{a}_1}{\begin{bmatrix}1\\0\end{bmatrix}} + \underset{\vec{a}_2}{\begin{bmatrix}0\\1\end{bmatrix}}$$

$$\underset{x_1}{1}\cdot\vec{a}_1 + \underset{x_2}{1}\cdot\vec{a}_2 - \underset{x_3}{1}\cdot\vec{a}_3 = \vec{0}$$

E11 bp3

Seeing things:

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly independent

$$\iff \underset{\substack{\mathbb{A} \\ m\times n}}{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}} \underset{n\times 1}{\vec{x}} = \vec{0}$$

has only $\vec{x} = \vec{0}$ as a solution

$$\iff N(\mathbb{A}) = \{\vec{0}\}$$

so exciting...

Defn: A spanning set that is linearly independent is called a **basis**

(plural: bases) ← "bay sieze"

Note: Bases are not unique (see ex(1) & (3) above) but some bases are better than others and some are totally awesome.

General goodness: Bases give us a unique representation of every point in the space they span.

Defn:

The dimension of a space is the number of vectors in any basis for that space

E11cp1

Why the dimension of $C(\mathbb{A})$ is the rank of $\mathbb{A}$, r

menu
- Including a second way to find $C(\mathbb{A})$
- Inigo & Fezzik

Claim: $\dim C(\mathbb{A}) = r = \#$ pivot columns in $\mathbb{R}_{\mathbb{A}}$

↑ RREF

Two key points:

#1 When we perform row operations on a matrix, the relationships between the columns do not change.

Fezzik:

Identity matrix

$$\mathbb{A} = \begin{array}{cccc} P & F & P & F \\ \hline \end{array}\begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix} \sim \begin{array}{cccc} P & F & P & F \\ \hline \end{array}\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbb{R}_{\mathbb{A}}$$

Observations:

x_1 & x_3 are pivot variables
x_2 & x_4 are free variables

$c_2 = 2c_1$ in both $\mathbb{A}$ & $\mathbb{R}_{\mathbb{A}}$

$c_4 = -c_1 + 2c_3$ " " " " "

easiest to see relationships in $\mathbb{R}_{\mathbb{A}}$

⇓

Identity matrix in pivot columns is key

#2 Follows that in $\mathbb{A}$, the free columns can be made out of the pivot columns, and the pivot columns have to be linearly independent.

⇒ The pivot columns of $\mathbb{A}$ form a basis for $C(\mathbb{A})$

⇒ Because there a r pivot columns, then $\dim C(\mathbb{A}) = r$.

Second way of finding $C(\mathbb{A})$

Fezzik:

$$C(\mathbb{A}) = \left\{ \vec{y} \in \mathbb{R}^3 \;\middle|\; \vec{y} = c_1\begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix} \right\}$$

$c_1, c_2 \in \mathbb{R}$

basis for $C(\mathbb{A})$: $\left\{ \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix} \right\}$

Inigo:

$$\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \sim \mathbb{R}_{\mathbb{A}} = \begin{array}{c} \begin{matrix} P & F & F \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\uparrow$ pivot column

Basis for $C(\mathbb{A}) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\dim C(\mathbb{A}) = r = 1$

Notes: $C(\mathbb{A}) \underset{\substack{\uparrow \\ \text{in general}}}{\neq} C(\mathbb{R}_{\mathbb{A}})$

$\underset{\substack{\uparrow \\ \text{about } \vec{x}\text{'s}}}{N(\mathbb{A})} = N(\mathbb{R}_{\mathbb{A}})$

$\left(\mathbb{A}\vec{x} = \vec{b} \text{ has same solutions as } \mathbb{R}_{\mathbb{A}}\vec{x} = \vec{d} \right)$

Why the dimension of N(A) is n−r

- See /E9fp1 (p59ish)

Big Deal:

Our one true method of finding nullspace always produces a set of vectors that are linearly independent and are therefore a basis for N(A)

← vectors span N(A)

Inigo: $\mathbb{N} = \begin{array}{c|cc} P & -2 & -1 \\ F & 1 & 0 \\ F & 0 & 1 \end{array} = \begin{bmatrix} -2 & -1 \\ \multicolumn{2}{c}{I} \end{bmatrix}$

I appears in free variable rows

Fezzik: $\mathbb{N} = \begin{array}{c|cc} P & -2 & 1 \\ F & 1 & 0 \\ P & 0 & -2 \\ F & 0 & 1 \end{array} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\underbrace{\qquad}_{n-r \,=\, \#\text{ free variables}}$

monks

b/c we express pivot variables in terms of the free variables when finding N(A) (always)

/E11dp1

$\mathbb{N}$ always has $n-r$ columns that are linearly independent

$\Rightarrow$ form a basis for N(A)

$\Rightarrow \dim N(A) = n-r.$

Inigo: $\dim N(A) = 3-1 = 2$ ✓

Fezzik: $\dim N(A) = 4-2 = 2$ ✓

"It came from Row Space!"

- the row space of $\mathbb{A}$ is a thing
- what this means for $\mathbb{A}\vec{x}=\vec{b}$
- Many big deals
- The Big Picture

Story: Row Space of $\underset{m\times n}{\mathbb{A}}$ = all linear combinations of the rows of $\mathbb{A}$.

= subspace of $\underset{\text{where } N(\mathbb{A}) \text{ lives}}{\mathbb{R}^n}$

Contrast: $C(\mathbb{A})$ = subspace of $\mathbb{R}^m$

Big Deal

BD #1: If $\vec{x}\in\mathbb{A}$'s Row Space, then

$$\mathbb{A}\vec{x}\neq\vec{0} \text{ unless } \vec{x}=\vec{0}$$

$\vec{x}\notin N(\mathbb{A})$

Example

Inigo:

$$\mathbb{A}=\begin{bmatrix}1&2&1\\2&4&2\end{bmatrix} \Rightarrow \text{clearly, Row Space of } \mathbb{A}$$

$$=\left\{\vec{x}\in\mathbb{R}^m \;\middle|\; \vec{x}=c\begin{bmatrix}1\\2\\1\end{bmatrix},\ c\in\mathbb{R}\right\}$$

$$\begin{bmatrix}1&2&1\\2&4&2\end{bmatrix}\times c\begin{bmatrix}1\\2\\1\end{bmatrix}=c\begin{bmatrix}6\\12\end{bmatrix}=6c\begin{bmatrix}1\\2\end{bmatrix}\in C(\mathbb{A})$$

has to be true

Recall $\mathbb{A}\vec{x}=\vec{b}$

$$\vec{x}=\underset{\text{particular}}{\vec{x}_p}+\underset{\text{homogeneous}}{\vec{x}_h}=\underset{\text{row}}{\vec{x}_r}+\underset{\text{null}}{\vec{x}_n}$$

note $\vec{x}_p\neq\vec{x}_r$ necessarily

E12ap1

$$\mathbb{A}(\vec{x}_p+\vec{x}_h)=\underbrace{\mathbb{A}\vec{x}_p}_{\vec{b}}+\underbrace{\mathbb{A}\vec{x}_h}_{\vec{0}}$$

may be infinitely many $\in N(\mathbb{A})\neq\{\vec{0}\}$

$\vec{x}_p$ must partly live in row space of $\mathbb{A}$.

BD #2

Any $\vec{x}$ in Row Space of $\mathbb{A}$ is $\perp$/orthogonal/at right angles to any $\vec{x}$ in Null space of $\mathbb{A}$.

$$\underset{\mathbb{A}}{\begin{bmatrix}1&2&1\\2&4&2\end{bmatrix}}\left(\underbrace{c_1\begin{bmatrix}-2\\1\\0\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\in N(\mathbb{A}) \text{ from before}}\right)=\vec{0}$$

BD#3 The row space of $\mathbb{A}$ is the same as the row space of $\mathbb{R}_{\mathbb{A}}$

$\Rightarrow$ beautiful basis for row space of $\mathbb{A}$

= non-zero rows of $\mathbb{R}_{\mathbb{A}}$.

ex Inigo:

$$\mathbb{R}_{\mathbb{A}}=\begin{bmatrix}1&2&1\\0&0&0\end{bmatrix}\rightarrow \text{basis is } \left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$$

Fezzik:

$$\mathbb{R}_{\mathbb{A}}=\begin{bmatrix}1&2&0&-1\\0&0&1&2\\0&0&0&0\end{bmatrix}\rightarrow \text{basis is } \left\{\begin{bmatrix}1\\2\\0\\-1\end{bmatrix},\begin{bmatrix}0\\0\\1\\2\end{bmatrix}\right\}$$

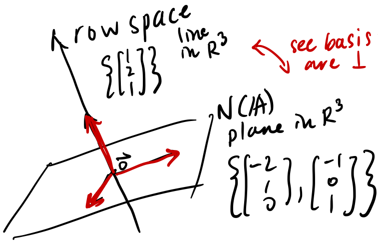
rows are linearly independent b/c of $\mathbb{I}$ sitting in pivot columns

BD#4 Dim Row Space of $\mathbb{A}$ } because samesies
$= \text{Dim Row Space of } \mathbb{R}_{\mathbb{A}}$
$= r$ = non-zero rows

(same as dim $C(\mathbb{A})$) amazing!!

BD#5 Dims of Row Space of $\mathbb{A}$ and Nullspace of $\mathbb{A}$ add up to n $(= r + (n-r))$.

Inigo:



see basis vectors are ⊥ to each other

dim RowSpace of $\mathbb{A} = r = 1$

dim $N(\mathbb{A})$ $= n - r = 3-1$ $= 2$

$1+2=3$

soon: "othogonal complements")

(Imagine loud organ music and lightning)

Row Space & $N(\mathbb{A})$ neatly divide up R^n

BD#6

E12ap2

Consider $\mathbb{A}^T$ for Inigo

$$\mathbb{A}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$$

⇒ now see Row Space of $\mathbb{A}$ is also the Column Space of $\mathbb{A}^T$

wow!

Notation:

$C(\mathbb{A}^T)$ = Row Space of $\mathbb{A}$

cat in a box:

repurpose b/c deep connection.

BD#7: We find a 3rd and final and totally bestest way for finding $C(\mathbb{A})$.

Find $\mathbb{R}_{\mathbb{A}^T}$ and then read off basis vectors for $C(\mathbb{A})$

awesome!!

row space of $\mathbb{A}^T$ = column space of $\mathbb{A}$

Note: eek!!

$\mathbb{R}_{\mathbb{A}^T} \neq (\mathbb{R}_{\mathbb{A}})^T$

in general

E12ap3

ex: $\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$ $\quad \mathbb{A}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$

Inigo:

$$\mathbb{A}^T \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$R_2' = R_2 - \left(\frac{2}{1}\right) R_1$

$R_3' = R_3 - \left(\frac{1}{1}\right) R_1$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $C(\mathbb{A})$

Fezzik:

$$\mathbb{A}^T = \begin{bmatrix} 2 & 2 & 6 \\ 4 & 4 & 12 \\ 3 & 6 & 12 \\ 4 & 10 & 18 \end{bmatrix} \overset{\text{row ops + tears}}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $C(\mathbb{A})$

$\mathbb{I}$ matrix now always appears

The Known Unknowns of Left Nullspace

- What Left Nullspace is
- Connection to the other three subspaces

We have $C(A)$, $N(A)$, and $\underbrace{C(A^T)}_{\text{row space of } A}$

What about $N(A^T)$?

↑ Left Nullspace of A

Reason: if $\vec{y} \in N(A^T) \subset R^m$

then $\underbrace{\underbrace{A^T}_{m\times n}}_{n\times m} \underbrace{\vec{y}}_{m\times 1} = \underbrace{\vec{0}}_{n\times 1}$ ← defn.

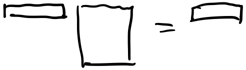
$\in R^m$ $\in R^n$ (where the $\vec{x}$'s are)

Take transpose of both sides:

$$(A^T\vec{y})^T = (\vec{0})^T \qquad \leftarrow (BC)^T = C^T B^T$$

$$\underbrace{\vec{y}^T}_{\substack{1\times m \\ \text{row vector}}} \underbrace{A}_{m\times n} = \underbrace{\vec{0}^T}_{\substack{1\times n \\ \text{row vector}}}$$

} $\vec{y}^T$ multiplies A from the left



So in fact $N(A)$ is the Right Nullspace of A

$$A\vec{x} = \vec{0}$$

↑ on the right.

Know immediately:

$$\dim N(A^T) = \#\text{ columns of } A^T - \text{rank of } A^T = m - r$$

(for $N(A)$, we have $n-r$)

We find $N(A^T)$ just as we would find $N(A)$

Solve $A^T\vec{y} = \vec{0}$

$\vec{y}$ in R^m

Ex: Intro.

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} P & F & \\ 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$ A^T $\qquad$ $\uparrow$ $\vec{0}$

Express pivot vars in terms of free

$$y_1 + 2y_2 = 0 \Rightarrow y_1 = -2y_2$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2y_2 \\ y_2 \end{bmatrix} = y_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \; y_2 \in R$$

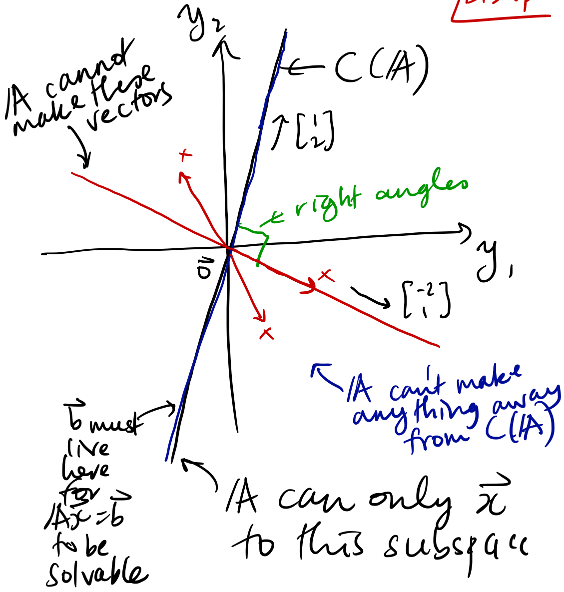
basis vector

Just as $N(A)$ & $C(A^T)$ divide up R^n so too do $N(A^T)$ & $C(A)$

basis $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ $\qquad$ basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

see these are 1-d orthogonal.

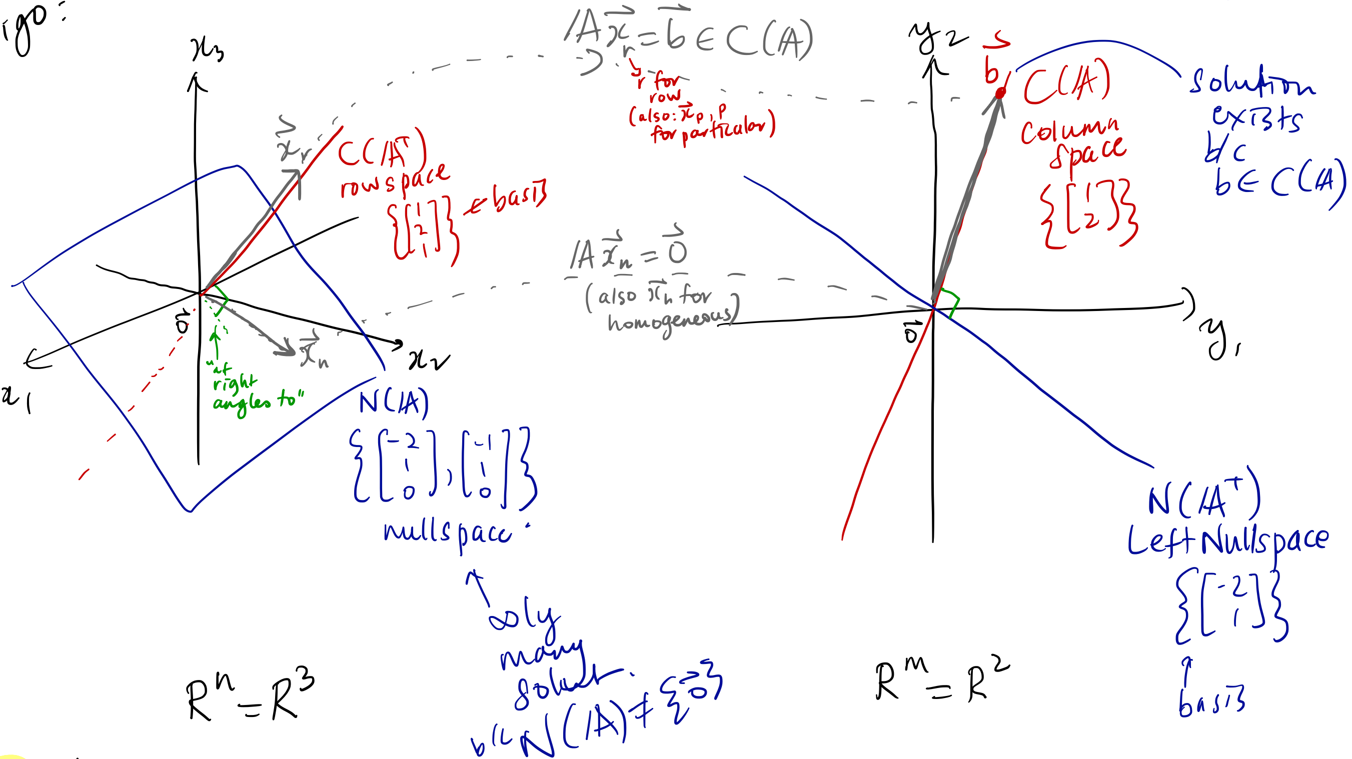
E13ap2



The Fundamental Theorem of Matrixology (almost)

$$A\vec{x} = A(\vec{x}_r + \vec{x}_n)$$
$$= \vec{b}$$

Big picture for Inigo:



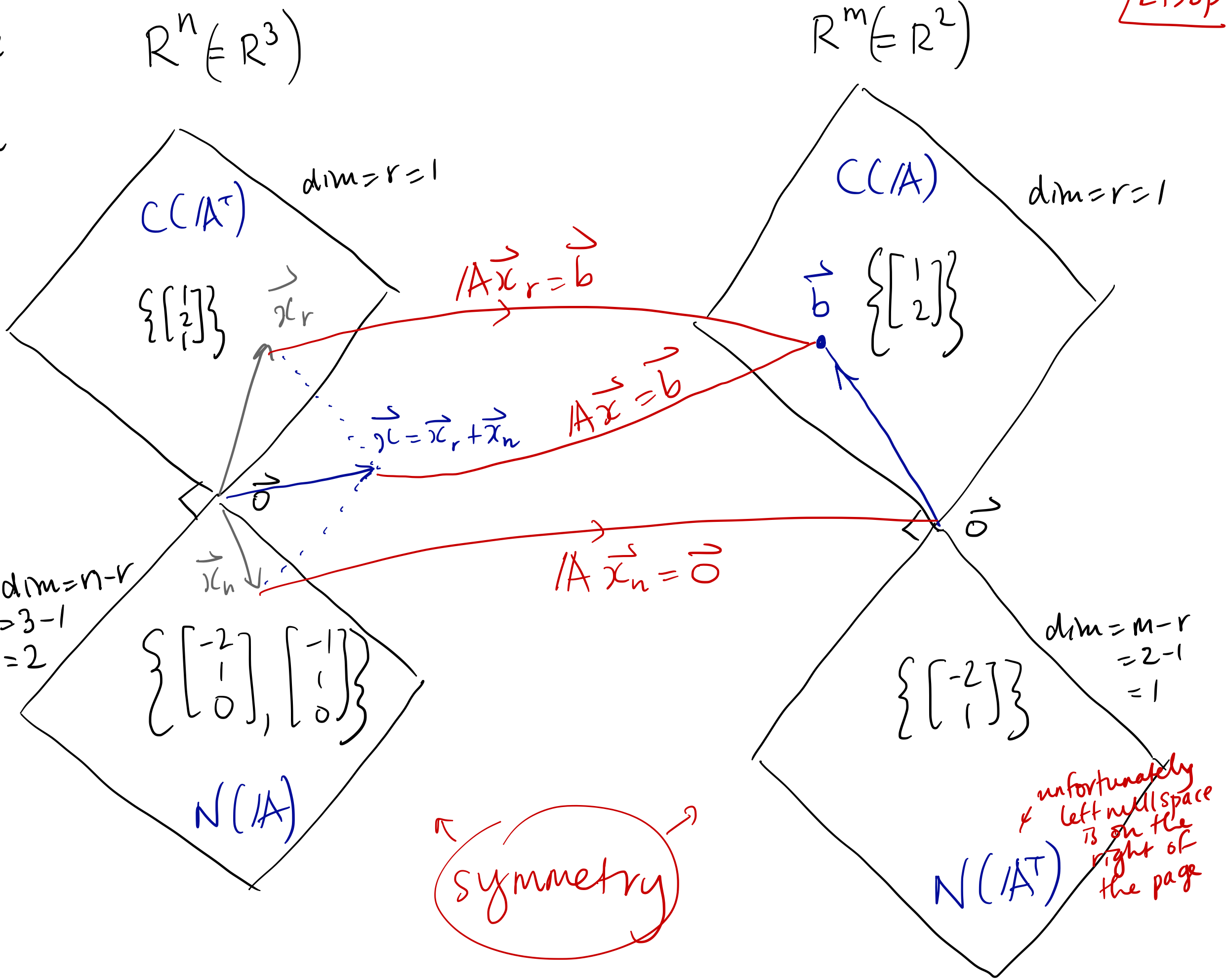
key:

Inigo sends a line to a line

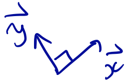
Later: see Inigo ~ $\sqrt{30}$

E13bp2

Abstract Big picture with Inigo's structure



Definitions we need:



(1) (old) if $\underset{\vec{y}^T\vec{x}}{\underset{\parallel}{\vec{x}^T\vec{y}}} = 0$ we say $\vec{x}$ & $\vec{y}$ are orthogonal

(2) We say two supspaces S_1 & S_2 are orthogonal if all vectors in S_1 are orthogonal to all vectors in S_2

(3) If two subspaces $\overset{S_1 \& S_2}{,}$ in a vector space $\overset{\dim n}{V}$ are orthogonal and their dimensions add to n, we say S_1 & S_2 are orthogonal complements of each other

Notation: S and $S^\perp$

and $S \oplus S^\perp = V$

⇑
any vector in V can be written as a sum of a vector in S and a vector in $S^\perp$

E13bp3

Fundamental Theorem of Matrixology (mostly):

• $\dim C(A) = r$	column space
• $\dim N(A^T) = m - r$	left null space
• $\dim C(A^T) = r$	row space
• $\dim N(A) = n - r$	nullspace

- $C(A)$ and $N(A^T)$ are orthogonal complements in R^m
- $C(A^T)$ and $N(A)$ are orthogonal complements in R^n
- The bases of $C(A)$ & $N(A^T)$ combine to give a basis of R^m
- The bases of $C(A^T)$ & $N(A)$ combine to give a basis of R^n

More near the end of course

The Man in Black, Westley:

Matrix-fu

E14ap1

$$\underset{2\times 2}{\mathbb{A}} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$$

2×2: m, n

m=2 rows
n=2 cols

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2' = R_2 - \left(\frac{3}{1}\right)R_1$
$R_1' = R_1$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \mathbb{R}_{\mathbb{A}}$$

P F

see rank $r=1$

x_1 is a pivot variable
x_2 is a free variable

$m=2,\ n=2,\ r=1$

$$\mathbb{A}^T = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$$

$R_2' = R_2 - \left(\frac{-2}{1}\right)R_1$

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \mathbb{R}_{\mathbb{A}^T}$$

y_1 y_2

Dimensions:

$$\dim \underset{\text{column space}}{C(\mathbb{A})} = \underset{1}{r} = \dim \underset{\text{row space}}{C(\mathbb{A}^T)}$$

$\dim N(\mathbb{A}) = n - r = 2 - 1 = 1.$

$\dim N(\mathbb{A}^T) = m - r = 2 - 1 = 1.$

Left Nullspace

(Right) Nullspace

Bases:

Nullspaces:

$\mathbb{A}\vec{x} = \vec{0} \iff \mathbb{R}_{\mathbb{A}}\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\underset{P}{x_1} - 2\underset{F}{x_2} = 0$

$\Rightarrow x_1 = 2x_2$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

replace pivot variables

$x_2 \in \mathbb{R}$

$N(\mathbb{A}) =$

$$\left\{ \vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ x_2 \in \mathbb{R} \right\}$$

∞ly many points

A basis for $N(\mathbb{A})$ is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Also good

$\begin{bmatrix} 2\infty \\ \infty \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

unit vector

Left Nullspace:

Solve $\mathbb{A}^T\vec{y}=\vec{0}$

$\iff \mathbb{R}_{\mathbb{A}^T}\vec{y}=\vec{0}$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \underset{\overset{\uparrow}{P}}{y_1}+3\underset{\overset{\uparrow}{F}}{y_2}=0.$

$y_1=-3y_2$

$\vec{y}=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} -3y_2 \\ y_2 \end{bmatrix}=\underset{\overset{\uparrow}{\text{free}}}{y_2}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$N(\mathbb{A}^T)=\left\{\vec{y}=y_2\begin{bmatrix} -3 \\ 1 \end{bmatrix}, y_2\in\mathbb{R}\right\}$

A basis for $N(\mathbb{A}^T)$ is

$$\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}.$$

Column space $C(\mathbb{A})$.

① of 3 Solve $\mathbb{A}\vec{x}=\vec{b}$ for general $\vec{b}$.

$$\left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 3 & -6 & b_2 \end{array}\right]$$

$\sim$ $R_2'=R_2-\left(\frac{3}{1}\right)R_1$ $\left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 0 & 0 & b_2-3b_1 \end{array}\right]$

$0=b_2-3b_1$ must hold if $\vec{b}\in C(\mathbb{A})$.

$\Rightarrow b_2=3b_1$

$\vec{b}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}=\begin{bmatrix} b_1 \\ 3b_1 \end{bmatrix}=b_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$C(\mathbb{A})=\left\{\vec{y}=b_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}, b_1\in\mathbb{R}\right\}$

1-d

basis: $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

② of 3 find pivot columns through $\mathbb{R}_{\mathbb{A}}$

$\Rightarrow$ Same columns in $\mathbb{A}$ form a basis for $C(\mathbb{A})$

here 1st column: $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$

③ of 3. Take non-zero rows of $\mathbb{R}_{\mathbb{A}^T}=\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Again: $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

best way

Row space: take non-zero rows of $\mathbb{R}_{\mathbb{A}}=\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$

E14ap2

$$\mathbb{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$$

Bases $\overset{R^m}{\rightarrow} C(\mathbb{A}): \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ N(\mathbb{A}^T): \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$R^n \rightarrow C(\mathbb{A}^T): \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ N(\mathbb{A}): \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\mathbb{A}\vec{x}$: ① $\mathbb{A}$ sends any multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ to some multiple of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

② $\mathbb{A}$ sends any multiple of $\underset{N(\mathbb{A})}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

③ $\mathbb{A}$ cannot make any vector which has some non-zero part of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

~~Complementary~~

Orthogonality of Subspaces

$$C(\mathbb{A}) \oplus N(\mathbb{A}^T) = R^2 \leftarrow m=2$$

$$C(\mathbb{A}^T) \oplus N(\mathbb{A}) = R^2 \leftarrow n=2,$$

$$\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0.$$

how $\mathbb{A}$ functions:

$$\mathbb{A} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\uparrow$ in C(A)

length $= \sqrt{1^2 + (-2)^2}$
$= \sqrt{5}$.

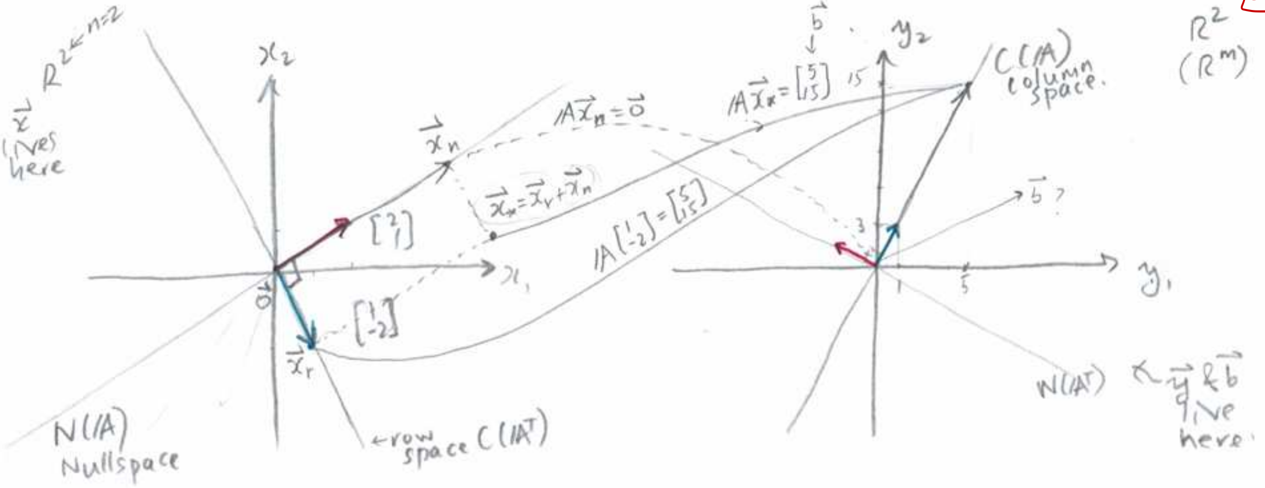
length
$5\sqrt{1+3^2}$
$= 5\sqrt{10}$

Stretch factor: $\dfrac{5\sqrt{10}}{\sqrt{5}} = \sqrt{5}\sqrt{10} = 5\sqrt{2}$.

So Westley is like $y = \underset{}{\overset{1\times 1 \leftarrow r=1}{5\sqrt{2}}}\, x$

but only for vectors in row & column space $\leftarrow$ 1-d subspace.

$\mathbb{A}$ is $\underset{r\times r}{{}^{\vee}}$invertible in these subspaces



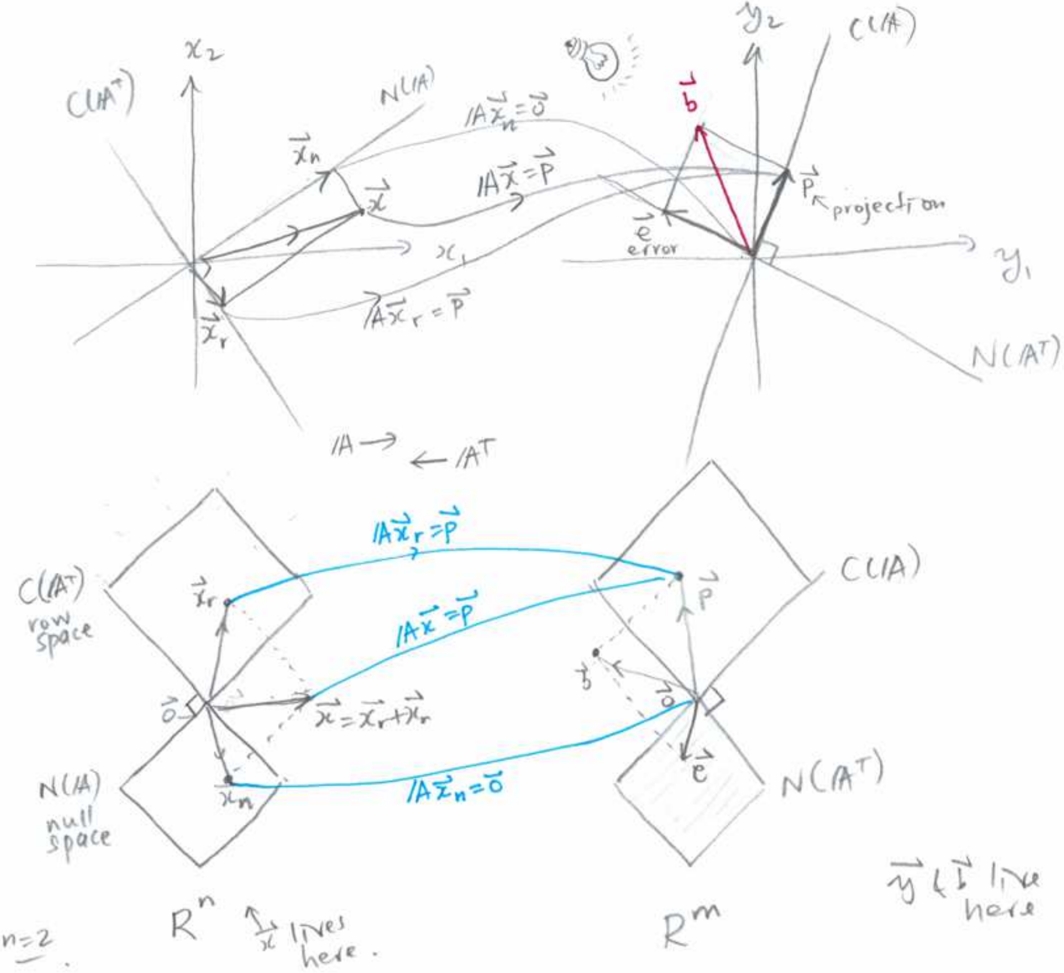
- $\underline{A}\vec{x} = \vec{b}$ is solvable if $\vec{b} \in C(A)$
- if $\vec{b} \in C(A)$, then there is one solutn if $N(A) = \{\vec{0}\}$ and ∞ly many otherwise. $\dim N(A) \geq 1$.

Ex// if $\vec{b} = 5\begin{bmatrix}1\\3\end{bmatrix}$ then $\vec{x} = \underset{C(A^T)}{\overset{\vec{x}_r}{\begin{bmatrix}1\\-2\end{bmatrix}}} + \underline{c}\underset{N(A)}{\overset{\vec{x}_n}{\begin{bmatrix}2\\1\end{bmatrix}}}$, $c \in R$.

Ex// if $\vec{b} = 5\begin{bmatrix}1\\3\end{bmatrix} + 4\underset{\in N(A^T)}{\begin{bmatrix}-3\\1\end{bmatrix}} = \begin{bmatrix}-7\\19\end{bmatrix}$, no solutn

#inconceivable

E14ap5



x_2
$C(\mathbb{A}^T)$
$N(\mathbb{A})$
$\vec{x}_n$
$\vec{x}$
$\mathbb{A}\vec{x}_n = \vec{0}$
$\mathbb{A}\vec{x} = \vec{p}$
x_1
$\mathbb{A}\vec{x}_r = \vec{p}$
$\vec{x}_r$
$\vec{b}$
y_2
$C(\mathbb{A})$
$\vec{p}$ ↖ projection
$\vec{e}$ error
y_1
$N(\mathbb{A}^T)$
$\mathbb{A}$ →
← $\mathbb{A}^T$
$\mathbb{A}\vec{x}_r = \vec{p}$
$C(\mathbb{A}^T)$ row space
$\vec{x}_r$
$\mathbb{A}\vec{x} = \vec{p}$
$\vec{0}$
$\vec{x} = \vec{x}_r + \vec{x}_n$
$N(\mathbb{A})$ null space
x_n
$\mathbb{A}\vec{x}_n = \vec{0}$
$C(\mathbb{A})$
$\vec{p}$
$\vec{y}$
$\vec{0}$
$\vec{e}$
$N(\mathbb{A}^T)$
R^n
$\vec{x}$ lives here.
n=2.
R^m
$\vec{y}$ & $\vec{b}$ live here

Inigo:

$$\underset{2\times3}{\mathbb{A}} = \begin{bmatrix}1&2&1\\2&4&2\end{bmatrix} \qquad m=2 \quad n=3 \qquad \underset{3\times2}{\mathbb{A}^T} = \begin{bmatrix}1&2\\2&4\\1&2\end{bmatrix}$$

$$\mathbb{R}_{\mathbb{A}} = \begin{array}{c}\begin{bmatrix}1&2&1\\0&0&0\end{bmatrix}\\ \begin{matrix}P&F&F\end{matrix}\end{array} \quad \underline{r=1} \qquad \mathbb{R}_{\mathbb{A}^T} = \begin{array}{c}\begin{bmatrix}1&2\\0&0\\0&0\end{bmatrix}\\ \begin{matrix}P&F\end{matrix}\end{array}$$

Bases

$C(\mathbb{A})$: $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

$C(\mathbb{A}^T)$: $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$

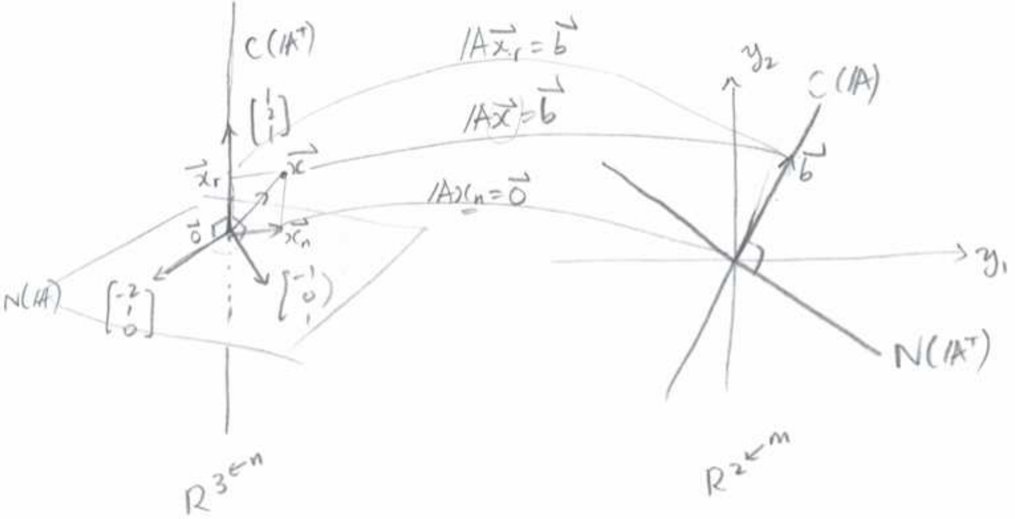
$m-r=2-1=1$

$N(\mathbb{A}^T)$: $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$.

$N(\mathbb{A})$: $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

↗ dim: $n-r$ $= 3-1 = 2$

not a beautiful basis #morelater.



row space ⟷ colspace.

Inigo is "$\underset{r\times r}{1\times 1}$" matrix, equivalent to $\sqrt{30}$.

E14cp1

Fezzik:

$$\underset{3\times4}{A} = \begin{bmatrix} 2 & 4 & 3 & 4 \\ 2 & 4 & 6 & 10 \\ 6 & 12 & 12 & 18 \end{bmatrix} \quad m=3 \quad n=4$$

$$\underset{4\times3}{A^T} = \begin{bmatrix} 2 & 2 & 6 \\ 4 & 4 & 12 \\ 3 & 6 & 12 \\ 4 & 10 & 18 \end{bmatrix}$$

$$R_A = \begin{array}{c}\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} P & F & P & F \end{matrix}\end{array} \quad r=2$$

$$R_{A^T} = \begin{array}{c}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} P & P & F \end{matrix}\end{array}$$

bases

$$C(A): \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

dim = r = 2

$$N(A^T): \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

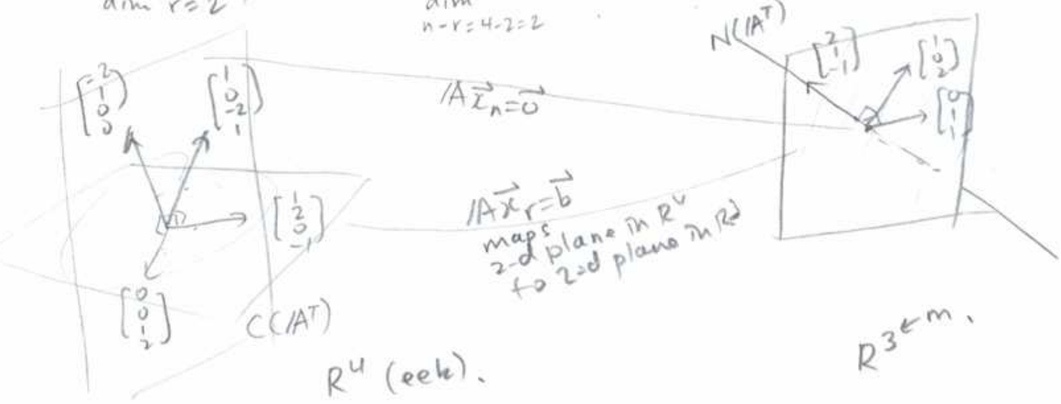
dim
m−r
= 3−2 = 1

$$C(A^T): \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

dim r = 2

$$N(A): \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

dim
n − r = 4 − 2 = 2



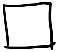
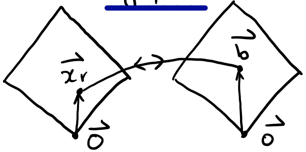
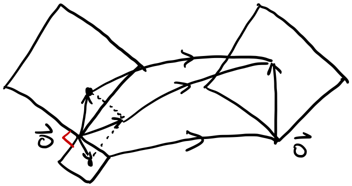
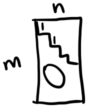
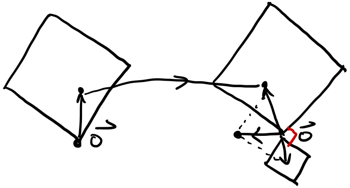
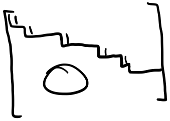
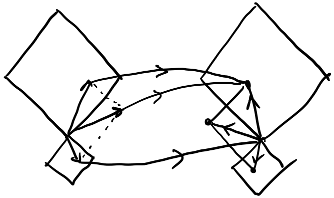
E14dp1

Everything hinges on $\mathbb{R}_{A}$ & $\mathbb{R}_{A^T}$ // Four main kinds of A.

m, n, r

how many solutions ↙ (dim $N(A)$)

whether solution is possible ↙ (dim $N(A^T)$)

Shape/rank:	big picture:	# solutions to $A\vec{x}=\vec{b}$:	dim $N(A)$:	dim $N(A^T)$:
full rank ↓ $m=n=r$ square  invertible		1	0	0
$m=r$ $n>r$  wide		∞	$\geqslant 1$	0
 $m>r$ $n=r$ tall		1 or 0 ↕ $b\in C(A)$	0	$\geqslant 1$
 $m, n > r$		∞ or 0	$\geqslant 1$	$\geqslant 1$

Projections:

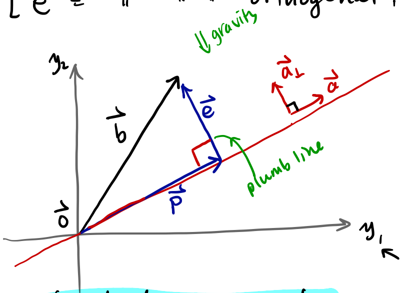
menu:
- Project a vector onto a line
- Notion of an error vector $\vec{e}$
- Goal: Handle $A\vec{x}=\vec{b}$ when no solutions are possible. Big idea: Best approximation

---

Idea: Given a vector $\vec{b}$ and a direction described by a vector $\vec{a}$, break $\vec{b}$ into two components $\vec{b}=\vec{p}+\vec{e}$:

[$\vec{p}$ = piece of $\vec{b}$ in direction of $\vec{a}$
$\vec{e}$ = " " " orthogonal to $\vec{a}$, in direction of $\vec{a}_\perp$

Picture:
(for R^2 but works in R^m)



[Big Deal: really projecting onto a subspace

↖ suspicious use of y's

[$\vec{p}$ = projected component
$\vec{e}$ = error

One reason for doing this:

In solving $A\vec{x}=\vec{b}$, if $\vec{b}\notin C(A)$, we can still solve $A\vec{x}_*=\vec{p}$ where $\vec{p}$ is $\vec{b}$'s projection onto Column Space.

- Best Approximation • Left Nullspace will matter!

How to find $\vec{p}$ & $\vec{e}$ given $\vec{b}$ & $\vec{a}$:

E15ap1

We want $\underbrace{\vec{p}\parallel\vec{a}}$ and $\underbrace{\vec{e}\perp\vec{a}}$

Mathematically: $\vec{p}=x_*\vec{a}$
↑ some number ∈ R

$\vec{e}^T\vec{a}=\vec{a}^T\vec{e}=0$
↖ inner (dot) product

Monks whisper: "Use the orthogonality..."

$$\vec{b}=\vec{p}+\vec{e}$$

Sneakiness:

$$\underset{\checkmark}{\vec{a}^T}(\underset{\checkmark}{\vec{b}}) = \underset{\checkmark}{\vec{a}^T}(\underset{?}{\vec{p}}+\underset{?}{\vec{e}})$$

$$\underbrace{\vec{a}^T\vec{b}}_{\text{number}} = \vec{a}^T\vec{p}+\cancel{\vec{a}^T\vec{e}}^{\,0}$$

$$= \vec{a}^T(x_*\vec{a})$$

$$= x_*\underbrace{\vec{a}^T\vec{a}}_{\text{number}}$$

$$\Rightarrow x_*=\frac{(\vec{a}^T\vec{b})}{(\vec{a}^T\vec{a})}$$

[$\vec{p}=x_*\vec{a}=\underbrace{\frac{(\vec{a}^T\vec{b})}{(\vec{a}^T\vec{a})}}_{\text{some scaling of }\vec{a}}\vec{a}$

$\vec{e}=\vec{b}-\vec{p}$

done.

E15ap2

Example:

project $\vec b = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ onto $\vec a = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{}$

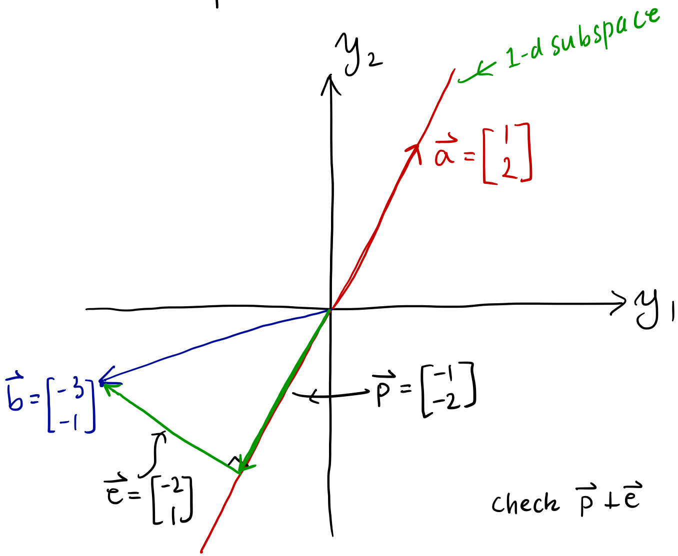
direction is all that matters

$$x_* = \frac{\vec a^T \vec b}{\vec a^T \vec a} = \frac{\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \frac{-5}{5} = -1$$

$$\Rightarrow \vec p = x_* \vec a = (-1)\vec a = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \vec e = \vec b - \vec p = \begin{bmatrix} -3 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

note: $\vec p \perp \vec e$ as required



check $\vec p \perp \vec e$

More Sneakiness:

We have
$$\vec p = \underbrace{\left(\frac{\vec a^T \vec b}{\vec a^T \vec a}\right)}_{\substack{1\times1 \\ \text{a number}}} \underset{m\times1}{\vec a}$$

not great to have to recalculate for each $\vec b$

really want a gadget matrix that projects $\vec b$ as an operator

$$= \frac{\overbrace{(\vec a^T \vec b)}^{1\times1}}{\underbrace{(\vec a^T \vec a)}_{1\times1}} \underset{m\times1}{\vec a} = \frac{1}{\underbrace{(\vec a^T \vec a)}_{1\times1}} \cdot \underbrace{(\vec a^T \vec b)}_{1\times1} \underset{m\times1}{\vec a}$$

$$= \frac{1}{\underbrace{(\vec a^T \vec a)}_{1\times1}} \underset{m\times1}{\vec a} \underbrace{(\overset{1\times m}{\vec a^T} \overset{m\times1}{\vec b})}_{1\times1} = \frac{1}{\underbrace{(\vec a^T \vec a)}_{1\times1}} \underbrace{(\overset{m\times1}{\vec a} \overset{1\times m}{\vec a^T})}_{\substack{m\times m \\ \text{square}}} \underset{m\times1}{\vec b}$$

outer product

$$= \frac{1}{\underbrace{\|\vec a\|^2}_{(\text{length of } \vec a)^2}} (\vec a \vec a^T) \vec b = \underbrace{\left(\frac{\overset{m\times1}{\vec a}}{\|\vec a\|} \frac{\overset{1\times m}{\vec a^T}}{\|\vec a\|}\right)}_{m\times m} \underset{m\times1}{\vec b}$$

good: length of a cannot matter

$$= \underbrace{\overset{m\times1}{\hat a} \overset{1\times m}{\hat a^T}}_{m\times m} \underbrace{\vec b}_{m\times1} = \mathbb{P}_{\hat a} \vec b$$

unit vector

outer product

Projection Operator

Example again:

project $\vec{b} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ onto $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

makes unit vector

$$\hat{a} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{1^2+2^2}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbb{P}_{\hat{a}} = \hat{a}\hat{a}^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

symmetry is guaranteed

So: $\vec{p} = \mathbb{P}_{\hat{a}}\vec{b} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -5 \\ -10 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ ✓

$\vec{e} = \vec{b} - \vec{p}$ as before

E15pa3

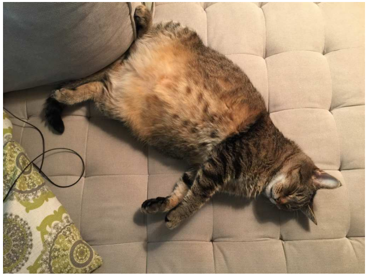
Bonus:

$\left(\overset{1\times 1}{1} - \overset{m\times m}{\mathbb{P}_{\hat{a}}}\right)$ wrong.

$$\vec{e} = \vec{b} - \vec{p} = \underset{m\times 1}{\vec{b}} - \underset{m\times m}{\mathbb{P}_{\hat{a}}}\underset{m\times 1}{\vec{b}}$$

$$= \underset{m\times m}{\mathbb{I}}\underset{m\times 1}{\vec{b}} - \underset{m\times m}{\mathbb{P}_{\hat{a}}}\underset{m\times 1}{\vec{b}} = \underbrace{\left(\underset{m\times m}{\mathbb{I}} - \underset{m\times m}{\mathbb{P}_{\hat{a}}}\right)}_{\text{Extracts } \vec{e} \text{ part of } \vec{a}.}\underset{m\times 1}{\vec{b}}$$

much happiness over $\mathbb{P}_{\hat{a}}$



The Amazing Normal Equation:

Menu:

- Find the best approximation to $A\vec{x} = \vec{b}$ when $\vec{b} \notin C(A)$

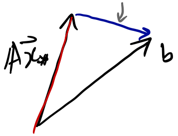
Before: We just gave up when $A\vec{x} = b$ had no solution ⇐ betrayed a lack of ticker

New plan: find $\vec{x}_*$ so that $A\vec{x}_*$ is as close to $\vec{b}$ as possible.

$_*$ denotes approximation

Mathematically: we want $\vec{x}^*$ that minimizes $\|\underbrace{\vec{b} - A\vec{x}_*}\|$

distance between $\vec{b}$ and $A\vec{x}_*$



Big idea: See $\vec{b} = \vec{p} + \vec{e}$ where

$$\left.\begin{array}{l}\vec{p} \in C(A) \\ \& \ \vec{e} \in N(A^T)\end{array}\right\} \vec{p} \perp \vec{e}$$

guarantees

We project $\vec{b}$ onto $C(A)$ and solve $\boxed{A\vec{x}_* = \vec{p}}$ instead

How?

Same approach as for simple projections:

We want

$$\underbrace{\vec{b} = \vec{p} + \vec{e}}_{①} \text{ where } \underbrace{A\vec{x}_* = \vec{p}}_{②} \ \& \ \underbrace{A^T\vec{e} = \vec{0}}_{③}$$

Start with $\underbrace{A^T\vec{e} = \vec{0}}_{③}$ ← Monks

$$\underset{③}{\vec{0}} = A^T\vec{e} \underset{①}{=} A^T(\vec{b} - \vec{p}) = A^T\vec{b} - A^T\vec{p}$$

$$\Rightarrow A^T\vec{b} = A^T\vec{p} \underset{②}{=} \overbrace{A^T(A\vec{x}_*)}^{\text{group.}}$$

$A\vec{x}_*$ means $\vec{p}$ is some linear combination of A's columns

Switch sides:

$$\underbrace{(\overbrace{\underbrace{A^T}_{n\times m}\underbrace{A}_{m\times n}}^{\text{square, symmetric = awesome}})}_{n\times n}\underbrace{\vec{x}_*}_{n\times 1} = \underbrace{\underbrace{A^T}_{n\times m}\underset{m\times 1}{\vec{b}}}_{n\times 1}$$

$\in R^n$ $\in R^n$

⇑ incredible! always solvable!

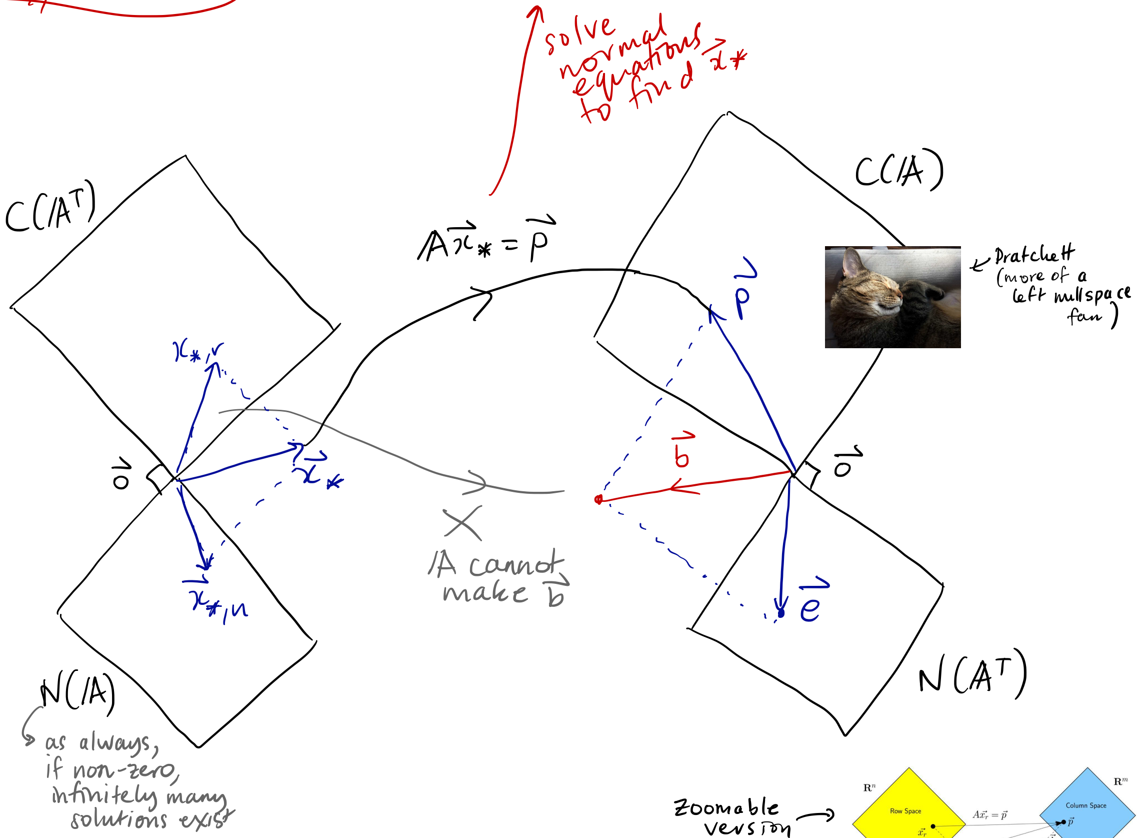
of the form:

$$\underset{n\times n}{A'}\ \underset{n\times 1}{\vec{x}_*} = \underset{n\times 1}{\vec{b}'}$$

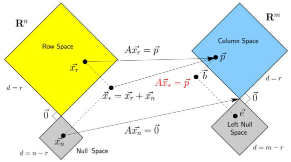
← prime

Abstract Big Picture

$$A^TA\vec{x}_* = A^T\vec{b}$$



Zoomable version



Example of using the Normal Equation

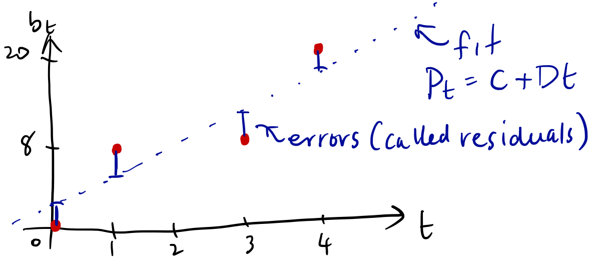
- fitting a straight line to a set of data points

← fundamental scientific activity!

Ex from Strang:

$$b_t = 0, 8, 8, 20$$

at times $t = 0, 1, 3, 4$



Want fit to be true:

$t=0 \qquad b_0 = 0 = 1C + D\times 0$

$t=1 \qquad b_1 = 8 = 1C + D\times 1$

$t=3 \qquad b_2 = 8 = 1C + D\times 3$

$t=4 \qquad b_3 = 20 = 1C + D\times 4$

E15cp1

Matrixify:

$\begin{bmatrix}0\\1\\9\\16\end{bmatrix}$ for $p_t = C + Dt + Et^2$

$$\underset{\checkmark A\ 4\times 2}{\begin{bmatrix}1 & 0\\1 & 1\\1 & 3\\1 & 4\end{bmatrix}} \underset{\vec{x}\ 2\times 1}{\begin{bmatrix}C\\D\\E\end{bmatrix}} = \underset{\checkmark \vec{b}\ 4\times 1}{\begin{bmatrix}0\\8\\8\\20\end{bmatrix}}$$

?

Clear $\vec{b}$ is not in A's Column Space

Solve $A^TA\vec{x}_* = A^T\vec{b}$:

$$\underset{A^T}{\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & 4\end{bmatrix}} \underset{A}{\begin{bmatrix}1 & 0\\1 & 1\\1 & 3\\1 & 4\end{bmatrix}} \underset{\vec{x}_*}{\begin{bmatrix}C\\D\end{bmatrix}} = \underset{A^T}{\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & 4\end{bmatrix}} \underset{\vec{b}}{\begin{bmatrix}0\\8\\8\\20\end{bmatrix}}$$

always 2×2

$$\Rightarrow \begin{bmatrix}4 & 8\\8 & 26\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}36\\112\end{bmatrix}$$

line fit for 10^6 (ex) data pts ⇒ still 2×2 problem

$[A^TA \mid A^T\vec{b}]$

$$R_2' = R_2 - \left(\frac{8}{4}\right)R_1 \qquad \left[\begin{array}{cc|c}4 & 8 & 36\\0 & 10 & 40\end{array}\right]$$

U

Back substitute: $4C_* + 8D_* = 36 \quad \leftarrow C_* = 1$

$10D_* = 40 \quad \leftarrow D_* = 4$

$$\vec{x}_* = \begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix} \qquad \text{Best fit line} = P_t = 1 + 4t$$

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix}0\\8\\8\\20\end{bmatrix} - \begin{bmatrix}1\\5\\13\\17\end{bmatrix} = \begin{bmatrix}-1\\3\\-5\\3\end{bmatrix}; \quad \|\vec{e}\|^2 = 1^2 + 3^2 + 5^2 + 3^2 = 44$$

The Normal Equation and the Man in Black

E15d p1

Westley, our hero: $A = \begin{bmatrix}1 & -2\\ 3 & -6\end{bmatrix}$

Solve $\begin{bmatrix}1 & -2\\ 3 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}5\\ 5\end{bmatrix} \leftarrow \vec{b} \notin C(A)$

$$\left[\begin{array}{cc|c}1 & -2 & 5\\ 3 & -6 & 5\end{array}\right] \underset{R_2' = R_2 - \left(\frac{3}{1}\right)R_1}{\sim} \left[\begin{array}{cc|c}1 & -2 & 5\\ 0 & 0 & -10\end{array}\right] \leftarrow \text{eek: } 0 = -10$$

inconceivable! (or: no solution)

Time for the Normal Equation: $A^T A \vec{x}_* = A^T \vec{b}$

$$\underbrace{\underset{n\times m}{A^T}\,\underset{m\times n}{A}}_{n\times n} = \begin{bmatrix}1 & 3\\ -2 & -6\end{bmatrix}\begin{bmatrix}1 & -2\\ 3 & -6\end{bmatrix} = \begin{bmatrix}10 & -20\\ -20 & 40\end{bmatrix}$$

A^TA is always symmetric

$$\underbrace{\underset{n\times m}{A^T}\,\underset{m\times 1}{\vec{b}}}_{n\times 1} = \begin{bmatrix}1 & 3\\ -2 & -6\end{bmatrix}\begin{bmatrix}5\\ 5\end{bmatrix} = \begin{bmatrix}20\\ -40\end{bmatrix}$$

Now Solve

$$\left[\begin{array}{cc|c}10 & -20 & +20\\ -20 & 40 & -40\end{array}\right] \underset{R_2' = R_2 - \left(\frac{-20}{10}\right)R_1}{\sim} \left[\begin{array}{cc|c}10 & -20 & +20\\ 0 & 0 & 0\end{array}\right]$$

all good as promised

$$\underset{R_1' = \frac{1}{10}R_1}{\sim} \left[\begin{array}{cc|c}1 & -2 & +2\\ 0 & 0 & 0\\ P & F & \end{array}\right]$$

$$\Rightarrow \underset{P}{x_{*,1}} - 2\underset{F}{x_{*,2}} = +2$$

$$\Rightarrow \underset{P}{x_{*,1}} = +2 + 2\underset{F}{x_{*,2}}$$

replace pivot vars in terms of free vars

$$\vec{x}_* = \begin{bmatrix}\overset{P}{x_{*,1}}\\ \underset{F}{x_{*,2}}\end{bmatrix} = \begin{bmatrix}+2+2x_{*,2}\\ 0+x_{*,2}\end{bmatrix} = \overset{\vec{x}_{*,p}}{\begin{bmatrix}+2\\ 0\end{bmatrix}} + \overset{\vec{x}_{*,h}}{x_{*,2}\begin{bmatrix}-2\\ 1\end{bmatrix}}$$

particular

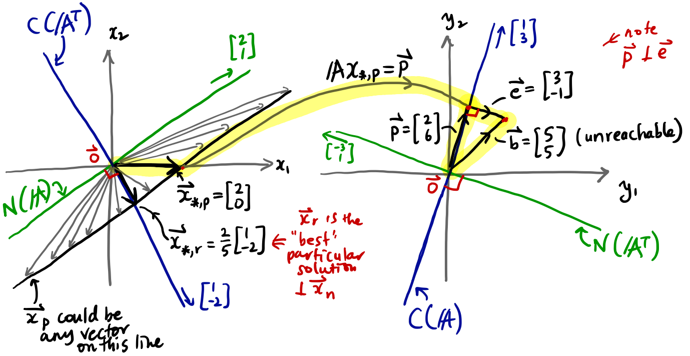
homogeneous

where $x_{*,2} \in R$

$$\vec{p} = A\vec{x}_* = \underbrace{\begin{bmatrix}1 & -2\\ 3 & -6\end{bmatrix}\begin{bmatrix}+2\\ 0\end{bmatrix}} + x_{*,2}\cancel{\begin{bmatrix}1 & -2\\ 3 & -6\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}}\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$= \begin{bmatrix}2\\ 6\end{bmatrix}$$

particular solution does the work



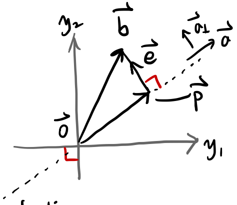
- Note $\vec{x}_p \neq \vec{x}_r$ in general; $\vec{x}_r$ is the special $\vec{x}_p$ that lives in A's row space
- How $\vec{b}$ was built:

$$\vec{b} = \underbrace{2\begin{bmatrix}1\\ 3\end{bmatrix}}_{\vec{p}} \underset{+}{-} \underbrace{1\begin{bmatrix}-3\\ 1\end{bmatrix}}_{\vec{e}} = \begin{bmatrix}5\\ 5\end{bmatrix}$$

Projecting a vector $\vec{b} \in R^m$ onto a subspace of R^m

#excitement

We know how to project a vector $\vec{b}$ onto a line defined by a vector $\vec{a}$:

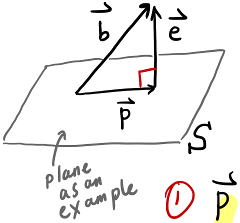


$$\vec{p} = \underbrace{\underset{m\times 1}{\hat{a}}\ \underset{1\times m}{\hat{a}^T}}_{m\times m}\vec{b} = \mathbb{P}_{\hat{a}}\vec{b}$$

outer product of unit vectors

Projection matrix/operator

Now:
Generalize to an r-dim subspace of R^m



We have ~~a~~ some basis for subspace S:

$\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_r\}$

$\vec{p} \subset C(\mathbb{A})$

① $\underset{m\times 1}{\vec{p}} = \underset{m\times r}{\mathbb{A}}\ \underset{r\times 1}{\vec{x}_*} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_r \\ | & | & & | \end{bmatrix}\vec{x}_*$

linearly independent because $\vec{a}_i$ form a basis

② $\mathbb{A}^T\vec{e} = \vec{0}$ $\leftarrow \vec{e} \in N(\mathbb{A}^T)$

③ $\vec{b} = \vec{p} + \vec{e}$

Monks:

$$\vec{0} \overset{②}{=} \mathbb{A}^T\vec{e} \overset{③}{=} \mathbb{A}^T(\vec{b} - \vec{p}) = \mathbb{A}^T\vec{b} - \mathbb{A}^T\vec{p}$$

$\vec{p} = \mathbb{A}\vec{x}_*$ ①

$\Rightarrow$
$$\vec{0} = \mathbb{A}^T\vec{b} - \mathbb{A}^T\mathbb{A}\vec{x}_*$$

$$\Rightarrow \mathbb{A}^T\mathbb{A}\vec{x}_* = \mathbb{A}^T\vec{b}$$

Solve for $\vec{x}_*$ then find $\vec{p}$ as $\vec{p} = \mathbb{A}\vec{x}_*$

E15ep1

Special deal: extra tofu knives

b/c $\mathbb{A}$'s columns are linearly independent, $\mathbb{A}^T\mathbb{A}$ is invertible

reason to follow

$$\Rightarrow \underbrace{(\mathbb{A}^T\mathbb{A})^{-1}(\mathbb{A}^T\mathbb{A})}_{\mathbb{I}}\vec{x}_* = (\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T\vec{b}$$

premultiply both sides by inverse

$$\Rightarrow \vec{x}_* = (\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T\vec{b}$$

$$\Rightarrow \vec{p} = \mathbb{A}\vec{x}_* = \underset{m\times r}{\mathbb{A}}\underbrace{(\underset{r\times m}{\mathbb{A}^T}\underset{m\times r}{\mathbb{A}})^{-1}}_{r\times r}\underbrace{\mathbb{A}^T}_{r\times m}\underbrace{\vec{b}}_{m\times 1}$$

$$\equiv \underset{m\times m}{\mathbb{P}}\ \underset{m\times 1}{\vec{b}}$$

Projection Matrix (good for low dimensions)

More goodness: expect $\mathbb{P}^2\vec{b} = \mathbb{P}^3\vec{b} = \cdots = \vec{p}$

or

$\mathbb{P} = \mathbb{P}^n$ for all $n \geq 1$

Check: $\mathbb{P}^2 =$

$$\mathbb{A}\underbrace{(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T\mathbb{A}}_{\mathbb{I}}(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T$$

$$= \mathbb{A}(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T = \mathbb{P}$$

cool! (right?)

LE15ep2

Our Projection Matrix when we have a basis for $\mathbb{S}$:

$$\mathbb{P} = \mathbb{A}(\mathbb{A}^T\mathbb{A})^{-1}\mathbb{A}^T; \quad \mathbb{P}\vec{b} = \vec{p} \in \mathbb{S}$$

Warning!

$$\underbrace{(\mathbb{A}^T\mathbb{A})^{-1}}_{\text{A may be rectangular!!!}} \underset{\text{may be equal sometimes}}{\neq} \mathbb{A}^{-1}(\mathbb{A}^T)^{-1} \text{ generally}$$

$\overbrace{\overset{r\times m}{\mathbb{A}^T}\overset{m\times r}{\mathbb{A}}}^{}$ $\underset{r\times r}{}$ is always square and symmetric

Important Truth:

$\mathbb{A}^T\mathbb{A}$ is invertible iff "if and only if"
$\mathbb{A}$'s columns are linearly independent

$N(\mathbb{A}^T\mathbb{A}) = \{\vec{0}\}$, $\overset{r\times r}{\mathbb{A}^T\mathbb{A}}$ is full rank r.

$\iff$ $\mathbb{A}\vec{x} = \vec{0}$ only has $\vec{x} = \vec{0}$ as a solution

$\iff$ $N(\mathbb{A}) = \{\vec{0}\}$

Plan: Show $\mathbb{A}^T\mathbb{A}$ & $\mathbb{A}$ have the same Nullspace always

Need to show that if $\vec{x} \in N(\mathbb{A})$ then $\vec{x} \in N(\mathbb{A}^T\mathbb{A})$ and vice versa

Assume $\vec{x} \in N(\mathbb{A})$: $\mathbb{A}\vec{x} = \vec{0}$

$\Rightarrow \mathbb{A}^T(\mathbb{A}\vec{x}) = \mathbb{A}^T(\vec{0})$

$\Rightarrow (\mathbb{A}^T\mathbb{A})\vec{x} = \vec{0}$

So $\vec{x} \in N(\mathbb{A}^T\mathbb{A})$

Second, if $\vec{x} \in N(\mathbb{A}^T\mathbb{A})$ then

$\mathbb{A}^T\mathbb{A}\vec{x} = \vec{0}$ ↙ by definition

$\Rightarrow \overset{1\times r}{\vec{x}^T}(\mathbb{A}^T\mathbb{A}\vec{x}) = \vec{x}^T\overset{r\times 1}{(\vec{0})}$

$\Rightarrow \vec{x}^T\mathbb{A}^T\mathbb{A}\vec{x} = \underset{1\times 1}{0}$

$\Rightarrow (\mathbb{A}\vec{x})^T(\mathbb{A}\vec{x}) = 0$

use $(\mathbb{B}\mathbb{C})^T = \mathbb{C}^T\mathbb{B}^T$

$\Rightarrow \|\underbrace{\overset{m\times r}{\mathbb{A}}\overset{r\times 1}{\vec{x}}}_{m\times 1 \text{ vector}}\|^2 = 0$

if length $= 0$

$\Rightarrow \mathbb{A}\vec{x} = \vec{0}$

So $\vec{x} \in N(\mathbb{A})$ // done

Note we can do the same sort of thing for $\mathbb{A}\mathbb{A}^T$

Upshot: if $N(\mathbb{A}) = \{\vec{0}\}$ then $\mathbb{A}^T\mathbb{A}$ is invertible

$\mathbb{A}$ need not be square

square r×r matrix with rank r
r

Orthogonal and Orthonormal bases help us win friends and influence people

Menu:
- Motivation for orthogonality
- orthogonal Matrices

Next:
- the Gram-Schmidt Process
- What this all means for $A\vec{x}=\vec{b}$

Observation: We've been finding bases for our four fundamental subspaces, and we've so far taken whatever popped out.

ex: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ ← Basis for Fezzik's $C(A)$

↓

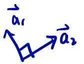
Describes 2-d subspace in R^3

Does the job BUT we really like orthogonality in our bases and $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 2 \neq 0$

not orthogonal

ex1 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ = basis for a plane in R^3

$\vec{a}_1 \quad \vec{a}_2$



$\vec{a}_1^T\vec{a}_2 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = -2+0+2 = 0$ ✓

We call such a basis **Orthogonal**

E16ap1

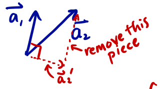
ex2. (from Monks) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \right\}$

$\vec{a}_1 \quad \vec{a}_2$

$\vec{a}_1^T\vec{a}_2 = 1+3+14 = 18 \neq 0$

sneakiness

$\vec{a}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$



← $\vec{a}_2$ contains some of $\vec{a}_1$

Big idea: Systematically turn a basis into an orthogonal basis by removing non-orthogonal pieces

Everything is connected: Projections will do the work for us.

Claim: Orthogonality makes a basis a happy basis

math

- Main reason: Representation of vectors is very clean.

Information contained in each basis vector is distinct.

- Later: We will see we get orthogonal bases for free when working with a certain kind of Totally Awesome Matrices

- Bonus: A set of orthogonal vectors is automatically linearly independent and therefore must form a basis for the subspace they span

"Obvious" but proof is nutritious
↖ dangerous word

Given $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$ with $\vec{a}_i^T \vec{a}_j = 0$ for all $1 \le i, j \le n$, $i \ne j$ & $\vec{a}_i \in R^m$

$$\underset{m\times n}{\mathbb{A}} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

Monks: Presume linear dependence

$\iff$ $\mathbb{A}\vec{x} = \vec{0}$ has a non-zero solution $\vec{x}$

$\iff$ $N(\mathbb{A}) \ne \{\vec{0}\}$

Must have

$$0 = \underset{1\times m}{\vec{0}^T} \underset{m\times 1}{\vec{0}} = (\mathbb{A}\vec{x})^T(\mathbb{A}\vec{x})$$

use $(\mathbb{B}\mathbb{C})^T = \mathbb{C}^T\mathbb{B}^T$

$$= \vec{x}^T \mathbb{A}^T \mathbb{A} \vec{x}$$

$$= \underset{\vec{x}^T}{[x_1 \cdots x_n]} \begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ \vdots & & \vdots \\ - & \vec{a}_n^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$$= [x_1 \ \cdots \ x_n] \begin{bmatrix} \|\vec{a}_1\|^2 & & \bigcirc \\ & \|\vec{a}_2\|^2 & \\ \bigcirc & & \ddots \ \|\vec{a}_n\|^2 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

↙ all zeros

diagonal matrix

$$= x_1^2 \underbrace{\|\vec{a}_1\|^2}_{\ne 0} + x_2^2 \underbrace{\|\vec{a}_2\|^2}_{\ne 0} + \cdots + x_n^2 \underbrace{\|\vec{a}_n\|^2}_{\ne 0}$$

$$= 0 \text{ only if } x_1 = x_2 = \cdots = x_n = 0$$

Contradiction

$\Rightarrow N(\mathbb{A}) = \{\vec{0}\}$

E16ap2

Extra happy kind of basis:

An Orthonormal Basis

$\equiv$ Orthogonal Basis made up of unit vectors

Observation: Easy to do!

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

orthogonal basis

divide by lengths (easy)

orthonormal basis

Next: How to create an orthogonal basis in the first place

↙ hard part

Transmuting a basis into an orthogonal one

Menu
- The Gram-Schmidt Process

Idea: Turn $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ into $\{\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n\}$

basis ⊂ R^m

orthogonal basis

and then $\{\hat{q}_1, \hat{q}_2, \dots, \hat{q}_n\}$ → $\hat{q}_i = \frac{1}{\|\vec{q}_i\|}\vec{q}_i$

orthonormal basis

by incrementally removing parts of vectors.

n=3 general formula:

3d subspace in R^m

① Set $\vec{q}_1 = \vec{a}_1$

② $\vec{q}_2 = \vec{a}_2 - \frac{\vec{q}_1^T \vec{a}_2}{\vec{q}_1^T \vec{q}_1}\vec{q}_1$ ← projection of $\vec{a}_2$ onto direction described by $\vec{q}_1$

③ $\vec{q}_3 = \vec{a}_3 - \frac{(\vec{q}_1^T \vec{a}_3)}{(\vec{q}_1^T \vec{q}_1)}\vec{q}_1 - \frac{(\vec{q}_2^T a_3)}{(\vec{q}_2^T \vec{q}_2)}\vec{q}_2$

projections

ⓝ $\vec{q}_n = \vec{a}_n - (\dots \quad \dots)$

n-1 projections of $\vec{a}_n$ onto $\vec{q}_1, \vec{q}_2, \dots, \vec{q}_{n-1}$

We know $\underbrace{\frac{(\vec{q}_1^T \vec{a}_2)}{(\vec{q}_1^T \vec{q}_1)}}_{\text{number}}\vec{q}_1 = \overset{P_{\hat{q}_1}}{\underbrace{\underset{m\times 1}{\hat{q}_1}\,\underset{1\times m}{\hat{q}_1^T}}_{m\times m}}\,\underset{m\times 1}{\vec{a}_2}$

number

outer product

So above 3 steps can be rewritten as:

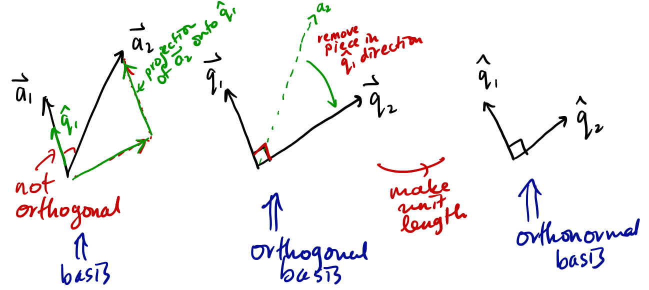
① $\vec{q}_1 = \vec{a}_1 \Rightarrow \hat{q}_1 = \frac{1}{\|\vec{q}_1\|}\vec{q}_1$

② $\vec{q}_2 = \vec{a}_2 - \hat{q}_1 \hat{q}_1^T \vec{a}_2 \Rightarrow \hat{q}_2 = \frac{1}{\|\vec{q}_2\|}\vec{q}_2$

③ $\vec{q}_3 = \vec{a}_3 - \hat{q}_1 \hat{q}_1^T \vec{a}_3 - \hat{q}_2 \hat{q}_2^T a_3 \Rightarrow \hat{q}_3 = \frac{1}{\|\hat{q}_3\|}\hat{q}_3$

good for theory

What's happening:



Example calculation:

$$\left\{ \underset{\vec a_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\vec a_2}{\begin{bmatrix}4\\2\\0\end{bmatrix}}, \underset{\vec a_3}{\begin{bmatrix}5\\-4\\-7\end{bmatrix}} \right\}$$

① $\vec q_1 = \vec a_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

orthogonal

② $\vec q_2 = \vec a_2 - \dfrac{\vec q_1^{\,T}\vec a_2}{\vec q_1^{\,T}\vec q_1}\vec q_1$

$$= \begin{bmatrix}4\\2\\0\end{bmatrix} - \frac{\overset{6}{[1\ 1\ 1]\begin{bmatrix}4\\2\\0\end{bmatrix}}}{\underset{3}{[1\ 1\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix}}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\2\\0\end{bmatrix} - \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix}$$

③ $\vec q_3 = \vec a_3 - \dfrac{\vec q_1^{\,T}\vec a_3}{\vec q_1^{\,T}\vec q_1}\vec q_1 - \dfrac{\vec q_2^{\,T}\vec a_3}{\vec q_2^{\,T}\vec q_2}\vec q_2$

$$= \begin{bmatrix}5\\-4\\-7\end{bmatrix} - \frac{\overset{-6}{\cancel{[1\ 1\ 1]\begin{bmatrix}5\\-4\\-7\end{bmatrix}}}}{\underset{+3}{\cancel{[1\ 1\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix}}}}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{\overset{24}{\cancel{[2\ 0\ -2]\begin{bmatrix}5\\-4\\-7\end{bmatrix}}}}{\underset{8}{\cancel{[2\ 0\ -2]\begin{bmatrix}2\\0\\-2\end{bmatrix}}}}\begin{bmatrix}2\\0\\-2\end{bmatrix}$$

$$= \begin{bmatrix}5\\-4\\-7\end{bmatrix} - \overset{+2}{\frac{\cancel{-6}}{\cancel{3}}}\begin{bmatrix}1\\1\\1\end{bmatrix} - \overset{-3}{\frac{\cancel{24}}{\cancel{8}}}\begin{bmatrix}2\\0\\-2\end{bmatrix} = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

E166p2

Normalize:

$$\hat q_1 = \frac{1}{\sqrt3}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \hat q_2 = \frac{1}{\underset{2\sqrt2}{\sqrt8}}\begin{bmatrix}2\\0\\-2\end{bmatrix}, \quad \hat q_3 = \frac{1}{\sqrt6}\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

$$\hat q_2 = \frac{1}{\sqrt2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Note: Gram-Schmidt method tends to produce many square roots

Check $\hat q_1^{\,T}\hat q_2 = \hat q_2^{\,T}\hat q_3 = \hat q_3^{\,T}\hat q_1 = 0$

- test every pair of basis vectors
- must be orthogonal

Next: See $\vec a_i$'s can be rebuilt from $\hat q_i$'s $\Rightarrow \underset{m\times n}{A} = \underset{m\times n}{Q}\,\underset{n\times n}{R}$

E16cp1

A new factorization: $\underset{m\times n}{\mathbb{A}} = \underset{m\times n}{\mathbb{Q}}\,\underset{n\times n}{\mathbb{R}}$

Q's shape matches $\mathbb{A}$'s

unfortunate namespace overlap → NOT $\mathbb{R}/\mathbb{A}$

Idea: We love to use matrices to encode our methods

empowering

ex $\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U} \equiv$ Gaussian Elimination

So: let's turn the Gram-Schmidt Process into a factorization of $\mathbb{A}$

From a few pages back:

①	$\vec{q}_1 = \vec{a}_1$	$\Rightarrow \hat{q}_1 = \frac{1}{\lVert\vec{q}_1\rVert}\vec{q}_1$
②	$\vec{q}_2 = \vec{a}_2 - \overbrace{\hat{q}_1\hat{q}_1^T}^{m\times m\ \mathbb{P}_{\hat{q}_1}}\vec{a}_2$	$\Rightarrow \hat{q}_2 = \frac{1}{\lVert\vec{q}_2\rVert}\vec{q}_2$
③	$\vec{q}_3 = \vec{a}_3 - \overbrace{\hat{q}_1\hat{q}_1^T}^{m\times m}\vec{a}_3 - \overbrace{\hat{q}_2\hat{q}_2^T}^{m\times m}a_3$	$\Rightarrow \hat{q}_3 = \frac{1}{\lVert\vec{q}_3\rVert}\hat{q}_3$

Monks say: Express the $\vec{a}_i$ in terms of the $\hat{q}_i$ using a column picture approach

$\Rightarrow$ Connect $\mathbb{A}$ to $\mathbb{Q} = \begin{bmatrix} | & | & & | \\ \hat{q}_1 & \hat{q}_2 & \cdots & \hat{q}_n \\ | & | & & | \end{bmatrix}$

Rearrange above so $\vec{a}_i = \cdots$:

put $\vec{a}_i$'s on left by themselves

① $\vec{a}_1 = \vec{q}_1$

② $\vec{a}_2 = \vec{q}_2 + \hat{q}_1\hat{q}_1^T\vec{a}_2$

③ $\vec{a}_3 = \vec{q}_3 + \hat{q}_1\hat{q}_1^T\vec{a}_3 + \hat{q}_2\hat{q}_2^T\vec{a}_3$

need these to look the same

Sneakiness: See $\vec{q}_i$ as projection of $\vec{a}_i$ onto $\hat{q}_i$ direction

ex ③ above:

$$\underbrace{(\hat{q}_3\hat{q}_3^T)}_{\mathbb{P}_{\hat{q}_3}}\vec{a}_3 = (\hat{q}_3\hat{q}_3^T)(\vec{q}_3 + \overset{\perp}{\hat{q}_1\hat{q}_1^T\vec{a}_3} + \overset{\perp}{\hat{q}_2\hat{q}_2^T\vec{a}_3})$$

$$= \hat{q}_3(\underbrace{\hat{q}_3^T\vec{q}_3}_{\lVert\vec{q}_3\rVert}) + 0 + 0$$

$$= \vec{q}_3$$

① $\vec{a}_1 = \hat{q}_1\hat{q}_1^T\vec{a}_1$

② $\vec{a}_2 = \hat{q}_2\hat{q}_2^T\vec{a}_2 + \hat{q}_1\hat{q}_1^T\vec{a}_2$

③ $\vec{a}_3 = \hat{q}_3\hat{q}_3^T\vec{a}_3 + \hat{q}_1\hat{q}_1^T\vec{a}_3 + \hat{q}_2\hat{q}_2^T\vec{a}_3$

Makes sense
$\vec{a}_3$ has components in $\hat{q}_1$, $\hat{q}_2$, & $\hat{q}_3$ directions

Reorder:

numbers (inner products)

① $$\vec{a}_1 = \underset{m\times 1}{\hat{q}_1}\,\underset{1\times m}{\hat{q}_1^T}\,\underset{m\times 1}{\vec{a}_1}$$

② $$\vec{a}_2 = \hat{q}_1 \hat{q}_1^T \vec{a}_2 + \hat{q}_2 \hat{q}_2^T \vec{a}_2$$

③ $$\vec{a}_3 = \hat{q}_1 \hat{q}_1^T \vec{a}_3 + \hat{q}_2 \hat{q}_2^T \vec{a}_3 + \hat{q}_3 \hat{q}_3^T \vec{a}_3$$

Column picture:

① $$\vec{a}_1 = \begin{bmatrix} | & | & | \\ \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \hat{q}_1^T \vec{a}_1 \\ 0 \\ 0 \end{bmatrix}$$

② $$\vec{a}_2 = \begin{bmatrix} | & | & | \\ \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \hat{q}_1^T \vec{a}_2 \\ \hat{q}_2^T \vec{a}_2 \\ 0 \end{bmatrix}$$

③ $$\vec{a}_3 = \begin{bmatrix} | & | & | \\ \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \hat{q}_1 \vec{a}_3 \\ \hat{q}_2 \vec{a}_3 \\ \hat{q}_3 \vec{a}_3 \end{bmatrix}$$

TRIUMPHANCY:

Q ↗ A cleaned up

R ↑ upper triangular "combining" matrix

$$A = \underset{m\times n}{\begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}} = \underset{m\times n}{\begin{bmatrix} | & | & | \\ \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ | & | & | \end{bmatrix}} \underset{n\times n}{\begin{bmatrix} \hat{q}_1^T \vec{a}_1 & \hat{q}_1^T \vec{a}_2 & \hat{q}_1^T \vec{a}_3 \\ 0 & \hat{q}_2^T \vec{a}_2 & \hat{q}_2^T \vec{a}_3 \\ 0 & 0 & \hat{q}_3^T \vec{a}_3 \end{bmatrix}}$$

E16cp2

- $A = QR$ will help with $A\vec{x} = \vec{b}$ (next)
- Delicious way to find R:

$$\underset{\substack{\uparrow \\ \text{premultiply} \\ \text{by } Q^T}}{Q^T} A = \underbrace{Q^T Q}_{\substack{I \leftarrow \text{because} \\ \text{Q's columns} \\ \text{are unit vectors}}} R \Rightarrow R = Q^T A$$

Return to example:

$$\left\{ \underset{\vec{a}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{a}_2}{\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}}, \underset{\vec{a}_3}{\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}} \right\} \Rightarrow \left\{ \underset{\hat{q}_1}{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\hat{q}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}, \underset{\hat{q}_3}{\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} \right\}$$

$$\underset{A}{\begin{bmatrix} 1 & 4 & 5 \\ 1 & 2 & -4 \\ 1 & 0 & -7 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{3} & 2\sqrt{3} & -2\sqrt{3} \\ 0 & 2\sqrt{2} & 6\sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix}}$$

check $A = QR$ works

check $R = Q^T A$

Find R by either computing inner products $\hat{q}_i^T \vec{a}_j,\ i \leq j$

or $R = Q^T A$ ↳ do this!

$$\underset{m\times n}{\mathbb{A}}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}} \quad \& \quad \underset{m\times n}{\mathbb{A}} = \underset{m\times n}{\mathbb{Q}}\,\underset{n\times n}{\mathbb{R}}:$$

← presume $r=n$

Solve the normal equation using $\mathbb{A}=\mathbb{Q}\mathbb{R}$:

$\vec{b}\notin C(\mathbb{A})$

$$\boxed{\mathbb{A}^T\mathbb{A}\vec{x}_* = \mathbb{A}^T\vec{b}}$$

$\mathbb{R}$ is upper triangular

$$\Rightarrow (\mathbb{Q}\mathbb{R})^T(\mathbb{Q}\mathbb{R})\vec{x}_* = (\mathbb{Q}\mathbb{R})^T\vec{b}$$

$$\Rightarrow \underset{n\times n}{\mathbb{R}^T}\,\underbrace{\overset{n\times m}{\mathbb{Q}^T}\,\overset{m\times n}{\mathbb{Q}}}_{\underset{n\times n}{\mathbb{I}}}\,\underset{n\times n}{\mathbb{R}}\,\underbrace{\vec{x}_*}_{n\times 1} = \underbrace{\underset{n\times n}{\mathbb{R}^T}\,\underset{n\times m}{\mathbb{Q}^T}\,\underset{m\times 1}{\vec{b}}}_{n\times 1}$$

$$\Rightarrow \underset{(\mathbb{R}^T)^{-1}\times}{\mathbb{R}^T}\,\mathbb{R}\,\vec{x}_* = \underset{(\mathbb{R}^T)^{-1}}{\mathbb{R}^T}\,\mathbb{Q}^T\vec{b}$$

$$\Rightarrow \mathbb{R}\vec{x}_* = \mathbb{Q}^T\vec{b} \quad \text{because } (\mathbb{R}^T)^{-1} \text{ exists}$$

square full rank all n pivots exist

$$\boxed{\underset{n\times n}{\mathbb{R}}\,\underset{n\times 1}{\vec{x}_*} = \underbrace{\mathbb{Q}^T}_{n\times m}\,\underset{m\times 1}{\vec{b}}}$$

$$\mathbb{A}'\vec{x}_* = \underset{n\times 1}{\vec{b}'}$$

← upper triangular system! Easy to solve!

c.f. $\mathbb{A}=\mathbb{L}\mathbb{U}$

But

$$\mathbb{A}\vec{x}=\vec{b}$$

$$\Rightarrow (\mathbb{Q}\mathbb{R})\vec{x}=\vec{b}$$

$$\Rightarrow \mathbb{Q}^T\mathbb{Q}\mathbb{R}\vec{x} = \mathbb{Q}^T\vec{b}$$

← premultiply both sides

↑ left inverse of $\mathbb{Q}$

$$\Rightarrow \underset{n\times n}{\mathbb{I}}\,\mathbb{R}\vec{x} = \mathbb{Q}^T\vec{b}$$

$$\Rightarrow \mathbb{R}\vec{x} = \mathbb{Q}^T\vec{b}$$

really $\mathbb{R}\vec{x}_* = \mathbb{Q}^T\vec{b}$.

↑ not $\vec{x}_*$ eek? ??

We are really solving the normal equation... because $\mathbb{Q}^T\vec{b} = \mathbb{Q}^T\vec{p}$

projection of $\vec{b}$ onto $C(\mathbb{A})$

$$\mathbb{Q}^T\vec{b} = \begin{bmatrix} - & \hat{q}_1^T & - \\ - & \hat{q}_2^T & - \\ & \vdots & \\ - & \hat{q}_n^T & - \end{bmatrix}\begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$

$\hat{q}_i$'s span $C(\mathbb{A})$

$$\underset{m\times m}{\mathbb{P}_{\hat{q}_i}} = \underset{m\times 1}{\hat{q}_i}\,\underset{1\times m}{\hat{q}_i^T}$$

← projection operator for direction $\hat{q}_i$

What's going on: $\mathbb{Q}^T\vec{b}$

$$= \mathbb{Q}^T(\underset{\in C(\mathbb{A})}{\vec{p}} + \underset{\in N(\mathbb{A}^T)}{\vec{e}})$$

$$= \mathbb{Q}^T\vec{p} + \underbrace{\cancel{\mathbb{Q}^T}\vec{e}}_{C(\mathbb{A})} = \vec{0}$$

really "orthonormal"

Orthogonal Matrices:

The Gram-Schmidt Process gave us $\underset{m\times n}{\mathbb{A}} = \underset{m\times n}{\mathbb{Q}}\,\underset{n\times n}{\mathbb{R}}$ ← upper triangular combining matrix

$$\begin{bmatrix} \ \end{bmatrix} \qquad \begin{bmatrix} \ \end{bmatrix}\begin{bmatrix} \ \end{bmatrix}$$

$\mathbb{Q}$'s columns $\{\hat{q}_i\}$ form an orthonormal basis for $\mathbb{A}$'s column space; $\hat{q}_i^T\hat{q}_j = \begin{cases} 0, & i\neq j \\ 1, & i=j \end{cases}$

[Note: $\mathbb{A}$'s columns are ideally linearly independent $(n=r)$

More on what $\mathbb{Q}$ type matrices can do for you:

left inverse for $\mathbb{Q}$

$$\mathbb{Q}^T\mathbb{Q} = \mathbb{I}$$

$$\underbrace{\begin{bmatrix} — & \hat{q}_1^T & — \\ — & \hat{q}_2^T & — \\ & \vdots & \\ — & \hat{q}_n^T & — \end{bmatrix}}_{n\times m} \underbrace{\begin{bmatrix} | & | & & | \\ \hat{q}_1 & \hat{q}_2 & \cdots & \hat{q}_n \\ | & | & & | \end{bmatrix}}_{m\times n} = \underbrace{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}_{n\times n} = \mathbb{I}$$

If $\mathbb{Q}$ is square, then $m=n=r$

So inverse exists $(N(\mathbb{Q}) = \{\vec{0}\})$

then

$$\left.\begin{matrix} \mathbb{Q}^T\mathbb{Q} = \mathbb{I} = \mathbb{Q}\mathbb{Q}^T \\ \mathbb{Q}^{-1}\mathbb{Q} = \mathbb{I} = \mathbb{Q}\mathbb{Q}^{-1} \end{matrix}\right\} \quad \boxed{\mathbb{Q}^{-1} = \mathbb{Q}^T}$$

Say $\mathbb{Q}$ is an orthogonal matrix

Many other groovy properties:

length

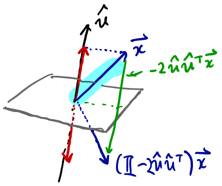
$$\|\mathbb{Q}\vec{x}\| = \|\vec{x}\|$$ ← length is preserved under transformation by $\mathbb{Q}$

$$(\mathbb{Q}\vec{x})^T(\mathbb{Q}\vec{y}) = \vec{x}^T\vec{y}$$ ← $\mathbb{Q}$ preserves angles

ex 1 $\mathbb{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ← rotation by θ ↺

ex 2 $\mathbb{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ← permutat.

ex 3 $\mathbb{Q} = \mathbb{I} - 2\,\overset{\mathbb{P}_{\hat{u}}}{\hat{u}\hat{u}^T}$ project onto $\hat{u}$ and flip



Three reasons to love arbitrary powers of square matrices

In our journey so far, we've spent a lot of time thinking about one of the Monks' favorite equations: $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$

Now: The Monks tell us to think about square matrices ($n\times n$) as gadgets, things that transform vectors into new vectors

✓ ✓

$$\boxed{\underset{n\times 1}{\vec{x}\,'} = \underset{n\times n}{A}\,\underset{n\times 1}{\vec{x}}}$$

A might $\left\{\begin{matrix}\text{flip}\\ \text{rotate}\\ \text{stretch}\\ \text{project}\end{matrix}\right\}\vec{x}$

Big Question: what happens if we use A to repeatedly transform a vector?

E17ap1

Start with $\vec{x}_0$

$\vec{x}_1 = A\vec{x}_0$, $\vec{x}_2 = A\vec{x}_1$, ..., $\vec{x}_k = A\vec{x}_{k-1}$, ...

$$\Rightarrow \boxed{\vec{x}_k = A^k\,\vec{x}_0}$$

$$\underset{n\times n}{A^k} = \overbrace{\underset{n\times n}{A}\cdot\underset{n\times n}{A}\times\cdots\times\underset{n\times n}{A}}^{k}$$

k th power of A

Difficulty: Mindless multiplication of many matrices works but is
(1) computationally expensive;
(2) doesn't give us any understanding of how A^k behaves

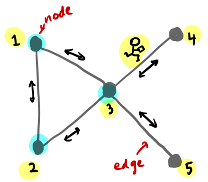
deep story

The Monks whisper that we must understand eigenthings...

vast and wonderful

But first: three example areas showing the excellence of $\underset{n\times n}{A^k}$....

(1) The distracted texter wandering randomly on a network:



columns must sum to 1

$$\underbrace{\begin{bmatrix} P_{t+1,1} \\ P_{t+1,2} \\ P_{t+1,3} \\ P_{t+1,4} \\ P_{t+1,5} \end{bmatrix}}_{\vec{P}_{t+1}} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 1 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix}}_{\mathbb{A}} \underbrace{\begin{bmatrix} P_{t,1} \\ P_{t,2} \\ P_{t,3} \\ P_{t,4} \\ P_{t,5} \end{bmatrix}}_{\vec{P}_t}$$

probability texter is at nodes 1,..,5 at time t+1

transition matrix

Natural question:

where is our texter likely to be as time goes on?

or, what is $\vec{P}_\infty$?

or what is $\mathbb{A}^k$ as $k \to \infty$?

Monks (and soon you) tell us that

$$\mathbb{A}\begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \vec{P}_\infty = \frac{1}{10}\begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

eigenvalue (later)

no change

eigenvector

normalized

great result: $P_{\infty,i}$ is proportional to the degree of node i

(2) Solving coupled linear differential equations

E17ap2

simple $\quad \dfrac{dx}{dt} = 3x \Rightarrow x(t) = \underline{x(0)}e^{3t}$

initial value at $t=0$

check this works

Coupled

$$\frac{dx_1}{dt} = 2x_1 - x_2$$

$$\frac{dx_2}{dt} = \underline{3x_1 + 2x_2}$$

change depends on current position

continuous & discrete math crushing it together

Rewrite with matrices:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \frac{d\vec{x}}{dt} = \mathbb{A}\vec{x}$$

Solution is of the form: $\underset{2\times1}{\vec{x}(t)} = \underbrace{e^{\overset{2\times2}{\mathbb{A}}t}}_{2\times2} \underset{2\times1}{\vec{x}(0)}$

what??

It's true!:

you can exponentiate matrices!!

Taylor expansion

$$e^{\mathbb{A}} = \mathbb{I} + \mathbb{A} + \frac{1}{2!}\mathbb{A}^2 + \frac{1}{3!}\mathbb{A}^3 + \ldots + \frac{1}{k!}\mathbb{A}^k + \ldots$$

we need to really understand all powers of $\mathbb{A}$ to cope with this...

scalar

$$e^{\mathbb{A}t} = \mathbb{I} + \mathbb{A}t + \frac{1}{2!}\mathbb{A}^2t^2 + \frac{1}{3!}\mathbb{A}^3t^3 + \ldots$$

$$e^{\varpi} = \mathbb{I}_{\varpi} + \varpi + \frac{1}{2!}\varpi^2 + \frac{1}{3!}\varpi^3 + \ldots$$

#awesome

(3) Solving super fun difference equations

ex $F_{k+2} = F_{k+1} + F_k$ with $\overbrace{F_0 = F_1 = 1}^{\text{initial conditions}}$

↖ Fibonacci sequence

Monks say try:

$$\vec{f}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

↖ stack consecutive Fibonacci numbers

$$\vec{f}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \underset{\substack{\uparrow \\ \text{unbelievable} \\ \text{matrix...}}}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\substack{\uparrow \\ \vec{f}_{k-1}}}{\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \vec{f}_{k-2}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \vec{f}_0$$

So if we can compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k$ for all k in a clever way, we'll have a formula for the Fibonacci numbers.

↘ later.

E17ap3

Again, understanding and calculating $\mathbb{A}^k$ is made possible through the magic of eigenthings

eigen → values, vectors, spaces, functions

The Magic of Eigenthings

an introduction to happiness

Scene:

A Monk hands us a parchment with

$$\mathbb{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and other strange symbols written on it, smiles, and then mysteriously disappears...

note $\vec{v}_1 \perp \vec{v}_2$ ....

Let's try some things...

2x2

$$\mathbb{A}\vec{v}_1 = \begin{bmatrix} 1 & 1/2 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{3}{2}\vec{v}_1$$

$\mathbb{A}$ symmetric $\vec{v}_1$

only direction matters

$\vec{v}_1$ comes back with a $3/2$ stretch factor

$$\mathbb{A}^2\vec{v}_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^2\vec{v}_1$$

$$\mathbb{A}^k\vec{v}_1 = \begin{bmatrix} 1 & 1/2 \\ \frac{1}{2} & 1 \end{bmatrix}^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{3}{2}\right)^k\vec{v}_1$$

grows

$\mathbb{A}$ "likes" the direction of $\vec{v}_1$

↳ "eigenvector"

German for "own"

And we'll call $\lambda_1 = 3/2$ the "eigenvalue" associated with $\vec{v}_1$

$$\mathbb{A}\vec{v}_2 = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\vec{v}_2$$

$\vec{v}_2$ is also an eigenvector of $\mathbb{A}$

$\lambda_2 = \frac{1}{2}$ is the associated eigenvalue

- Note again: only direction of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ matters

$$\mathbb{A}\underset{\uparrow\, 4\vec{v}_1}{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} = \frac{3}{2}\begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \& \quad \mathbb{A}\underset{\uparrow\, 17\vec{v}_2}{\begin{bmatrix} 17 \\ -17 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} 17 \\ -17 \end{bmatrix}$$

- Also: $\mathbb{A}^k\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \left(\frac{1}{2}\right)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

vanishes

One more thing:

$$\mathbb{A}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

↑ not an eigenvector

← different direct

$$\mathbb{A}^2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \mathbb{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 2 \end{bmatrix}$$

← different again...

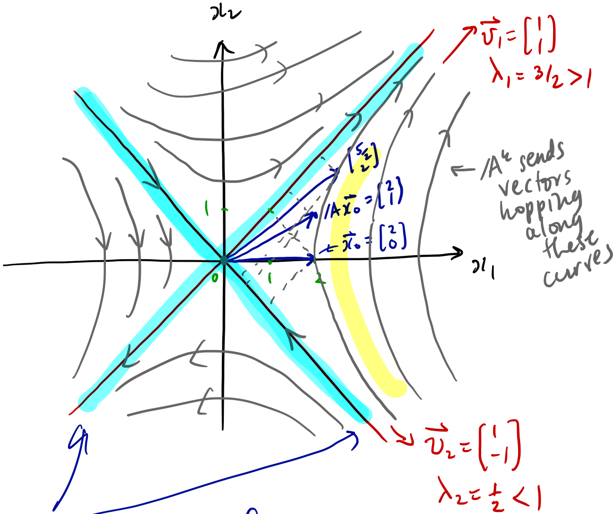
Better:

$$\mathbb{A}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \mathbb{A}\Bigg(\underbrace{\overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}}_{\text{huge}}\Bigg) = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\mathbb{A}$ is simple in eigenvector basis

huge: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is a great basis for $\mathbb{A}$

picture for A^k:



Eigenspaces of A
(1-d subspaces of $\mathbb{R}^2$)

possibilities for $A\vec{v} = \lambda\vec{v}$

$\lambda > 1$: growth

$\lambda = 1$: stays the same

$0 < \lambda < 1$: shrinkage

$\lambda < 0$: jumping back and forth across origin, $|\lambda|$ governs growth

λ complex : rotation

Big question: If monks aren't around, how do we find $\vec{v}$'s and λ's ? How many are possible if A is $n \times n$?

$\overset{\checkmark}{A}\overset{?}{\vec{x}} = \overset{\checkmark}{\vec{b}}$

Game is to solve the Eigenvalue Equation:

$$\boxed{\underset{n\times n}{\overset{\checkmark}{A}}\ \underset{n\times 1}{\overset{?}{\vec{v}}} = \underset{\uparrow \text{ scalar}}{\overset{?}{\lambda}}\ \underset{n\times 1}{\overset{?}{\vec{v}}}}$$

$$A\vec{v} - \lambda\vec{v} = \underset{n\times 1}{\vec{0}} \ \leftarrow \text{vector!!}$$

temptation

shame → $(\overset{n\times n}{A} - \overset{1\times 1}{\lambda})\vec{v} = \vec{0}$

no!!

sneakiness (Monks)

$$\underset{n\times n}{A}\ \underset{n\times 1}{\vec{v}} - \underset{\text{scalar}}{\lambda}\ \underset{n\times n}{I}\ \underset{n\times 1}{\vec{v}} = \underset{n\times 1}{\vec{0}}$$

$$\boxed{\underbrace{(A - \lambda I)}_{n\times n}\ \underset{n\times 1}{\vec{v}} = \underset{n\times 1}{\vec{0}}}$$

Nullspace Equation!!

λ ← lambda must be such that,

- $\text{rank}(A - \lambda I) < n$
- $N(A - \lambda I) \neq \{\vec{0}\}$
- $A - \lambda I$ has no inverse (A is singular)
- $\det(A - \lambda I) = |A - \lambda I| \equiv 0$

new thing

Solve $(\mathbb{A}-\lambda\mathbb{I})\vec{v}=\vec{0}$ for $\mathbb{A}=\begin{bmatrix}1 & 1/2\\ 1/2 & 1\end{bmatrix}$

$\lambda\mathbb{I}$ — $\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$

Usual way:

$$\begin{bmatrix}1-\lambda & \frac{1}{2}\\ \frac{1}{2} & 1-\lambda\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

subtracting λ from diagonal entries of $\mathbb{A}$

Augmented matrix

$$\left[\begin{array}{cc|c}1-\lambda & \frac{1}{2} & 0\\ \frac{1}{2} & 1-\lambda & 0\end{array}\right]\underset{R_2'=R_2-\left(\frac{1/2}{1-\lambda}\right)R_1}{\sim}\left[\begin{array}{cc|c}1-\lambda & \frac{1}{2} & 0\\ 0 & \left(1-\lambda-\frac{(\frac{1}{2})^2}{1-\lambda}\right) & 0\end{array}\right]$$

eek!!

See $(1-\lambda)-\frac{(\frac{1}{2})^2}{1-\lambda}=0$

for $r=1$, $N(\mathbb{A}-\lambda\mathbb{I})\neq\{\vec{0}\}$
rank
$\Rightarrow \vec{v}$ is healthy

$(1-\lambda)^2=(\frac{1}{2})^2$

$\Rightarrow 1-\lambda=\pm\frac{1}{2}$

$\Rightarrow \lambda=1\pm\frac{1}{2}$

$\lambda_1=3/2,\ \lambda_2=\frac{1}{2}$

just as we found...

$\Rightarrow$ next step: find $\vec{v}$ as nullspace vectors for $\mathbb{A}-\frac{3}{2}\mathbb{I}$ & $\mathbb{A}-\frac{1}{2}\mathbb{I}$

really: basis vectors

Unfortunately, preceding is a very messy way to handle $(\mathbb{A}-\lambda\mathbb{I})\vec{v}=\vec{0}$... There's a better, more illuminating way.

=

Set $\det(\mathbb{A}-\lambda\mathbb{I})=0$ and find λ... ← reason is coming...

$\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$ for 2×2s

$$\begin{vmatrix}1-\lambda & \frac{1}{2}\\ \frac{1}{2} & 1-\lambda\end{vmatrix}=(1-\lambda)^2-(\tfrac{1}{2})^2=0$$

same equation as before

- see next episodes for all things determinants
- we return to eigenthings after this strange excursion

E176p4

$\lambda_1 = 3/2$, $\lambda_2 = 1/2$

Find $\vec{v}_1$ & $\vec{v}_2$

$\lambda_1 = 3/2$:

$$(\mathbb{A} - \tfrac{3}{2}\mathbb{I})\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ← standard Nullspace Equation

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 1-3/2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1-3/2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} P & F & \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$R_2' = R_2 - \left(\frac{+\frac{1}{2}}{-\frac{1}{2}}\right) R_1$

$$-\tfrac{1}{2}v_1 + \tfrac{1}{2}v_2 = 0$$

$$\Rightarrow v_1 = v_2$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ ← Eigenspace for $\lambda_1 = 3/2$

Say $\lambda_1 = 3/2$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

↑ basis for eigenspace

$\lambda_2 = \frac{1}{2}$:

$$(\mathbb{A} - \tfrac{1}{2}\mathbb{I})\vec{v}_2 = \vec{0}$$

$$\left[\begin{array}{cc|c} 1-\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1-\frac{1}{2} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} P & F & \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{v}_2 = c\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad c \in \mathbb{R}$$ ← Eigenspace

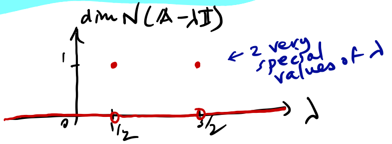
$\lambda_1 = \frac{1}{2}$ has eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Often, unit vectors are best

$$\hat{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \& \quad \hat{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_1 = 3/2$ $\qquad$ $\lambda_2 = 1/2$

↗ $\mathbb{A}$'s natural basis

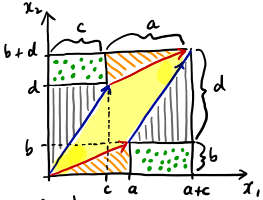


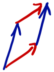
Determinants from the ground up:

2×2's first then n×n's

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

idea: Consider area of parallelogram formed by row vectors of A: $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$





Area of $= (a+c)(b+d)$

$- ab \quad - dc \quad - 2bc$

$= ab + ad + cb + cd$

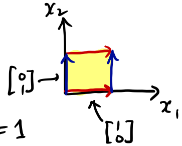
$\quad - ab - dc - 2bc$

$= ad - bc$ → call this A's determinant: $\det(A)$ or $|A|$

Three Observations about this determinant thing for 2×2's:

① $|I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

anchor



area = 1

E18ap1

② If we swap A's rows, we flip the sign of the determinant

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ but $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad-bc)$

straight lines for determinant

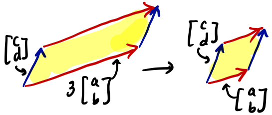
−ve area indicates ordering of vectors

③ $|A| = ad - bc$ is **multilinear** in the rows of A

Two pieces:

3.1 Area scales:

ex $\begin{vmatrix} 3a & 3b \\ c & d \end{vmatrix} = 3 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$



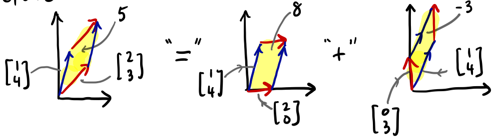
ex $\begin{vmatrix} 2a & 2b \\ 4c & 4d \end{vmatrix} = 2 \cdot 4 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3.2 Areas add when single rows add:

ex $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}$

formula: $\underbrace{2 \cdot 4 - 3 \cdot 1}_{5} = \underbrace{(2 \cdot 4 - 1 \cdot 0)}_{8} + \underbrace{(0 \cdot 4 - 1 \cdot 3)}_{-3}$

pictures:



Determinants from the ground up:

The plan: We assert Three Properties
for Determinants of $n \times n$ matrices

① $|\mathbb{I}| = 1.$

$\nwarrow$ $n \times n$ unit hypercube

[Determinant $= \pm$ volume
of parallelipiped
created by row
vectors of $\mathbb{A}$]

② Swapping any two rows of $\mathbb{A}$
changes the sign of the determinant.

③ Determinants are multilinear
in their rows.

Big Deal:
Can now connect $|\mathbb{A}|$ to $|\mathbb{I}| = 1$
and many, many good things will follow

ok: Let's fully connect $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$ to $|\mathbb{I}|$

$$|\mathbb{A}| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} \leftarrow \text{linear in row 1}$$

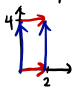
③

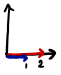
need to introduce 0's
to get to $\mathbb{I}$

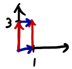
$$= \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix}$$

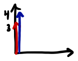
linear in row 2 ③

linear in row 2 ③







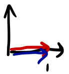


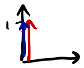
$$= 2 \cdot 4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 3.4 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

0 volume

0 volume

row swap





$$= 8 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 3(-1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (8-3) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 5\,|\mathbb{I}| = 5$$

Next: Show how our standard row operations
lead to many results $\swarrow \mathbb{PA} = \mathbb{LU}$
including $|\mathbb{A}| = \pm |\mathbb{U}|$
& $|\mathbb{A}\,\mathbb{B}| = |\mathbb{A}|\,|\mathbb{B}|$

#excitement

/E186p1

[Many results for determinants based on three properties:
① $|\mathbb{I}_{n\times n}|=1$, ② Row swap $\rightarrow \times(-1)$ (sign flip), ③ Multilinearity (rows)]

[notation $\det A = |A|$]

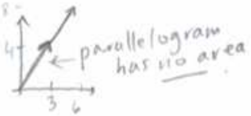
(a) $|t\overset{n\times n}{A}| = t^n|A|$ ← multilinearity, $t\in\mathbb{R}$

$t\begin{bmatrix}3&4\\2&1\end{bmatrix}=\begin{bmatrix}3t&4t\\2t&t\end{bmatrix}$

(b) If two of A's rows are the same, then $|A|=0$.

ex $\begin{vmatrix}3&4\\3&4\end{vmatrix} \underset{R_2\leftrightarrow R_1}{=} -\begin{vmatrix}3&4\\3&4\end{vmatrix}$ (property ②) so $\begin{vmatrix}3&4\\3&4\end{vmatrix}=0$

ex $\begin{vmatrix}3&6&6\\3&6&6\\2&-2&1\end{vmatrix} \underset{R_2\leftrightarrow R_1}{=} -\begin{vmatrix}3&6&6\\3&6&6\\2&-2&1\end{vmatrix}$ (property ②) so must be zilch



(c) Performing a step of standard row reduction doesn't change the value of the determinant. #bigdeal

$R_i' = 1R_i - (l_{ij})R_j$

Row reduction for a 2×2:

$A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix} \underset{R_2'=R_2-l_{21}R_1}{\sim} \begin{bmatrix}a_{11}&a_{12}\\a_{21}-l_{21}a_{11}&a_{22}-l_{21}a_{12}\end{bmatrix}$

tilde; a_{21}: zero this out; $\left(\frac{a_{21}}{a_{11}}\right)$; now find determinant of this new matrix

$\begin{vmatrix}a_{11}&a_{12}\\a_{21}-l_{21}a_{11}&a_{22}-l_{21}a_{22}\end{vmatrix} = $ ← multilinearity

$\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix} - l_{21}\begin{vmatrix}a_{11}&a_{12}\\a_{11}&a_{12}\end{vmatrix}$ (0)

original matrix; same rows (b)

☺ Above generalizes to n×n
"$\sim$" for solving $A\vec{x}=\vec{b} \Rightarrow$ "$=$" for determinants

(d) If $\underset{n\times n}{A}$'s rows are linearly dependent, then $|A|=0$ #bigdeal

Reason:

* We can use row ops to make one or more rows of zeros

* Now add one non-zero row to a zero row using one more row op
$\Rightarrow$ 2 rows the same (b) $\Rightarrow |A|=0$

(d) cont.

Or multiply zero row by any number c.

$\Rightarrow$ Multilinearity means determinant should scale by a factor of c

$\Rightarrow$ But row was unchanged $c\vec{0}^T = \vec{0}^T$

so $|\mathbb{A}| = 0$.

ex $$\begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ c[0 & 0 & 0] \end{vmatrix} = c \begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ 0 & 0 & 0 \end{vmatrix}$$

only possible if $\begin{vmatrix} 2 & -2 & 1 \\ 3 & 6 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Connections:

Applies if $\mathbb{A}$ is square only

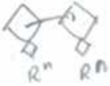
$\underset{n\times n}{|\mathbb{A}|} = 0$

$\Leftrightarrow$ $\mathbb{A}$'s rows are linearly dependent

$\Leftrightarrow$ Rank of $\mathbb{A}$, r, is less than n

$\Leftrightarrow$ $N(\mathbb{A})$ & $N(\mathbb{A}^T)$ have dim $n-r \geq 1$

$\Leftrightarrow$ $\mathbb{A}$ has no inverse



$\Leftrightarrow$ If $\mathbb{A}\vec{x} = \vec{b}$ has a solution, then there are ∞ly many solutions

$\Leftrightarrow$ One or more sides of $\mathbb{A}$'s parallelipiped have 0 length.

Example calculation of $|\mathbb{A}|$ using row ops:

$$|\mathbb{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix}$$

freestyling

$$\underset{R_2 \Leftrightarrow R_3}{=} - \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix}$$

$$\underset{R_2' = R_2 - (\frac{2}{1})R_1}{=} - \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 3 & 2 \end{vmatrix}$$

$$\underset{R_3' = R_3 - (\frac{3}{-1})R_2}{=} - \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & \boxed{2} \end{vmatrix}$$

d_1 d_2 d_3 pivots!

$\mathbb{U}$ from $\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U}$

$$\underset{\substack{R_1' = R_1 - (\frac{1}{-1})R_2 \\ R_1'' = R_1' - (\frac{1}{2})R_3}}{=} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \underbrace{-(1)(-1)(2)}_{2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\mathbb{R}^{\text{echelon}}_{\mathbb{A}}$ new thing

$\mathbb{R}_{\mathbb{A}} = \mathbb{I}$

$$= 2\,|\mathbb{I}| = 2 \times 1 = 2$$

$\Leftrightarrow$ one or more pivots $= 0$

(e) $|\mathbb{A}| = \pm |\mathbb{U}|$ as in $\mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U}$

depends on number of row swaps

permutation

(f) $|\mathbb{A}| = \pm \prod_{i=1}^{n} d_i$ pivots of $\mathbb{A}$

if one or more pivots = 0

$\Rightarrow |\mathbb{A}| = 0.$

follows from row op goodness (c)

(g) $\underset{n\times n}{|\mathbb{A}}\ \underset{n\times n}{\mathbb{B}|} = \underset{n\times n}{|\mathbb{A}|}\ \underset{n\times n}{|\mathbb{B}|}$ ← #bigdeal

Various proofs exist.

Monks say use row reduction:

Know $\mathbb{E}\,\mathbb{P}\,\mathbb{A} = \mathbb{R}^{e}_{\mathbb{A}} = \mathbb{D}_{\mathbb{A}}$

reduced row echelon form with pivots still in

all elimination matrices

permutation matrix

pivot matrix for $\mathbb{A}$

(first presume all pivots $\neq 0$).

ok $|\mathbb{A}\,\mathbb{B}|$

det unchanged with row ops

$= \pm |\mathbb{E}\,\mathbb{P}\,\mathbb{A}\,\mathbb{B}| = \pm |\mathbb{D}_{\mathbb{A}}\,\mathbb{B}|$

depends on $\mathbb{P}$

$$= \pm \left| \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix} \begin{bmatrix} — \vec{b}_{1*} — \\ — \vec{b}_{2*} — \\ \vdots \\ — \vec{b}_{n*} — \end{bmatrix} \right|$$

$$= \pm \begin{vmatrix} — d_1 \vec{b}_{1*} — \\ — d_2 \vec{b}_{2*} — \\ \vdots \\ — d_n \vec{b}_{n*} — \end{vmatrix}$$

$= \pm \left(\prod_{i=1}^{n} d_i\right) |\mathbb{B}| = |\mathbb{A}|\,|\mathbb{B}|.$

multilinearity

Now if one or more pivots $= 0$, $|\mathbb{A}| = 0$

can see same row reductions leads to row of 0's $\Rightarrow |\mathbb{A}\,\mathbb{B}| = 0$ ✓

(h) $|\mathbb{A}^{-1}| = \dfrac{1}{|\mathbb{A}|}$ from (g)

reason $|\mathbb{A}\,\mathbb{A}^{-1}| = |\mathbb{A}|\,|\mathbb{A}^{-1}|$

$\overset{\shortparallel}{|\mathbb{I}|} = 1$

Note $|\mathbb{A}| = 0 \Rightarrow |\mathbb{A}^{-1}| = \infty$ ouch!

(i) If $\mathbb{A}$ is upper or lower triangular then $|\mathbb{A}|$ = product of entries on $\mathbb{A}$'s main diagonal.

Reason: row reduction on a triangular matrix requires no row swaps and does not change entries on main diag.
Plus: zero leads to a zero row $\Rightarrow |\mathbb{A}| = 0$.

ex

$$\begin{vmatrix} 4 & 77 & 16 \\ 0 & -3 & 17 \\ 0 & 0 & 2 \end{vmatrix} = (4)(-3)(2) = -24 \; ; \quad \begin{vmatrix} 4 & 0 & 0 \\ 77 & 1 & 0 \\ 13 & 99 & 2 \end{vmatrix} = (4)(1)(2) = 8.$$

$$\begin{vmatrix} 4 & 77 & 16 \\ 0 & 0 & 17 \\ 0 & 0 & 2 \end{vmatrix} = (4)(0)(2) = 0$$

last

(j) $|\mathbb{A}| = |\mathbb{A}^T|$ ← #groovy ← #bigdeal

Means: can use "column ops" in the same way as row ops & all results for rows work for columns too.

Reason

Use monks $\underset{n\times n}{\mathbb{P}}\underset{n\times n}{\mathbb{A}} = \underset{n\times n}{\mathbb{L}}\underset{n\times n}{\mathbb{U}} \longrightarrow |\mathbb{P}||\mathbb{A}| = |\mathbb{L}||\mathbb{U}|$

E18bp4

$$(\mathbb{P}\mathbb{A})^T = (\mathbb{L}\mathbb{U})^T$$

C*

$$\mathbb{A}^T\mathbb{P}^T = \mathbb{U}^T\mathbb{L}^T$$

Take determinants of both sides

$$|\mathbb{A}^T||\mathbb{P}^T| = |\mathbb{U}^T||\mathbb{L}^T|$$

triangular determinant is unchanged by transpose

$$|\mathbb{P}^T||\mathbb{A}^T| = |\mathbb{L}||\mathbb{U}|$$

handle this

$\mathbb{P}$ is a permutation matrix, a shuffling of the identity matrix.

$\Rightarrow |\mathbb{P}| = \pm 1$ ← # row swaps

Also know

$$\mathbb{P}^{-1} = \mathbb{P}^T \text{ So } |\mathbb{P}^T\mathbb{P}| = |\mathbb{I}| = 1.$$

$$\overset{(g)}{=} |\mathbb{P}^T||\mathbb{P}|$$

$\Rightarrow$ either $|\mathbb{P}| = |\mathbb{P}^T| = 1$, or $|\mathbb{P}| = |\mathbb{P}^T| = -1$ ← they match.

C*

$\Rightarrow |\mathbb{P}||\mathbb{A}^T| = |\mathbb{L}||\mathbb{U}| \Rightarrow |\mathbb{A}^T| = |\mathbb{A}|.$

Computing determinants:
The way of the cofactor

* Recipe first, understanding later.

* Need a clean way to find determinants for eigenvalue problem $\mathbb{A}\vec{v} = \lambda\vec{v}$

* Row operations helped us with results about determinants but are messy...

The story:

* $n \times n$ determinants are sums of $\underline{n}$ $(n-1)\times(n-1)$ determinants

* 3×3 determinants are sums of 3 2×2 determinants. recursive.

Example to work with:

$$\mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}.$$

Defn: $\mathbb{M}_{ij}$ is the $(n-1)\times(n-1)$ matrix formed by deleting the i^{th} row and j^{th} column of $\mathbb{A}$; there are n^2 of these **"minor matrices"**

exs

$$\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{0} & 3 & 2 \\ \cancel{2} & 1 & 2 \end{bmatrix} \quad \mathbb{M}_{11} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} \ldots \\ 0 & 3 & \cancel{2} \\ 2 & 1 & \cancel{2} \end{bmatrix} \quad \mathbb{M}_{13} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cancel{1} & 1 \\ \cancel{0} & \cancel{3} & \cancel{2} \\ 2 & \cancel{1} & 1 \end{bmatrix} \quad \mathbb{M}_{\underline{22}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

← $\underset{3\times3}{\mathbb{A}}$ has $3\times3=9$ minor matrices

Defn: $\overset{\text{det}}{|\mathbb{M}_{ij}|}$ is the ij^{th} **minor** of $\mathbb{A}$

Defn: $C_{ij} = (-1)^{i+j}|\mathbb{M}_{ij}|$ is the ij^{th} **cofactor** of $\mathbb{A}$

$(-1)^{i+j} \Rightarrow$ checkerboard of +'s & −'s

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \\ + & - & + & - & \\ - & + & - & + & \\ \vdots & & & & \ddots \end{bmatrix}$$

Theorem:

The Determinant of $\mathbb{A}$ is given by the dot product of $\mathbb{A}$'s cofactors and $\mathbb{A}$'s entries along <u>any</u> <u>one</u> row or column.

ex Using row 1

$$|\mathbb{A}| \overset{\downarrow}{=} \sum_{j=1}^{n} C_{1j}\, a_{1j}$$

$$= \sum_{i=1}^{n} C_{i2}\, a_{i2}$$

column 2

#crazytown bananapants

$$\text{Ex}\quad \mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$$

Let's first try row 1

$\Rightarrow$ Compute C_{11}, C_{12}, C_{13} using $C_{ij} = (-1)^{i+j}|\mathbb{M}_{ij}|$

$\mathbb{M}_{11} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$; $|\mathbb{M}_{11}| = 3\cdot2 - 2\cdot1 = 4$; $C_{11} = (-1)^{1+1}\cdot 4 = 4$

$\mathbb{M}_{12} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$; $|\mathbb{M}_{12}| = 0\cdot2 - 2\cdot2 = -4$; $C_{12} = (-1)^{1+2}\cdot(-4) = 4$

$\mathbb{M}_{13} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$; $|\mathbb{M}_{13}| = 0\cdot1 - 3\cdot2 = -6$; $C_{13} = (-1)^{1+3}(-6) = -6$

$$|\mathbb{A}| = C_{11}a_{11} + C_{12}a_{12} + C_{13}a_{13}$$ ← dot product

$$= (4)(1) + (4)(1) + (-6)(1) = 2 \;\; /\!/ .$$

(as for row ops)

Now We can choose any row or column so let's do all <u>of them at</u> once. #crazy

Create cofactor matrix $\mathbb{C}$:

$$\mathbb{C} = \begin{bmatrix} 4 & 4 & -6 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$$

← first row as above

} exercise

$$[a_{ij} C_{ij}] = \begin{bmatrix} 1\times4 & 1\times4 & 1\times(-6) \\ 0\times(-1) & 3\times0 & 2\times1 \\ 2\times(-1) & 1\times(-2) & 2\times3 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -6 \\ 0 & 0 & 2 \\ -2 & -2 & 6 \end{bmatrix}$$

direct product of elements

Magic:

$$\begin{array}{ccc|c} & & & \text{row sums} \\ 4 & 4 & -6 & ② \text{ as above} \\ 0 & 0 & 2 & ② \\ -2 & -2 & 6 & ② \\ ② & ② & ② & \end{array}$$

column sums →

#inconceivable

Cofactor method enables excellent. sneakiness:

Choose row or column with most zeros:

ex $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix}$ ← best choice

$$= \underset{\uparrow \atop a_{11}}{1} \underbrace{(-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}_{C_{11}} = 1 \cdot 1 \cdot (6+1) = \underline{7}$$

$a_{12} = a_{13} = 0$

ex row 2 $\begin{vmatrix} 2 & 7 & 0 \\ 0 & 3 & 2 \\ -1 & 4 & 0 \end{vmatrix}$ col 3

$$= \underset{\uparrow \atop a_{23}}{2} \underbrace{(-1)^{2+3}\begin{vmatrix} 2 & 7 \\ -1 & 4 \end{vmatrix}}_{C_{23}} = 2 \cdot (-1) \cdot (\underline{8+7})$$

$$= \underline{-30}.$$

No need to compute cofactors associated with 0's in A. → avoid trauma

Fun: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ← go along top row

$$= a_{11} \underbrace{(-1)^{1+1}|a_{22}|}_{C_{11}} + a_{12} \underbrace{(-1)^{1+2}|a_{21}|}_{C_{12}}$$

det of a 1×1 = length

$$= a_{11}a_{22} - a_{12}a_{21} \checkmark$$

↑ everything works

One more example: 4×4

$$\begin{vmatrix} 2 & 2 & 3 & -1 \\ 0 & 7 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 4 & 0 & 3 \end{vmatrix}$$ ← most 0's

$$= \underset{\uparrow \atop a_{22}}{7} \cdot (-1)^{2+2} \begin{vmatrix} 2 & 3 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$ ← most 0's

$$= 7 \cdot \left(3 \cdot (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \right)$$

$$= 7 \cdot 3 \, (\underbrace{2+3}_{5}) = 7 \cdot 3 \cdot 5 = \underline{\underline{105}}$$

* Starting with, say, row 1 would have really hurt...

Determinants & $\underset{n\times n}{\mathbb{A}}\vec{x}=\vec{b}$
Cramer's rule and a formula for the inverse of $\mathbb{A}$

#inconceivable

Monks say try this for 3×3 matrices:

$$\mathbb{A}\begin{bmatrix}| & 0 & 0\\ \vec{x} & 1 & 0\\ | & 0 & 1\end{bmatrix}=\begin{bmatrix}| & & \\ \mathbb{A}\vec{x} & \mathbb{A}\begin{bmatrix}0\\1\\0\end{bmatrix} & \mathbb{A}\begin{bmatrix}0\\0\\1\end{bmatrix}\\ | & & \end{bmatrix}$$

$\mathbb{I}$ with $\vec{x}$ in first column

$$=\begin{bmatrix}| & | & |\\ \vec{b} & \vec{a}_2 & \vec{a}_3\\ | & | & |\end{bmatrix}=\mathbb{B}_1$$

↑ call this

$\mathbb{A}$ with first column replaced by $\vec{b}$.

Similarly

$$\mathbb{A}\begin{bmatrix}1 & | & 0\\ 0 & \vec{x} & 0\\ 0 & | & 1\end{bmatrix}=\overset{\mathbb{B}_2}{\begin{bmatrix}| & | & |\\ \vec{a}_1 & \vec{b} & \vec{a}_3\\ | & | & |\end{bmatrix}}\ ;\ \mathbb{A}\begin{bmatrix}1 & 0 & |\\ 0 & 1 & \vec{x}\\ 0 & 0 & |\end{bmatrix}=\overset{\mathbb{B}_3}{\begin{bmatrix}| & | & |\\ \vec{a}_1 & \vec{a}_2 & \vec{b}\\ | & | & |\end{bmatrix}}$$

Monks whisper "take determinants"....

$$|\mathbb{A}|\underbrace{\begin{vmatrix}| & 0 & 0\\ \vec{x} & 1 & 0\\ | & 0 & 1\end{vmatrix}}_{=\,x_1}=|\mathbb{B}_1|$$

(use row reduction on transpose)

$$\Rightarrow\ x_1=\frac{|\mathbb{B}_1|}{|\mathbb{A}|},\ x_2=\frac{|\mathbb{B}_2|}{|\mathbb{A}|},\ x_3=\frac{|\mathbb{B}_3|}{|\mathbb{A}|}$$

↑ wait! we just solved $\underset{n\times n}{\mathbb{A}}\vec{x}=\vec{b}$!!

$$\vec{x}=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=\frac{1}{|\mathbb{A}|}\begin{bmatrix}|\mathbb{B}_1|\\ |\mathbb{B}_2|\\ \vdots\\ |\mathbb{B}_n|\end{bmatrix}\quad !!!$$

Problems: ① only works for $n\times n$ ← ok for normal equations

② computing determinants is horribly slow

③ must recompute for new $\vec{b}$

Main utility: theoretical.

E18dp1

Let's use Cramer's rule to find $\mathbb{A}^{-1}$:

Monks say solve these special $\mathbb{A}\vec{x}=\vec{b}$ problems:

$$\mathbb{A}\vec{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \leftarrow \vec{b}$$

$$\mathbb{A}\vec{x}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \Rightarrow \mathbb{A}\begin{bmatrix} | & | & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ | & | & | \end{bmatrix} \leftarrow \text{so must have } \mathbb{A}^{-1} \text{ here}$$

← generalizes to n×n

$$\mathbb{A}\vec{x}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad = \begin{bmatrix} | & | & | \\ \mathbb{A}\vec{x}_1 & \mathbb{A}\vec{x}_2 & \mathbb{A}\vec{x}_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \mathbb{I}.$$

Use Cramer's rule and work with example $\mathbb{A} = \begin{bmatrix}1&1&1\\0&3&2\\2&1&2\end{bmatrix}$:

Solve $\mathbb{A}\vec{x} = \overset{\vec{b}}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$ first with $\vec{x} = \frac{1}{|\mathbb{A}|}\begin{bmatrix}|\mathbb{B}_1|\\|\mathbb{B}_2|\\|\mathbb{B}_3|\end{bmatrix}$

$$\mathbb{B}_1 = \begin{bmatrix}1&1&1\\0&\boxed{3}&\boxed{2}\\0&\boxed{1}&\boxed{2}\end{bmatrix}, \quad \mathbb{B}_2 = \begin{bmatrix}1&1&1\\\boxed{0}&0&\boxed{2}\\\boxed{2}&0&\boxed{2}\end{bmatrix}, \quad \mathbb{B}_3 = \begin{bmatrix}1&1&1\\\boxed{0}&\boxed{3}&0\\\boxed{2}&\boxed{1}&0\end{bmatrix}$$

$\uparrow \mathbb{M}_{11}$ $\quad \uparrow \mathbb{M}_{12}$ $\quad \uparrow \mathbb{M}_{13}$

$|\mathbb{B}_1| = C_{11}$ $\quad |\mathbb{B}_2| = C_{12}$ $\quad |\mathbb{B}_3| = C_{13}$

← cofactors of $\mathbb{A}$! →

$$\begin{bmatrix}1\\0\\0\end{bmatrix}: \quad \vec{x}_1 = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{11}\\C_{12}\\C_{13}\end{bmatrix} \leftarrow \text{top row of } \mathbb{C}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}: \quad \vec{x}_2 = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{21}\\C_{22}\\C_{23}\end{bmatrix} \leftarrow \text{middle row of } \mathbb{C}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}: \quad \vec{x}_3 = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{31}\\C_{32}\\C_{33}\end{bmatrix} \leftarrow \text{bottom row of } \mathbb{C}$$

Combine: $\mathbb{A}^{-1} = \begin{bmatrix} | & | & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ | & | & | \end{bmatrix} = \frac{1}{|\mathbb{A}|}\begin{bmatrix}C_{11}&C_{21}&C_{31}\\C_{12}&C_{22}&C_{32}\\C_{13}&C_{23}&C_{33}\end{bmatrix}$

see transpose of $\mathbb{C}$

$$\Rightarrow \boxed{\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|}\mathbb{C}^T}$$

#inconceivable

Using earlier calculations

$$\mathbb{A}^{-1} = \underset{\uparrow\, |\mathbb{A}|}{\frac{1}{2}}\begin{bmatrix}4&-1&-1\\4&0&-2\\-6&1&3\end{bmatrix} \leftarrow \mathbb{C}^T$$

Check

$$\frac{1}{2}\begin{bmatrix}4&-1&-1\\4&0&-2\\-6&1&3\end{bmatrix}\begin{bmatrix}1&1&1\\0&3&2\\2&1&2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix} = \mathbb{I} \quad ✓$$

E18ep1

Algebraic & Geometric Multiplicity of eigenvalues

"Some matrices are bad matrices"
— traditional matrix-fu saying.

ex

$$\mathbb{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$(4-\lambda)(7-\lambda)^2 = 0$

Solve $\mathbb{A}\vec{v} = \lambda\vec{v}$:

$\lambda_1 = 4, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\boxed{\begin{matrix}\lambda_2 = 7 \\ \lambda_3 = 7\end{matrix}}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

↑ y-z plane

$(\mathbb{A} - 7\mathbb{I})\vec{v} = \vec{0}$ gives a plane of vectors.

↑

$\dim N(\mathbb{A} - 7\mathbb{I}) = 2$

Defn: Algebraic Multiplicity is # times an eigenvalue appears as a root of $|\mathbb{A} - \lambda\mathbb{I}| = 0$

↖ characteristic equation of $\mathbb{A}$

Defn: Geometric Multiplicity is the dimension of the eigenspace associated with an eigenvalue λ

$\Leftrightarrow \dim N(\mathbb{A} - \lambda\mathbb{I})$

ex a.m. of $\lambda = 4$ is 1, g.m. is 1

a.m. of $\lambda = 7$ is 2, g.m. is 2 ← healthy

Observation $1 \leq g.m. \leq a.m.$

ex

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Find eigenvalues: Solve $|\mathbb{A} - \lambda\mathbb{I}| = 0$

$$0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = \underline{(1-\lambda)}^3 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$$

$\Rightarrow \lambda = 1$ has algebraic multiplicity of $\underline{3}$.

Find evectors: Solve $(\mathbb{A} - \overset{\lambda}{(1)}\mathbb{I})\underline{\underline{\vec{v}}} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ P & P & F & \end{array}\right] \Rightarrow \vec{v} = c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ only}$$

↑ 1-d $c \in \mathbb{R}$.

$\Rightarrow \lambda = 1$ has geometric multiplicity of 1

* $N(\mathbb{A} - \lambda\mathbb{I})$ not big enough... $\underline{\dim = 1}$

* $\mathbb{A}$ is a bad matrix...
and does not have a full complement of eigenvectors

↑ basis for eigenspace

Sneaky Monk Tricks (SMTs) for Eigenstuff:

All about $\underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}} = \underset{\#}{\lambda}\,\underset{n\times 1}{\vec{v}}$

Recap: Solve by

① Finding λs as roots of $|A-\lambda I| = 0$ Characteristic Equation

use cofactor method

&

② For each distinct λ, solving the nullspace equation $(A-\lambda I)\vec{v} = \vec{0}$ for λ's eigenspace

Our helper example:

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \qquad \lambda_1 = \frac{3}{2} > 1 \qquad \vec{v}_1 \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} < 1 \qquad \vec{v}_2 \propto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

symmetry will be meaningful

SMT #1

$$|A| = \prod_{i=1}^{n} \lambda_i$$

The determinant of A is equal to the product of its eigenvalues

E19ap1

Check:

$$|A| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = 1\cdot 1 - \frac{1}{2}\cdot\frac{1}{2} = 3/4$$

$$\lambda_1 \cdot \lambda_2 = \frac{3}{2}\cdot\frac{1}{4} = 3/4 \quad ✓$$

general characteristic equation

ex $\left(\frac{3}{2}-\lambda\right)\left(\frac{1}{2}-\lambda\right)$

Why? $|A - \cancel{\lambda I}| = (\underset{\#}{\lambda_1} - \cancel{\lambda})(\underset{\#}{\lambda_2} - \cancel{\lambda}) \cdots (\lambda_n - \cancel{\lambda})$

set $\lambda = 0$

$$\Rightarrow |A| = \prod_{i=1}^{n} \lambda_i$$

from before: $|A| = \pm \prod_{i=1}^{n} d_i$ ← pivots

SMT #2

Defn Trace of $A = \mathrm{Tr}(A)$ = sum A's main diagonal elements $= \sum_{i=1}^{n} a_{ii}$

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$$

Our example: $\mathrm{Tr}\left(\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\right) = 1 + 1 = 2$ ✓

$$\sum_{i=1}^{n} \lambda_i = \frac{3}{2} + \frac{1}{2} = 2$$

General 2x2

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (\lambda_1 - \lambda)(\lambda_2 - \lambda)$$

$$= (a-\lambda)(d-\lambda) - b.c$$

$$(-\lambda)^2 + (a+d)(-\lambda) + ad - bc$$

$$= (-\lambda)^2 + (\lambda_1 + \lambda_2)(-\lambda) + \lambda_1\lambda_2$$

matching

$\lambda_1 + \lambda_2 = a + d = \mathrm{Tr}(A)$

$\lambda_1 \lambda_2 = ad - bc = |A|$

General $n\times n$:

$$|\mathbb{A}-\lambda\mathbb{I}| = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

$$= (-\lambda)^n + \left(\sum_{i=1}^{n}\lambda_i\right)(-\lambda)^{n-1} + \cdots$$

$$= (-\lambda)^n + (\mathrm{Tr}\,\mathbb{A})(-\lambda)^{n-1} + \cdots$$

One thing: Check $\underbrace{\mathrm{Tr}}_{\text{easy}}(\mathbb{A}) = \sum_{i=1}^{n}\lambda_i$

SMT3

$$|\mathbb{A}| = \underset{\substack{\uparrow\\ \text{depends on row swaps required to uncover } \mathbb{U}}}{\pm}\prod_{i=1}^{n} d_i = \prod_{i=1}^{n}\lambda_i$$

SMT4

If $\overset{n\times n}{\mathbb{A}}\vec{v} = \lambda\vec{v}$ then $\mathbb{A}^k\vec{v} = \lambda^k\vec{v}$

$$\mathbb{A}^k\vec{v} = \mathbb{A}^{k-1}(\mathbb{A}\vec{v}) = \lambda\mathbb{A}^{k-1}\vec{v} = \cdots = \lambda^k\vec{v}$$

SMT5

If $\mathbb{A}\vec{v} = \lambda\vec{v}$ then $(\mathbb{A}+t\mathbb{I})\vec{v} = (\lambda+\underbrace{t}_{\text{shift}})\vec{v}$

$$(\mathbb{A}+t\mathbb{I})\vec{v} = \mathbb{A}\vec{v} + t\mathbb{I}\vec{v} = \lambda\vec{v} + t\vec{v} = (\lambda+t)\vec{v}$$

SMT6

If $\mathbb{A}\vec{v} = \lambda\vec{v}$ then $\mathbb{A}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$

if $\mathbb{A}^{-1}$ exists

$$\underbrace{\cancel{\mathbb{A}^{-1}\mathbb{A}}}^{\mathbb{I}}\vec{v} = \lambda\mathbb{A}^{-1}\vec{v}$$

$$\mathbb{A}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$

matches SMT4 for $k=-1$

E19ap2

SMT7

If $\mathbb{A}$'s eigenvalues are all different from each other then $\mathbb{A}$'s eigenvectors are linearly independent and form a basis for $\mathbb{R}^n$.

Reason:

Assume λ's are distinct and look at $\vec{v}_1$ & $\vec{v}_2$

If dependent, $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ for some $c_1, c_2 \neq 0$

(a) $\mathbb{A}(c_1\vec{v}_1 + c_2\vec{v}_2) = \mathbb{A}\vec{0}$

$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$ ...... (1)

(b) $\lambda_2\times(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda_2\times\vec{0}$

$c_1\lambda_2\vec{v}_1 + c_2\lambda_2\vec{v}_2 = 0$ .... (2)

(2) − (1); $c_1(\lambda_2-\lambda_1)\vec{v}_1 = \vec{0}$

$\Rightarrow \lambda_1 \neq \lambda_2$

] build up from here for $n\times n$'s

SMT8

eigenvalues & eigenvectors of $\mathbb{A}$ & $\mathbb{B}$ are not simply related to those of $\mathbb{A}\mathbb{B}$ & $\mathbb{A}+\mathbb{B}$

- If $\mathbb{A}$ & $\mathbb{B}$ share an eigenvector $\vec{v}$ with eigenvalues $\lambda_{\mathbb{A}}$ & $\lambda_{\mathbb{B}}$ then $(\mathbb{A}+\mathbb{B})\vec{v} = (\lambda_{\mathbb{A}}+\lambda_{\mathbb{B}})\vec{v}$ & $(\mathbb{A}\mathbb{B})\vec{v} = \lambda_{\mathbb{A}}\lambda_{\mathbb{B}}\vec{v}$

but this is generally not the case.

Why diagonal matrices make us happy

E20ap1

ex

$$\mathbb{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 17 \end{bmatrix} \Rightarrow \mathbb{A}^k = \begin{bmatrix} 3^k & 0 & 0 \\ 0 & (-7)^k & 0 \\ 0 & 0 & (17)^k \end{bmatrix}$$

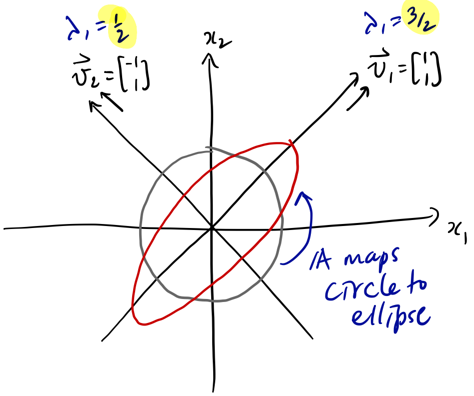
$$\mathbb{A}\vec{x} = \begin{bmatrix} 3x_1 \\ -7x_2 \\ 17x_3 \end{bmatrix}$$

how A changes $\vec{x}$ is simple

$\lambda_1 = 3$, $\lambda_2 = -7$, $\lambda_3 = 17$

$$\underbrace{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{natural or standard basis for } \mathbb{R}^3}$$

$$\mathbb{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$



- If we could rotate the axes, A's action would be simple
- Big idea: change from standard basis to eigenvector basis and find happiness

Diagonalization is just the best

- Let's assume $\underset{n\times n}{A}$ has n $\underbrace{\text{linearly independent eigenvectors}}_{\text{form a basis for } R^n}$

- know $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i=1,\dots,n$

Monks whisper

Create a new matrix with A's eigenvectors:

$$\underset{n\times n}{S} = \underset{n\times n.}{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}}$$

Consider:

$$\underset{n\times n}{A}\,\underset{n\times n}{S} \overset{\text{matrix-fu}}{=} \begin{bmatrix} | & | & & | \\ \underbrace{\underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}_1}}_{n\times 1} & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix} \overset{\text{sneaky}}{=} \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= \underset{n\times n}{S}\,\underset{n\times n}{\Lambda} \leftarrow \text{capital } \lambda$$

Let's assume A is a good matrix meaning its eigenvectors form a basis for R^n

E20bp1

$\Leftrightarrow S^{-1}$ exists

$$\Rightarrow A\underset{\substack{\uparrow \\ \text{postmultiply} \\ \text{by } S^{-1}}}{S} = S\Lambda \Rightarrow$$

Diagonalization:

$$\boxed{\underset{n\times n}{A} = \underset{n\times n}{S}\,\underset{n\times n}{\Lambda}\,\underset{n\times n}{S^{-1}}}$$

↑ an amazing factorization

- We say A and Λ are similar ↑ similarity transformation

- We begin to see how $A\vec{x}$ works:

more soon

$$A\vec{x} = \underset{\substack{③ \text{ changes back to} \\ \text{standard basis} \\ \text{representation}}}{S}\;\underset{\substack{② \text{ simple} \\ \text{multiplication} \\ \text{because } \Lambda \\ \text{is diagonal}}}{\Lambda}\;\underset{\substack{① \text{ changes representation} \\ \text{of } \vec{x} \text{ from standard basis} \\ \text{to A's eigenvector basis}}}{S^{-1}}\vec{x}$$

Big deal: If A is diagonalizable,
then A is really a diagonal
matrix when viewed in
the right way.

Example: $\mathbb{A} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ note: symmetry

$\lambda_1 = \frac{3}{2}$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{1}{2}$ $\quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

our choice (any multiple would work)

$$\Rightarrow S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

note: $S^{-1} = S^T$

$$\Rightarrow S^{-1} = \underset{|S|=2}{\frac{1}{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

use $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\Rightarrow \mathbb{A} = S \Lambda S^{-1}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

E206p2

Let's see how this is super useful for ① $\mathbb{A}\vec{x}$ & ② $\mathbb{A}^k$

① Examine what happens for $\vec{x} = 2\vec{v}_1 + 2\vec{v}_2$

$$\vec{x} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$\mathbb{A}\vec{x}$ in 3 ways $\Downarrow$

(i) $\mathbb{A}\vec{x} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ← no great understanding

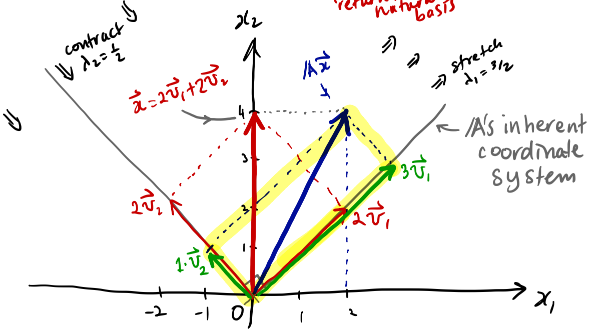
(ii) $\mathbb{A}\vec{x} = \mathbb{A}\Big(\overbrace{2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}}^{\text{may not know this}}\Big) = \frac{3}{2}\cdot 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\cdot 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ✓

(iii) $\mathbb{A}\vec{x} = S\Lambda S^{-1}\vec{x} = S\Lambda\left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right) = S\Lambda \underset{\text{representation of } \vec{x} \text{ in eigenvector basis}}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$

$= S\begin{bmatrix} 3/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = S \underset{\text{in eigenvector basis still}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ✓

return to natural basis



E206p3

② A^k for $k = 0, \pm 1, \pm 2, \ldots$

$$A^2 = (S\Lambda \underbrace{S^{-1})(S}_{I}\Lambda S^{-1}) = S\Lambda^2 S^{-1}$$

$$A^3 = (S\Lambda \underbrace{\cancel{S^{-1})(S}}_{I}\Lambda \underbrace{\cancel{S^{-1})(S}}_{I}\Lambda S^{-1})$$

$$= S\Lambda^3 S^{-1}$$

super easy!!

$$\boxed{A^k = S\Lambda^k S^{-1}}$$

Super easy to compute!

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ & & \ddots \\ 0 & & & \lambda_n^k \end{bmatrix}$$

→ clearly important

$|\lambda_i| < 1$

$|\lambda_i| = 1$

$|\lambda_i| > 1$

can see that largest eigenvalue will dominate Λ^k and A^k

Ex/ $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}^{523}$

$$= \overset{S}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}} \overset{\Lambda^{523}}{\begin{bmatrix} \overset{\text{blows up}}{(3/2)^{523}} & 0 \\ 0 & (\frac{1}{2})^{523} \to 0 \end{bmatrix}} \frac{1}{2} \overset{S^{-1}}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$$

$$\simeq \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^{523} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

More goodness;

$2^0 = 1$

- $A^0 = S\Lambda^0 S^{-1} = S I S^{-1} = I$
- $A^{-1} = S\Lambda^{-1}S^{-1}$ works:

$$(S\overset{A^{-1}}{\Lambda^{-1}}\underbrace{S^{-1})(S}_{I}\overset{A}{\Lambda}S^{-1}) = I$$

$A^{\frac{1}{2}} = S\Lambda^{1/2}S^{-1}$ works too!!!

$A^{1/2}A^{1/2} = A^1$ ✓

Fibonacci number finder

note: clearly a monk

E20cp1

From before:

$$F_{k+2} = F_{k+1} + F_k \quad \text{with} \quad F_0 = F_1 = 1$$

← Fibonacci sequence

$$\vec{f}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \vec{f}_0$$

note symmetry...

$\vec{f}_{k-1}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Mission: Diagonalize $\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

usual thing

$$|\mathbb{A} - \lambda \mathbb{I}| = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$$

← very famous

Find $\lambda_1 = \frac{1+\sqrt{5}}{2} = \rho \Rightarrow \vec{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ ← good hygiene

← golden ratio

$\lambda_2 = \frac{1-\sqrt{5}}{2} \qquad \vec{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ ← keep $\sqrt{5}$'s at bay

Note: $\lambda_1 + \lambda_2 = 1 = \mathrm{Tr}(\mathbb{A})$

$\lambda_1 \lambda_2 = -1 = |\mathbb{A}|$

Three pieces:

$$\mathbb{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\mathbb{S} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$$

$$\mathbb{S}^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

aside

$$\rho = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}$$

← most irrational number

continued fraction

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k}_{\mathbb{A}} = \underbrace{\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}}_{\mathbb{S}} \underbrace{\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}}_{\mathbb{\Lambda}^k} \underbrace{\frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}}_{\mathbb{S}^{-1}}$$

← grows

decays

So:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} (\lambda_1^{k+1} - \lambda_2^{k+1}) \\ (\lambda_1^k - \lambda_2^k) \end{bmatrix}$$

2x1 (not a 2x2)

$$F_k = \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^k - \lambda_2^k)$$

$$= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right)$$

← disappears

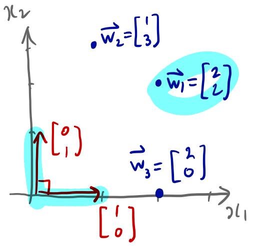
dominates

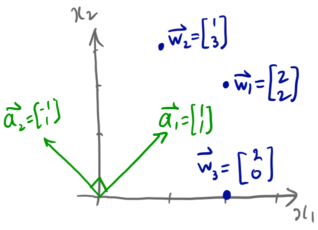
#delicious

Cool beans; $\frac{F_{k+1}}{F_k} \rightarrow \frac{1+\sqrt{5}}{2} = \rho$ as $k \rightarrow \infty$

$= \lambda_1$

The gentle art of changing basis:





So far, we've expressed all vectors in terms of the standard (or natural) basis.

ex $\vec{w}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- What's the representation of $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ in terms of the new basis $\{\vec{a}_1, \vec{a}_2\}$?
- How do we do this systematically?

By solving an $\mathbb{A}\vec{x} = \vec{b}$ problem!!!

The set up for $\vec{w}_1$:

$$\vec{w}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \underline{c_1}\,\vec{a}_1 + \underline{c_2}\,\vec{a}_2 = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

↑ in natural basis

← column picture

← coordinates of $\vec{w}_1$ in $\{\vec{a}_1, \vec{a}_2\}$ basis

$$= \mathbb{A}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \equiv \mathbb{M}\vec{w}_1^{(a)}$$

← often used for basis transformation

$$\Rightarrow \vec{w}_1^{(a)} = \mathbb{M}^{-1}\vec{w}_1$$

← inverse takes us from natural to new basis

E21ap1

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \underline{\tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

↑ $\vec{w}_1$ in $\{\vec{a}_1, \vec{a}_2\}$ basis.

Similarly: $\vec{w}_2^{(a)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{w}_3^{(a)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To change back: $\vec{w}_i = \mathbb{M}\vec{w}_i^{(a)}$

We say:

In basis $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, $\vec{w}_1$ is represented as $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

In basis $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, $\vec{w}_1$ is represented as $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

specify same point/vector in space

Big deal: The vector $\vec{w}_1$ never changes but our representation does.

Big deal:

$$\mathbb{A} = \overset{\mathbb{M}}{\mathbb{S}}\,\overset{\text{does the real work}}{\Lambda}\,\overset{\mathbb{M}^{-1}}{\mathbb{S}^{-1}}$$

$\mathbb{S}$: change basis only (from eigenvector to normal)

Λ: "change" vector

$\mathbb{S}^{-1}$: ← change basis only (from normal to eigenvector)

Symmetry and the Spectral Theorem

We know:

- Diagonalization is joyous and empowering
- $A_{n \times n}$ can only be diagonalized if it has n linearly independent eigenvectors
- Trouble potentially arises when eigenvalues are repeated (algebraic multiplicity > 1)
  - ↳ May not end up with a full eigenspace

Bonus truths:

- If one or more eigenvalues $= 0$, A^{-1} does not exist
  - ↳ $|A| = 0$
- But A may still be diagonalizable
  - ↳ depends on eigenvectors

An amazing matrix truth:

If $A_{n \times n}$ is real & symmetric, i.e. $A = A^T$, then (+ a_{ij} is real for all i,j)

A always has n linearly independent eigenvectors and is therefore always diagonalizable

even better

① All of A's eigenvalues are real (no complex numbers ⇒ no rotations)

② A's eigenvectors form an orthogonal basis for R^n !!!

proofs later

Ezzap1

We get so excited, we replace $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$ with $Q = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \end{bmatrix}$

unit vectors

because we realize we have an orthogonal matrix.

And because $Q^{-1} = Q^T$ ← saves a lot of trouble, our diagonalization takes on a new level of majesty:

$$\boxed{A = Q \Lambda Q^T}$$

Wow!!

More amazingness:

$$A = Q \Lambda Q^T = \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} \begin{bmatrix} - & \hat{v}_1^T & - \\ - & \hat{v}_2^T & - \\ & \vdots & \\ - & \hat{v}_n^T & - \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & \lambda_1 \hat{v}_1^T & - \\ - & \lambda_2 \hat{v}_2^T & - \\ & \vdots & \\ - & \lambda_n \hat{v}_n^T & - \end{bmatrix}$$

$A\vec{x}$ broken into clean pieces

$$= \lambda_1 \hat{v}_1 \hat{v}_1^T + \lambda_2 \hat{v}_2 \hat{v}_2^T + \cdots + \lambda_n \hat{v}_n \hat{v}_n^T = \sum_{i=1}^{n} \lambda_i \hat{v}_i \hat{v}_i^T$$

outer products

projection operators!!

for $A\vec{x}$, each one chops out a piece of $\vec{x}$ and then scales by λ.

Spectral Theorem for Symmetric Matrices

E22ap2

Example:

$$\mathbb{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} = \mathbb{A}^T$$

← symmetric!

Use unit vectors for eigenvectors

$$\mathbb{S} = \mathbb{Q} = \begin{bmatrix} | & | \\ \hat{v}_1 & \hat{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbb{S}^{-1} = \mathbb{Q}^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

take transpose! easy!

$$\mathbb{A} = \overset{\mathbb{S}}{\mathbb{Q}}\,\overset{\Lambda}{\Lambda}\,\overset{\mathbb{S}^{-1}}{\mathbb{Q}^T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$\mathbb{A} = \sum_{i=1}^{n} \lambda_i \hat{v}_i \hat{v}_i^T$$

$$= \underset{\lambda_1}{\left(\frac{3}{2}\right)} \underset{\hat{v}_1}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \underset{\hat{v}_1^T}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}} + \underset{\lambda_2}{\left(\frac{1}{2}\right)} \underset{\hat{v}_2}{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \underset{\hat{v}_2^T}{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \end{bmatrix}}$$

$\mathbb{A}\vec{x}$: breaks $\vec{x}$ into two orthogonal pieces for which $\mathbb{A}\hat{v}_1$ & $\mathbb{A}\hat{v}_2$ are very simple. and then recombines.

Why the spectral theorem works

① All of A's eigenvalues are real

no rotations

Assume $A = A^T$ and A's entries are real

Given $A\vec{v} = \lambda\vec{v}$, we test to see if λ can be complex: $\lambda = a + bi$ $\quad b \neq 0$

$i = \sqrt{-1}$

Denote complex conjugate by over bar:

Result: $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ $\qquad \overline{a+bi} = a - bi$

$$\underset{\underset{n\times n}{[\ ]}}{A}\,\underset{[]}{\vec{v}} = \underset{\#}{\lambda}\,\underset{[]}{\vec{v}} \xrightarrow{\text{monks}} \overline{A}\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

↓ real

$$A\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

take transpose

$$\overline{\vec{v}}^T A^T = \overline{\lambda}\,\overline{\vec{v}}^T$$

↓ symmetry

$$\overline{\vec{v}}^T A = \overline{\lambda}\,\overline{\vec{v}}^T$$

↓ postmultiply by $\vec{v}$

$$\overline{\vec{v}}^T A \vec{v} = \overline{\lambda}\,\overline{\vec{v}}^T\vec{v}$$

Monk sneakiness

monks ↓ premultiply by $\overline{\vec{v}}^T$

$$\overline{\vec{v}}^T A \vec{v} = \lambda\,\overline{\vec{v}}^T\vec{v}$$

See everything matches except λ & $\overline{\lambda}$

$\Rightarrow$ $\lambda = \overline{\lambda}$ so λ is real

E22 bp1

② A's eigenvectors form an orthogonal basis for $\mathbb{R}^n$

crazy!

- Again have $A = A^T$ and A is real
- We want to show $\vec{v}_i^T\vec{v}_j = 0$ if $i \neq j$
- Work up to full story...

First If A's eigenvalues are all distinct (i.e., each has algebraic multiplicity 1):

$$\underset{1\times n}{\vec{v}_1^{\,T}}\,\underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}_2} = \vec{v}_1^{\,T}(A\vec{v}_2)$$

a number

More Monk Sneakiness

$$\vec{v}_1^{\,T}A\vec{v}_2 = (A^T\vec{v}_1)^T\vec{v}_2 = \lambda_1\underbrace{\vec{v}_1^{\,T}\vec{v}_2}$$

$A^T = A$

$$\vec{v}_1^{\,T}(A\vec{v}_2) = \vec{v}_1^{\,T}(\lambda_2\vec{v}_2) = \lambda_2\underbrace{\vec{v}_1^{\,T}\vec{v}_2}$$

what we're interested in...

$$\lambda_1\vec{v}_1^{\,T}\vec{v}_2 = \lambda_2\vec{v}_1^{\,T}\vec{v}_2$$

but $\lambda_1 \neq \lambda_2$ so these can only be equal if $\vec{v}_1^{\,T}\vec{v}_2 = 0$

OK

- What if an eigenvalue is repeated?
- We're worried we won't have enough eigenvectors....

E22bp2

A suggestive pair of examples:

Not symmetric:

$$\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \mathbb{A}^T$$

$\lambda_1 = \lambda_2 = 1$ repeated

only one dimension for eigenspace

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ #sadness

Symmetric:

$$\mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{A}^T$$

$\lambda_1 = \lambda_2 = 1$ repeated

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2-d eigenspace healthy

Tweaks:

$$\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{bmatrix} \neq \mathbb{A}^T$$

small

now distinct

$\lambda_1 = 1, \lambda_2 = 1+\epsilon$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓, $\vec{v}_2 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$ ✓

See as $\epsilon \to 0$, eigenvectors become the same

$$\mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1+\epsilon \end{bmatrix} = \mathbb{A}^T$$

distinct

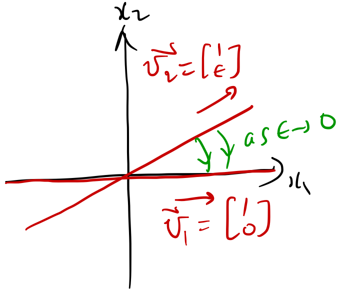
$\lambda_1 = 1, \lambda_2 = 1+\epsilon$

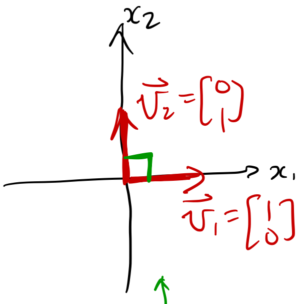
$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

as before

$\vec{v}_1^{\,T} \vec{v}_2 = 0$

Eigenvectors do not budge as $\epsilon \to 0$





true for all $\epsilon \neq 0$

Idea: Smooth change in tweaks ($\epsilon \to 0$) cannot lead to eigenvectors snapping into orthogonal directions

⇒ orthogonality is preserved

//.

Requires more work to show in general but we have the basic story here.

Surprising things about traces

E22c p1

Defn Trace of $\underset{n\times n}{\mathbb{A}} = \operatorname{Tr}(\mathbb{A})$

= Sum of the entries of $\mathbb{A}$'s main diagonal

$= \sum_{i=1}^{n} a_{ii}$

ex $\operatorname{Tr}\left(\begin{bmatrix} 3 & 0 & 2 \\ 2 & -1 & -1 \\ 1 & 2 & 4 \end{bmatrix}\right) = 3 + (-1) + 4 = 6$

From earlier: $\operatorname{Tr}(\mathbb{A}) = \sum_{i=1}^{n} \lambda_i$

Now, two more things:

(1) $\operatorname{Tr}(\underset{n\times n}{\mathbb{A}}\ \underset{n\times n}{\mathbb{B}}) = \operatorname{Tr}(\underset{n\times n}{\mathbb{B}}\ \underset{n\times n}{\mathbb{A}})$

Reason: $\operatorname{Tr}(\mathbb{A}\mathbb{B}) = \sum_{i=1}^{n} (\mathbb{A}\mathbb{B})_{ii}$

$= \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} a_{ij} b_{ji}}_{\text{inner product of } i\text{th row of } \mathbb{A} \text{ \& } j\text{th column of } \mathbb{B}}$ ← from defn of multiplication

swap everything →

$= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} = \operatorname{Tr}(\mathbb{B}\mathbb{A})$

Generalizes:

$\operatorname{Tr}(\mathbb{A}\mathbb{B}\mathbb{C}) = \operatorname{Tr}((\mathbb{A}\mathbb{B})(\mathbb{C})) = \operatorname{Tr}(\mathbb{C}(\mathbb{A}\mathbb{B}))$ ← two matrices

$= \operatorname{Tr}((\mathbb{C}\mathbb{A})(\mathbb{B})) = \operatorname{Tr}((\mathbb{B})(\mathbb{C}\mathbb{A}))$

↑ any cycling leaves Trace unchanged

(2) If $\mathbb{A} = \mathbb{S}\Lambda\mathbb{S}^{-1}$ ← sadly not possible for all matrices

then $\operatorname{Tr}(\mathbb{A}) = \operatorname{Tr}(\mathbb{S}\Lambda\mathbb{S}^{-1})$ ← cycle to front

$= \operatorname{Tr}(\underbrace{\mathbb{S}^{-1}\mathbb{S}}_{\mathbb{I}}\Lambda) = \operatorname{Tr}(\Lambda)$

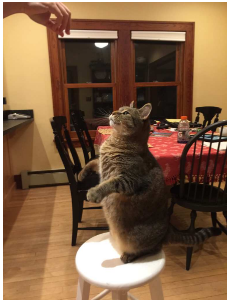
$= \sum_{i=1}^{n} \lambda_i$

#delicious

So: a very enjoyable proof of $\operatorname{Tr}(\mathbb{A}) = \sum_{i=1}^{n} \lambda_i$

(but does not work if $\mathbb{A}$ is not diagonalizable)

Pratchett does tricks for treats...



Positive Definite Matrices

matrices that are really sure about themselves

defn: A Positive Definite Matrix is a real, symmetric matrix with positive eigenvalues, i.e., $\lambda_i > 0, i = 1, \dots, n$

If a matrix is real and symmetric with $\lambda_i \geq 0$ and at least one eigenvalue equal to zero, then we say it is Semi-Positive Definite

We recall with alacrity that real, symmetric matrices always have

$A = Q \Lambda Q^T$

(1) Real eigenvalues $\leq$ flipping stretching & shrinking

(2) Eigenvectors that form an orthonormal basis for R^n

Turns out that also having $\lambda_i > 0$ or $\lambda_i \geq 0$ is an excellent bonus feature ...

Menu
- i/ How to spot a PDM ← surprising results from monks pivots ⟺ eigenvalues
- ii/ Why we like PDMs (and SPDMs)

Places we'll go, things we'll see:
- $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1 \Rightarrow$ matrices
- what elimination really does for symmetric matrices
- Completing the square

Three example 2x2 matrices:

$$\mathbb{A}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}; \quad \mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}; \quad \mathbb{A}_3 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

$\lambda_1 = +3$
$\lambda_2 = +1$

computing happens elsewhere

PDM ☺

$\lambda_1 = \sqrt{5}$
$\lambda_2 = -\sqrt{5}$

☹

$\lambda_1 = -1$
$\lambda_2 = -3$

☹

Problem: Finding eigenvalues can be pretty hard for real matrices

- We only want to know signs of the eigenvalues
- Could there be a sneaky way?

especially one that helps computers

SMT #37

If $\underset{n\times n}{\mathbb{A}} = \mathbb{A}^T$ & $\mathbb{A}$ is real then:

\# positive eigenvalues = # positive pivots
\# negative eigenvalues = # negative pivots
\# zero eigenvalues = zero pivots

↑
\#crazytownbananapants

- Very peculiar: eigenvalues and pivots come from very different parts of matrixology
- Recall we already know for general $\underset{n\times n}{\mathbb{A}}$ that $|\mathbb{A}| = \prod_{i=1}^{n} \lambda_i = \pm \prod_{i=1}^{n} d_i$
- SMT #37 says more for real symmetric matrices

Big deal: $\mathbb{A}$ is a PDM if all $d_i > 0$
Pivots are much easier to compute than eigenvalues

E23ap2

Beautiful reason:

Consider $\mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$ $\lambda_1 = \sqrt{5}$ $\lambda_2 = -\sqrt{5}$

find pivots using $\mathbb{L}\mathbb{U}$ decomposition

$R_2' = R_2 - \underset{l_{21}}{\left(-\frac{1}{2}\right)} R_1$

$$\mathbb{A}_2 = \mathbb{L}\mathbb{U} = \overset{\mathbb{L}}{\begin{bmatrix} 1 & 0 \\ \underset{l_{21}}{\boxed{-\frac{1}{2}}} & 1 \end{bmatrix}} \overset{\mathbb{U}}{\begin{bmatrix} \overset{d_1}{2} & -1 \\ 0 & \underset{d_2}{-5/2} \end{bmatrix}}$$

Because $\mathbb{A}_2$ is symmetric, we can go further:
$(\mathbb{A}_2 = \mathbb{A}_2^T)$

$$\mathbb{A}_2 = \mathbb{L}\mathbb{D}\mathbb{L}^T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Let's think about this parametrized matrix:

$$\mathbb{B}(l_{21}) = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & l_{21} \\ 0 & 1 \end{bmatrix}$$

When $l_{21} = -\frac{1}{2}$, we have $\mathbb{B}(-\frac{1}{2}) = \mathbb{A}_2$

Next
What happens as we move from $l_{21} = -\frac{1}{2}$ to $l_{21} = 0$?

$$\mathbb{B}(0) = \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\mathbb{I}}\underbrace{\begin{bmatrix}2 & 0\\ 0 & -\frac{5}{2}\end{bmatrix}}_{\mathbb{D}}\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\mathbb{I}} = \begin{bmatrix}\overset{d_1}{2} & 0\\ 0 & \underset{d_2}{-\frac{5}{2}}\end{bmatrix} = \mathbb{D}$$

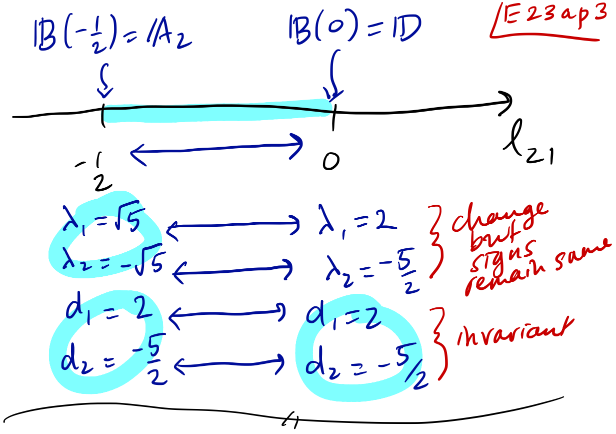
Observations:

- For diagonal matrices, pivots $\equiv$ eigenvalues
- $\mathbb{B}(\ell_{21})$ has pivots $d_1 = 2$ and $d_2 = -\frac{5}{2}$
  independent of ℓ_{21}
- $\det(\mathbb{B}(\ell_{21})) = d_1 \cdot d_2 = (2)(-\frac{5}{2}) = -5$
  again independent of ℓ_{21}

Big connections:

- We also know $\det(\mathbb{B}(\ell_{21})) = \overset{\sqrt{5}}{\lambda_1} \cdot \overset{-\sqrt{5}}{\lambda_2}$ for $\ell_{21} = -\frac{1}{2}$ must $= -5$ for all ℓ_{21}
- As ℓ_{21} changes, the eigenvalues change BUT they cannot pass through 0 as then the determinant would be 0 $(\neq -5)$
- When $\ell_{21} = 0$, $\mathbb{B}(0) = \mathbb{D}$ is diagonal and the pivots and eigenvalues match up: $d_1 \equiv \lambda_1$, $d_2 \equiv \lambda_2$
- Therefore as ℓ_{21} moves away from 0, the eigenvalues must maintain the same signs as the pivots
- Argument assumes all pivots $\neq 0$; proof is tweakable

E23ap3



General argument:

Given $\underset{n\times n}{\mathbb{A}} = \mathbb{L}\mathbb{D}\mathbb{L}^T$ create $\hat{\mathbb{L}}(t) = \mathbb{I} + t(\mathbb{L} - \mathbb{I})$

$t = 0$: $\hat{\mathbb{L}}(0) = \mathbb{I}$

$t = 1$: $\hat{\mathbb{L}}(1) = \mathbb{L}$

$$\left[\begin{array}{l}\mathbb{B}(t) = \hat{\mathbb{L}}(t)\,\mathbb{D}\,\hat{\mathbb{L}}^T(t)\\ \mathbb{B}(0) = \mathbb{D} \quad \& \quad \mathbb{B}(1) = \mathbb{A}\end{array}\right.$$

- As before, pivots don't change as we vary t from 1 to 0
- Same story: Eigenvalues cannot change sign as t varies
- Signs of eigenvalues must match signs of pivots

Positive Definite Matrices in the Wild:

Menu for 23b,c,d:

- $\vec{x}^T \mathbb{A} \vec{x}$ and ellipses and other functions
- Completing the Square
- Cholesky factorization

Idea: re-express polynomial functions using matrices.

← especially PDMs

Key construct: $\vec{x}^T \mathbb{A} \vec{x}$ where $\mathbb{A} = \mathbb{A}^T$

$$\underbrace{\underset{1\times n}{[\quad]}\underset{n\times n}{\begin{bmatrix} & \\ & \end{bmatrix}}\underset{n\times 1}{\begin{bmatrix} \\ \end{bmatrix}}}_{1\times 1 \leftarrow \text{a scalar}}$$

$$= (\mathbb{A}^T\vec{x})^T\vec{x}$$
$$= (\mathbb{A}\vec{x})^T\vec{x}$$

General 2×2 example:

$$\vec{x}^T \mathbb{A} \vec{x} = \underset{1\times 2}{[x_1 \; x_2]} \underset{2\times 2}{\begin{bmatrix} a & b \\ b & c \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

← $\mathbb{A} = \mathbb{A}^T$

$$= \underset{1\times 2}{[x_1 \; x_2]} \underset{2\times 1}{\begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}}$$

inner product

$$= ax_1^2 + bx_1x_2 + bx_1x_2 + cx_2^2$$

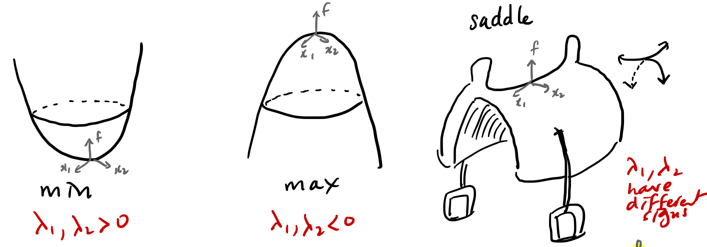
$$= ax_1^2 + 2bx_1x_2 + cx_2^2 = f(x_1, x_2)$$

← height

easy to go back this way

/E23bp1

$f(x_1, x_2)$ could be:



The Story:
f has a minimum at $x_1 = x_2 = 0$
iff $\mathbb{A}$ is Positive Definite

Why? (1) $\vec{x}^T \mathbb{A} \vec{x} = 0$ at $\vec{x} = 0$

(2) Consider what happens as $\vec{x}$ moves away from $\vec{0}$

Write $\vec{x} = \sum_{i=1}^{n} c_i \hat{v}_i$ ← $\mathbb{A}$'s evector basis

possible because $\mathbb{A} = \mathbb{A}^T$
⇒ eigenvectors form an orthonormal basis for R^n

$$\vec{x}^T \mathbb{A} \vec{x} = \left(\sum_{i=1}^{n} c_i \hat{v}_i^T\right) \mathbb{A} \left(\sum_{j=1}^{n} c_j \hat{v}_j\right) = \left(\sum_{i=1}^{n} c_i \hat{v}_i^T\right)\left(\sum_{j=1}^{n} c_j \overbrace{\mathbb{A}\hat{v}_j}^{\lambda_j \hat{v}_j}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_j c_i c_j \underbrace{\hat{v}_i^T \hat{v}_j}_{\substack{1 \text{ if } i=j \\ 0 \text{ otherwise}}} = \sum_{i=1}^{n} \lambda_i \underbrace{c_i^2}_{>0} > 0$$

for all $\{c_i\}$
iff
all $\lambda_i > 0$

LE23bp2

Ex 1.

Does $f(x_1,x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$ have a maximum at $x_1 = x_2 = 0$?

Answer: Yes if eigenvalues for f's A are both positive

$\iff$ A's pivots are both positive

(1) Construct $\vec{x}^T A \vec{x}$

$$f(x_1,x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$$

$$= [x_1 \quad x_2] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

split evenly

← A_1 from before

(2) Determine pivots

$d_1 = 2$

$d_2 = 3/2$

$\Rightarrow$ $\lambda_1 > 0$, $\lambda_2 > 0$ $\Rightarrow$ f has a minimum

Ex 2.

$$f(x_1,x_2) = 2x_1^2 - 2x_1x_2 - 2x_2^2$$

(1) construct $\vec{x}^T A \vec{x}$

$$f(x_1,x_2) = [x_1 \quad x_2] \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

← A_2 from before

(2) Determine pivots:

$d_1 = 2$

$d_2 = -5/2$

$\Rightarrow$ $\lambda_1 > 0$, $\lambda_2 < 0$ $\Rightarrow$ Saddle

Alternate definition:

$A = A^T$ is positive definite iff

$\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$

E23cp1

Completing the Square = Gaussian Elimination !!

for square symmetric matrices

Idea We could approach question of determining kinds of stationary points by creating clear squares and then looking at signs.

Ex
$$f(x_1,x_2) = \underbrace{2x_1^2 - 2x_1 x_2}_{\text{complete square here}} + 2x_2^2 = [x_1, x_2]\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

leave as constant

$\Uparrow$ $\mathbb{A}_1$ from before

$$= 2\left(x_1^2 - (x_2)\, x_1\right) + 2x_2^2$$

$$= 2\left(\left(x_1 - \frac{x_2}{2}\right)^2 - \frac{x_2^2}{4}\right) + 2x_2^2$$

$$= \underset{d_1}{2}\left(x_1 \underset{l_{21}}{-\tfrac{1}{2}} x_2\right)^2 + \underset{d_2}{\frac{3}{2}} x_2^2$$

$\Leftarrow$ from $\mathbb{A}_1 = \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$

$\leftarrow$ what!??

Ex
$$f(x_1,x_2) = 2x_1^2 - 2x_1x_2 - 2x_2^2 \quad \Leftarrow \mathbb{A}_2 = \begin{bmatrix}2 & -1\\ -1 & -2\end{bmatrix}$$

$$= \underset{d_1}{2}\left(x_1 \underset{l_{21}}{-\tfrac{1}{2}} x_2\right)^2 \underset{d_2}{-\frac{5}{2}} x_2^2$$

same thing

In general for 2x2s:

$$a x_1^2 + 2b x_1 x_2 + c x_2^2$$

$$= \underset{d_1}{a}\underbrace{\left(x_1 + \overset{}{\underset{l_{21}}{\frac{b}{a}}} x_2\right)}^2 + \underbrace{\left(\frac{ac - b^2}{a}\right)}_{d_2} x_2^2$$

see $x_1 + \frac{b}{a}x_2$ as a new variable...

Does completing the square always work like this?

Yes! for symmetric matrices

$$\vec{x}^T \mathbb{A} \vec{x} \quad \leftarrow \text{any quadratic in } n \text{ variables}$$

$\nwarrow \mathbb{A} = \mathbb{A}^T$

$$= \vec{x}^T\left(\mathbb{L}\mathbb{D}\mathbb{L}^T\right)\vec{x}$$

$$= \left(\mathbb{L}^T\vec{x}\right)^T \mathbb{D} \left(\mathbb{L}^T\vec{x}\right) \quad \leftarrow \text{see as a variable transformation}$$

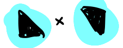
$$= \vec{y}^T \mathbb{D} \vec{y}$$

$$= d_1 y_1^2 + d_2 y_2^2 + \dots + d_n y_n^2$$

$\leftarrow$ if pivots > 0 then $\mathbb{A}$ is a PDM

Super bonus: if $\mathbb{A}$ is a PDM then

$$\mathbb{A} = \tilde{\mathbb{L}}\tilde{\mathbb{L}}^T \quad \text{with } \tilde{\mathbb{L}} = \mathbb{L}\mathbb{D}^{\frac{1}{2}}$$

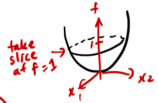


$\uparrow$ Cholesky Factorization

$\uparrow$ all real numbers lower triangular

- Even better for $\mathbb{A}\vec{x} = \vec{b}$

Principle Axis theorem



Consider $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$

Equation of an ellipse oriented at an angle to standard axes

Matrixify:

$$[x_1 \ \ x_2]\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 1$$

↓ use $A = Q\Lambda Q^T$ ← one of our magic friends

$$[x_1 \ \ x_2]\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}\overset{\hat{v}_1}{1} & \overset{\hat{v}_2}{-1}\\ 1 & 1\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}\overset{\lambda_1}{3} & 0\\ 0 & \underset{\lambda_2}{1}\end{bmatrix}}_{\Lambda}\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}_{Q^T}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 1$$

$\hat{v}_1^T$ $\hat{v}_2^T$

$\vec{y}^T = (Q^T\vec{x})^T$ $\quad$ $\vec{y} = Q^T\vec{x}$

$$\Rightarrow \underset{\vec{y}^T}{[y_1 \ \ y_2]}\overset{\Lambda}{\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}}\overset{\vec{y}}{\begin{bmatrix}y_1\\ y_2\end{bmatrix}} = 1$$

$$\Rightarrow 3y_1^2 + y_2^2 = 1$$

clearly an ellipse

Completely clear in $y_1 - y_2$ coordinate system

E23dp1

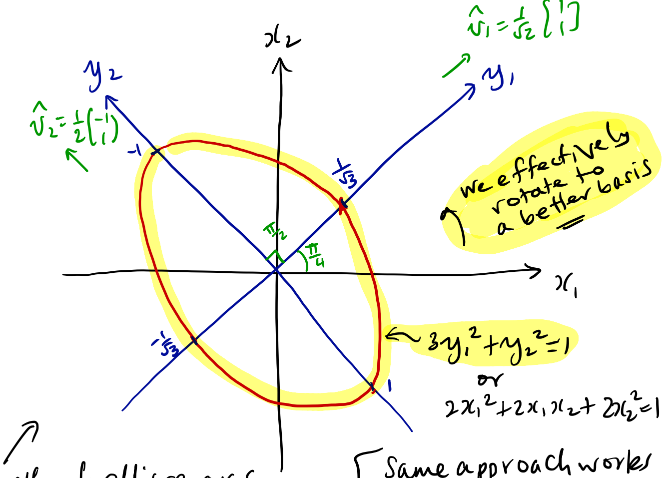
What's this new coordinate system?:

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = Q^T\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix}x_1+x_2\\ -x_1+x_2\end{bmatrix}$$

best

Also: See Q as IM ⇒ basis transformat.

$\left\{\underset{\hat{v}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}}, \underset{\hat{v}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}}\right\}$ = new basis



length of ellipse axes $= \frac{1}{\sqrt{\lambda_i}}$

Same approach works for higher dimensional football

Singular Value Decomposition

Insert ominous organ music

Big deal:

- Matrix Factorizations encode our understanding of problems and greatly enable our methods

$PA = LU$ ⇒ Simultaneous Equations

$A = QR$ ⇒

$\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$ rectangular

$$\left.\begin{array}{l} A = S\Lambda S^{-1} \\ A = Q\Lambda Q^{T} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \underset{n\times1}{\vec{x}'} = \overset{n\times n}{A}\,\underset{n\times1}{\vec{x}} \\ \& \quad A^{k} \\ A\vec{v} = \lambda\vec{v} \end{array}\right.$$

square only

- All have limitations
- We love diagonalization for example

but
1. (1) A must be $n\times n$
2. (2) A must have n linearly independent eigenvectors
3. (3) Eigenvector basis may not be orthogonal (only guaranteed if $A = A^{T}$)

E24ap1

In attempting to overcome these problems, we'll find a factorization that works for all matrices plus

- helps us identify the most important features of a system (pages on the web, supreme court decisions, data in general, building blocks of images, ...)
- Completes our "Big Picture" story for $A\vec{x} = \vec{b}$

↓

Fundamental Theorem of Linear Algebra

[Theoretical story first, then some nutritious examples

Eigen Story: $\underset{n\times n}{A}\vec{v} = \lambda\vec{v}$ ← eigenvectors may not form a basis :(

same direction :)

We give this up to (i) accommodate $m\times n$ matrices

& (ii) ensure orthogonality of bases

↑ want this

New plan:

$$\overset{\text{real}}{\underset{m\times n}{A}}\ \overset{\text{unit vector}}{\underset{n\times 1}{\hat{v}_i}} = \overset{\text{"singular value"}}{\sigma_i}\ \underset{m\times 1}{\hat{u}_i} \quad \leftarrow \text{singular vector}, \quad \hat{u}_i \leftarrow \text{unit vector}$$

where:

- $\hat{v}_i \perp \hat{v}_j$ if $i \neq j$, $\hat{v}_i \in R^n$ (row space + nullspace)
- $\hat{u}_i \perp \hat{u}_j$ if $i \neq j$, $\hat{u}_i \in R^m$ (column space + left nullspace)
- $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r > 0$ ← r = rank
- the $\hat{v}_i$ form an orthonormal basis for R^n
- the $\hat{u}_i$ form an orthonormal basis for R^m

How $A\vec{x}$ works:

(1) Transform $\vec{x}$ to $\{\hat{v}_i\}$ basis in R^n

(2) Send $\hat{v}_i$'s to $\hat{u}_i$'s and multiply by σ_i

(3) Transform from $\{\hat{u}_i\}$ basis to standard basis in R^m

Yet another great moment in Matrixology:

Singular Value Decomposition

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

← Monks tell us to believe

c.f. $A = Q\Lambda Q^T$

$$\underset{m\times m}{U} = \begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix} \leftarrow \vec{b}, \vec{p}, \vec{e}, \vec{y}$$

$$\underset{n\times n}{V} = \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \leftarrow \vec{x}, \vec{x}_r, \vec{x}_n, \vec{x}_p, \vec{x}_h$$

$$\underset{m\times n}{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

all zeros

← same shape as A

r = rank

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

Let's see how this all works:

Claim → $\mathbb{A} = \mathbb{U}\Sigma\mathbb{V}^T$ with $\mathbb{A}\hat{v}_i = \sigma_i \hat{u}_i$

Monks: "Try $\underbrace{\mathbb{A}^T\mathbb{A}}_{\uparrow}$, grasshopper"

symmetric for all $\mathbb{A}$

okay....

$$\underbrace{\underbrace{\mathbb{A}^T}_{n\times m}\underbrace{\mathbb{A}}_{m\times n}}_{n\times n} = (\mathbb{U}\Sigma\mathbb{V}^T)^T(\mathbb{U}\Sigma\mathbb{V}^T)$$

$$= \mathbb{V}^{T^T}\Sigma^T\underbrace{\mathbb{U}^T\mathbb{U}}_{\mathbb{I}}\Sigma\mathbb{V}^T$$

$$= \mathbb{V}\underbrace{\Sigma^T}_{n\times m}\underbrace{\Sigma}_{m\times n}\mathbb{V}^T$$

$$= \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1^2 & & & & & \\ & \sigma_2^2 & & & 0 & \\ & & \ddots & & & \\ & & & \sigma_r^2 & & \\ & 0 & & & 0 & \\ & & & & & \ddots\, 0 \end{bmatrix} \begin{bmatrix} - & \hat{v}_1^T & - \\ - & \hat{v}_2^T & - \\ & \vdots & \\ - & \hat{v}_n^T & - \end{bmatrix}$$

square

$n\times n$

key: know
$(\mathbb{A}^T\mathbb{A})^T = \mathbb{A}^T\mathbb{A}$
for any $\mathbb{A}$

$\mathbb{U}^{-1} = \mathbb{U}^T$
$\mathbb{V}^{-1} = \mathbb{V}^T$

Looks a lot like: $\mathbb{Q}\Lambda\mathbb{Q}^T$ (sneaky monks)

> But will $\mathbb{A}^T\mathbb{A}$ always be so wonderfully diagonalizable?

← drama

Monk Joy

$\underbrace{\underbrace{\mathbb{A}^T}_{n\times m}\underbrace{\mathbb{A}}_{m\times n}}_{n\times n}$ is real, symmetric

and therefore (1) eigenvalues are real
(2) eigenvectors form an orthonormal basis for R^n

Monk Joy

↑ augmented

$$\vec{x}^T(\mathbb{A}^T\mathbb{A})\vec{x}$$

$$= \underbrace{(\mathbb{A}\vec{x})}_{n\times 1}{}^T\underbrace{(\underbrace{\mathbb{A}}_{n\times n}\underbrace{\vec{x}}_{n\times 1})}_{n\times 1}$$

$$= \|\mathbb{A}\vec{x}\|^2 \geq 0$$

↖ length ← has to be true

⇒ $\mathbb{A}^T\mathbb{A}$ is Semi-Positive Definite

⇒ $\mathbb{A}^T\mathbb{A}$'s eigenvalues are all ≥ 0

⇒ $\sigma_i = \sqrt{\lambda_i} \geq 0$ is all good

Upshot: Diagonalize $\mathbb{A}^T\mathbb{A}$ to find σ_i's and $\hat{v}_i$'s

Monks chant: "$\mathbb{A}\mathbb{A}^T$! $\mathbb{A}\mathbb{A}^T$! $\mathbb{A}\mathbb{A}^T$!..."

$$\underbrace{\underbrace{\mathbb{A}}_{m\times n}\underbrace{\mathbb{A}^T}_{n\times m}}_{m\times m} = (\mathbb{U}\Sigma\mathbb{V}^T)(\mathbb{U}\Sigma\mathbb{V}^T)^{\textcircled{T}}$$

↗ symmetric

$$= \mathbb{U}\Sigma\underbrace{\mathbb{V}^T\mathbb{V}}_{\mathbb{I}}{}^{\cancel{TT}}\Sigma^T\mathbb{U}^T$$

$$= \mathbb{U}\underbrace{\Sigma\Sigma^T}_{m\times m}\mathbb{U}^T$$

$$= \begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1^2 & & & & & \\ & \sigma_2^2 & & & 0 & \\ & & \ddots & & & \\ & 0 & & \sigma_r^2 & & \\ & & & & 0 & \\ & & & & & \ddots\, 0 \end{bmatrix} \begin{bmatrix} - & \hat{u}_1^T & - \\ - & \hat{u}_2^T & - \\ & \vdots & \\ - & \hat{u}_n^T & - \end{bmatrix}$$

$m\times m$

surprising!!
↓
$\mathbb{A}^T\mathbb{A}$ & $\mathbb{A}\mathbb{A}^T$ must have same non-zero eigenvalues

More upshot:

Diagonalize $\mathbb{A}\mathbb{A}^T$ to find $\hat{u}_i$'s and, again, σ_i's

How do we know $\mathbb{A}\hat{v}_i = \sigma_i \hat{u}_i$?

We have: $\mathbb{A}^T\mathbb{A}\hat{v}_i = \sigma_i^2 \hat{v}_i$

(1) Monks: $\hat{v}_i^T(\mathbb{A}^T\mathbb{A}\hat{v}_i)$

$$(\mathbb{A}\hat{v}_i)^T(\mathbb{A}\hat{v}_i) = \|\underbrace{\mathbb{A}\hat{v}_i}_{m\times 1\ \text{vector}}\|^2$$

$$\hat{v}_i^T(\sigma_i^2\hat{v}_i) = \sigma_i^2\,\underbrace{\hat{v}_i^T\hat{v}_i}_{=1} = \sigma_i^2$$

$$\|\mathbb{A}\hat{v}_i\|^2 = \sigma_i^2$$

$\Rightarrow \|\mathbb{A}\hat{v}_i\|^2 = \sigma_i^2$

$\Rightarrow \|\mathbb{A}\hat{v}_i\| = \sigma_i$

So we have the right length ✓

E24ap4

Monks again

(2)

$$\mathbb{A}(\mathbb{A}^T\mathbb{A}\hat{v}_i) = \mathbb{A}(\underbrace{\sigma_i^2\hat{v}_i}_{\text{eigenvalue equat}})$$

$$= (\mathbb{A}\mathbb{A}^T)(\underbrace{\mathbb{A}\hat{v}_i}_{m\times 1\ \text{vector}}) \qquad = \sigma_i^2(\underbrace{\mathbb{A}\hat{v}_i}_{m\times 1\ \text{vector}})$$

$\Rightarrow$ $\mathbb{A}\hat{v}_i$ is an eigenvector of $\mathbb{A}\mathbb{A}^T$ with eigenvalue σ_i^2

$\Rightarrow$ $\mathbb{A}\hat{v}_i \propto \hat{u}_i$

(1) + (2) $\Rightarrow$ $\mathbb{A}\hat{v}_i = \sigma_i\hat{u}_i$

Important details:

- Choose $\hat{u}_i$'s direction to match $\mathbb{A}\hat{v}_i$ — makes sure $\sigma_i > 0$
- If we have found $\hat{v}_i$ already, $\boxed{\hat{u}_i = \frac{1}{\sigma_i}\mathbb{A}\hat{v}_i}$ is best way to compute $\hat{u}_i$
- $\mathbb{A}\hat{v}_i = \vec{0}$ for $\underbrace{i = r+1, r+2, \ldots, n}_{\text{null space basis}}$
- $\hat{u}_i$ for $i = r+1, r+2, \ldots, m$ = left nullspace basis

One last piece:

For $A = A^T$, we had $Q\Lambda Q^T$ and therefore

$$\underset{n\times n}{A} = \lambda_1 \hat{u}_1 \hat{u}_1^T + \lambda_2 \hat{u}_2 \hat{u}_2^T + \dots + \lambda_n \hat{u}_n \hat{u}_n^T$$

outer products = projection operators

A = sum of n rank 1 matrices.

For SVD:

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix}}_{m\times m} \underbrace{\begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & 0 & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & 0 & & & 0 & \\ & & & & & \ddots \; 0 \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} - & \hat{v}_1^T & - \\ - & \hat{v}_2^T & - \\ & \vdots & \\ - & \hat{v}_n^T & - \end{bmatrix}}_{n\times n}$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\ | & | & & | \end{bmatrix}}_{m\times m} \underbrace{\begin{bmatrix} - & \sigma_1 \hat{v}_1^T & - \\ - & \sigma_2 \hat{v}_2^T & - \\ & \vdots & \\ - & \sigma_r \hat{v}_r^T & - \\ - & \vec{0}^T & - \\ & \vdots & \\ - & \vec{0}^T & - \end{bmatrix}}_{m\times n} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} r \text{ rows non-zero} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m-r \text{ rows of zilchos} \end{matrix}$$

$$= \sigma_1 \underset{m\times 1}{\hat{u}_1} \underset{1\times n}{\hat{v}_1^T} + \sigma_2 \underset{m\times 1}{\hat{u}_2} \underset{1\times n}{\hat{v}_2^T} + \dots + \sigma_r \underset{m\times 1}{\hat{u}_r} \underset{1\times n}{\hat{v}_r^T}$$

$\boxed{\text{E24ap5}}$

$$\boxed{A = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^T}$$

See A as a superposition of r outer product rank 1 matrices of diminishing significance

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

- Each rank 1 matrix is a piece of Scottish Tartan
- SVD makes approximation of large matrices vigorous
- Speak of best rank 1, best rank 2, ... approximations

SVD Example Calculation #1:

For $\underset{\substack{2\times2\\m\times n}}{\mathbb{A}} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$ find $\mathbb{A} = \underset{\substack{m\times m\\2\times2}}{\mathbb{U}}\,\underset{\substack{m\times n\\2\times2}}{\Sigma}\,\underset{\substack{n\times n\\2\times2}}{\mathbb{V}^T}$

(1) Find $\hat{v}_i$'s and σ_i's using $\mathbb{A}^T\mathbb{A}$

- $\mathbb{A}^T\mathbb{A} = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 18 & -6 \\ -6 & 2 \end{bmatrix}$ ← note symmetry
- Solve $\left| \mathbb{A}^T\mathbb{A} - \lambda\mathbb{I} \right| = 0$

$$\Rightarrow 0 = \begin{vmatrix} 18-\lambda & -6 \\ -6 & 2-\lambda \end{vmatrix} = (18-\lambda)(2-\lambda) - 36$$

$$= \cancel{36} - 20\lambda + \lambda^2 - \cancel{36}$$

$$= \lambda(\lambda - 20)$$

$$\Rightarrow \begin{matrix} \lambda_1 = 20 = \sigma_1^2 \\ \lambda_2 = 0 = \sigma_2^2 \end{matrix} \Rightarrow \begin{matrix} \sigma_1 = \sqrt{20} \\ \text{"}\sigma_2\text{"} = 0 \end{matrix}$$

← row space

← null space $\mathbb{A}\vec{x} = \vec{0}$

- $\lambda_1 = 20$: Solve $(\mathbb{A}^T\mathbb{A} - 20\mathbb{I})\vec{v}_1 = \vec{0}$

$$\Rightarrow \left[\begin{array}{cc|c} -2 & -6 & 0 \\ -6 & -18 & 0 \end{array}\right] \Rightarrow \hat{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

rows must be multiples of each other for 2x2's

as promised, $\hat{v}_1 \perp \hat{v}_2$

- $\lambda_2 = 0$: Solve $(\mathbb{A}^T\mathbb{A} - \cancel{0}\mathbb{I})\vec{v}_2 = \vec{0}$

$$\Rightarrow \left[\begin{array}{cc|c} 18 & -6 & 0 \\ -6 & 2 & 0 \end{array}\right] \Rightarrow \hat{v}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

E246p1

So far:

$$\mathbb{V} = \begin{bmatrix} | & | \\ \hat{v}_1 & \hat{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 0 \end{bmatrix}$$

$\underset{\substack{\uparrow\\\text{rank}}}{r = 1}$ $\sigma_1 = \sqrt{20}$

- Need $\mathbb{U}$ as well

Either solve for eigenthings of $\underset{\substack{\parallel\\ \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}}}{\mathbb{A}\mathbb{A}^T}$ $\nearrow \lambda_1 = 20$, $\hat{u}_1$; $\searrow \lambda_2 = 0$, $\hat{u}_2$

- Better: Use $\mathbb{A}\hat{v}_i = \sigma_i \hat{u}_i$

$$\Rightarrow \hat{u}_i = \frac{1}{\sigma_i}\mathbb{A}\hat{v}_i$$

$$\hat{u}_1 = \underset{\sigma_1}{\frac{1}{\sqrt{20}}}\,\mathbb{A}\,\underbrace{\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}}_{\hat{v}_1} = \frac{1}{\sqrt{20}}\frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{20}}\frac{1}{\sqrt{10}}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{2}}\frac{1}{10}\begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

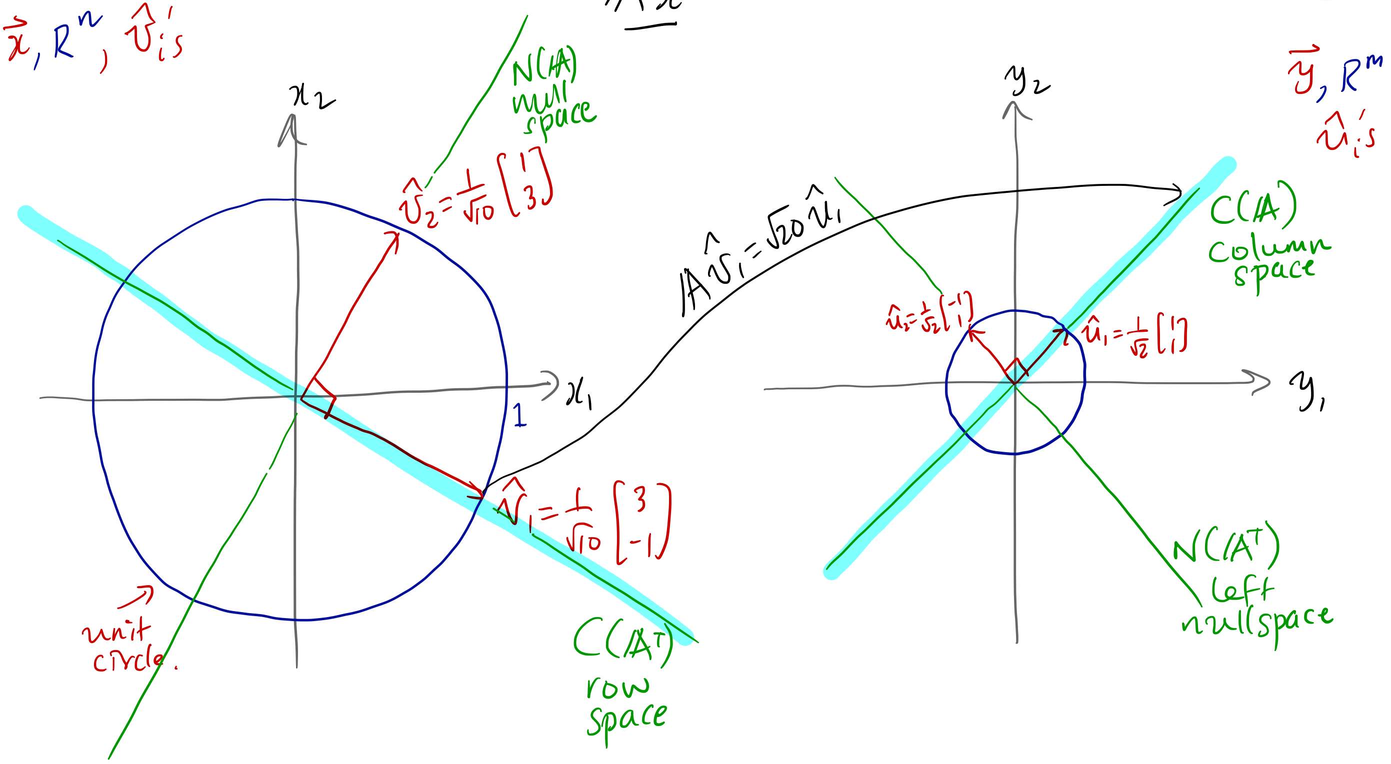
For $\hat{u}_2$, we just need a vector orthogonal to $\hat{u}_1$

By inspection: $\hat{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

better way to represent $\mathbb{A}$

$$\mathbb{A} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\mathbb{U}}\underbrace{\begin{bmatrix} \sqrt{20} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma}\underbrace{\frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}}_{\mathbb{V}^T}$$

$A\vec{x}$



- See A sends $C(A^T)$ to $C(A)$ with a stretch factor of $\sqrt{20}$.
- A's action between $C(A^T)$ & $C(A)$ is invertible

SVD Example Calculation #2

Factorize $A = \frac{1}{5}\begin{bmatrix}2 & 11\\ 10 & 5\end{bmatrix}$ as $U\Sigma V^T$

- Diagonalize A^TA

$$\underbrace{A^TA}_{\text{symmetric}} = \frac{1}{5}\begin{bmatrix}2 & 10\\ 11 & 5\end{bmatrix}\frac{1}{5}\begin{bmatrix}2 & 11\\ 10 & 5\end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix}104 & 72\\ 72 & 146\end{bmatrix} = \frac{2}{25}\begin{bmatrix}52 & 36\\ 36 & 73\end{bmatrix}$$

SMT #731

if $B\vec{v} = \lambda\vec{v}$
then
$cB\vec{v} = c\lambda\vec{v}$
$\Rightarrow B'\vec{v} = (c\lambda)\vec{v}$

If $\vec{v}$ is an eigenvector of B with eigenvalue λ then $\vec{v}$ is an eigenvector of cB with eigenvalue $c\lambda$

Find λ's for $\begin{bmatrix}52 & 36\\ 36 & 73\end{bmatrix}$

E24cp1

Solve $|A^TA - \lambda I| = 0$

$$0 = \begin{vmatrix}52-\lambda & 36\\ 36 & 73-\lambda\end{vmatrix} = (52-\lambda)(73-\lambda) - (36)^2$$

$$= 3796 - 125\lambda + \lambda^2 - 1296$$

$$= \lambda^2 - \underset{\overset{\uparrow}{25+100}}{125}\lambda + 2500$$

$$= (\lambda - 25)(\lambda - 100) \Rightarrow \begin{matrix}\lambda_1 = 100\\ \lambda_2 = 25\end{matrix} \nearrow \text{for } \tfrac{25}{2}A^TA$$

$$\times\frac{2}{25} \Rightarrow \begin{matrix}\lambda_1 = 8 = \sigma_1^2\\ \lambda_2 = 2 = \sigma_2^2\end{matrix} \Rightarrow \begin{matrix}\sigma_1 = \sqrt{8}\\ \sigma_2 = \sqrt{2}\end{matrix}$$

$\swarrow$ for A^TA

- $\lambda_1 = 8$: $\frac{2}{25}\left[\begin{array}{cc|c}-48 & 36 & 0\\ 36 & -27 & 0\end{array}\right] \Rightarrow \hat{v}_1 = \frac{1}{5}\begin{bmatrix}3\\ 4\end{bmatrix}$

Solve $(A^TA - \lambda I)\vec{v} = \vec{0}$

- $\lambda_2 = 2$: $\frac{2}{25}\left[\begin{array}{cc|c}27 & 36 & 0\\ 36 & 48 & 0\end{array}\right] \Rightarrow \hat{v}_2 = \frac{1}{5}\begin{bmatrix}-4\\ 3\end{bmatrix}$

(could choose $\hat{v}_2 = -\frac{1}{5}\begin{bmatrix}-4\\ 3\end{bmatrix}$)

now have

$$V = \frac{1}{5}\begin{bmatrix}3 & -4\\ 4 & 3\end{bmatrix} \quad \& \quad \Sigma = \begin{bmatrix}\sqrt{8} & 0\\ 0 & \sqrt{2}\end{bmatrix}$$

- Now find $\hat{u}_1$ & $\hat{u}_2$

$\hat{u}_i = \frac{1}{\sigma_i} A \hat{v}_i$ ← best way

$$\hat{u}_1 = \frac{1}{\sqrt{8}} \frac{1}{5} \begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{\sqrt{8}} \frac{1}{25} \begin{bmatrix} 50 \\ 50 \end{bmatrix} = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

unit vectors guaranteed

$$\hat{u}_2 = \frac{1}{\sqrt{2}} \frac{1}{5} \begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} \frac{1}{5} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{25} \begin{bmatrix} 25 \\ -25 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ ✓.

Could also diagonalize AA^T: E24cp2

$$AA^T = \frac{1}{5} \begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 2 & 10 \\ 11 & 5 \end{bmatrix}$$

$$= \frac{1}{25} \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

Find $\lambda_1 = 8$, $\lambda_2 = 2$

$$\hat{u}_1 = \pm\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \hat{u}_2 = \pm\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

not sure about signs

⇒ Still have to compute $\hat{u}_i = \frac{1}{\sigma_i} A \hat{v}_i$ ✓.

Overall:

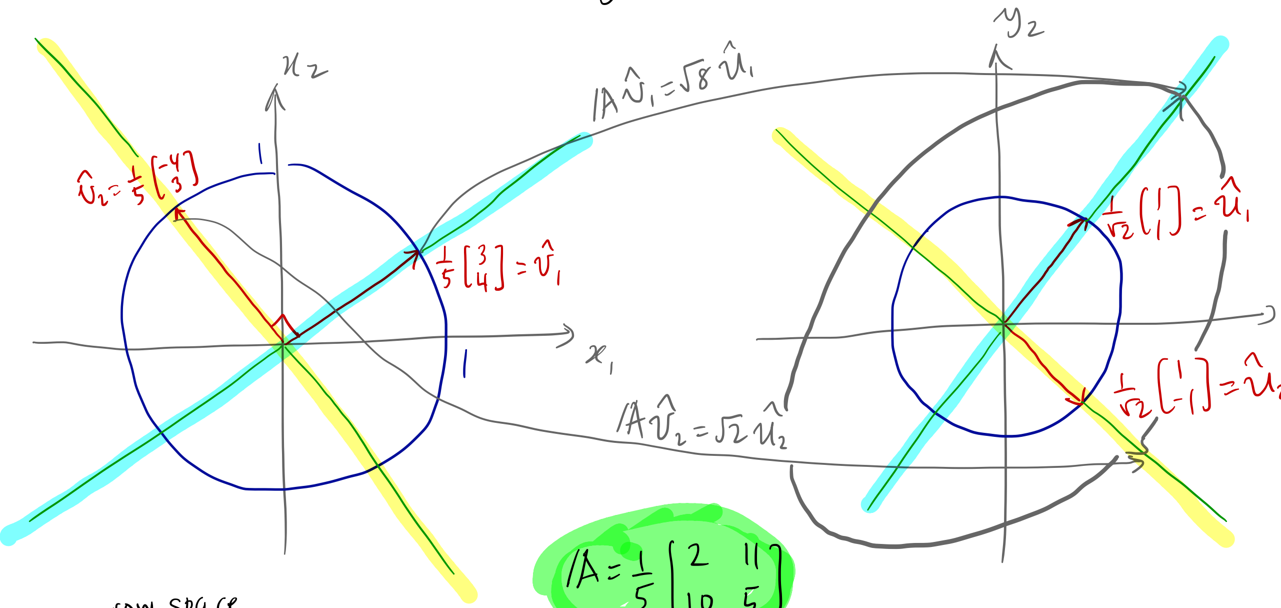
$$A = \frac{1}{5} \begin{bmatrix} 2 & 11 \\ 10 & 5 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}}_{V^T}$$

$\mathbb{R}^2, \mathbb{R}^n, \vec{x}$'s

$\mathbb{A}\vec{x} = \vec{b}$

or $\vec{y} = \mathbb{A}\vec{x}$

E24cp3



$$\mathbb{A} = \frac{1}{5}\begin{bmatrix}2 & 11\\10 & 5\end{bmatrix}$$

row space

$C(\mathbb{A}^T) \equiv \mathbb{R}^2$

$N(\mathbb{A}) = \{\vec{0}\}$

Big deal: Circle ↦ Ellipse

↑

generalizes to higher dimensions

alphabetic, u, v

$\longrightarrow$

note $\mathbb{U}\Sigma\mathbb{V}^T$

but wrong order of operations

how $\vec{y} = A\vec{x}$ works

$$A = U \Sigma V^T$$

③ Change from $\{\hat{u}\}$ to standard basis

② Does the work of A

Stretch/Shrink by σ_i factors in r dimensions

↗ $C(A^T)$ & $C(A)$

① Change $\vec{x}$'s representation from standard to $\{\hat{u}\}$ basis

Fundamental Theorem of Matrixology

From E13bp3 :

- $\dim C(A) = r$ ← rank — column space
- $\dim N(A^T) = m - r$ — left null space
- $\dim C(A^T) = r$ — row space
- $\dim N(A) = n - r$ — nullspace
- $C(A)$ and $N(A^T)$ are orthogonal complements in R^m $C(A) \oplus N(A^T)$
- $C(A^T)$ and $N(A)$ are orthogonal complements in R^n $C(A^T) \oplus N(A)$
- The bases of $C(A)$ & $N(A^T)$ combine to give a basis of R^m
- The bases of $C(A^T)$ & $N(A)$ combine to give a basis of R^n

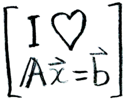
Now we also have:

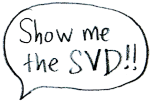
- Row space has a "natural" orthonormal basis $\{\hat{v}_1, \ldots, \hat{v}_r\}$, eigenvectors of A^TA
- Nullspace has a "natural" orthonormal basis $\{\hat{v}_{r+1}, \ldots, \hat{v}_n\}$, eigenvectors of A^TA
- Column Space has a "natural" orthonormal basis $\{\hat{u}_1, \ldots, \hat{u}_r\}$, eigenvectors of AA^T
- Left Nullspace has a "natural" orthonormal basis $\{\hat{u}_{r+1}, \ldots, \hat{u}_m\}$, eigenvectors of AA^T
- The transformation between the "best" bases for row space and column space is diagonal with positive entries:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}$$

with $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r$

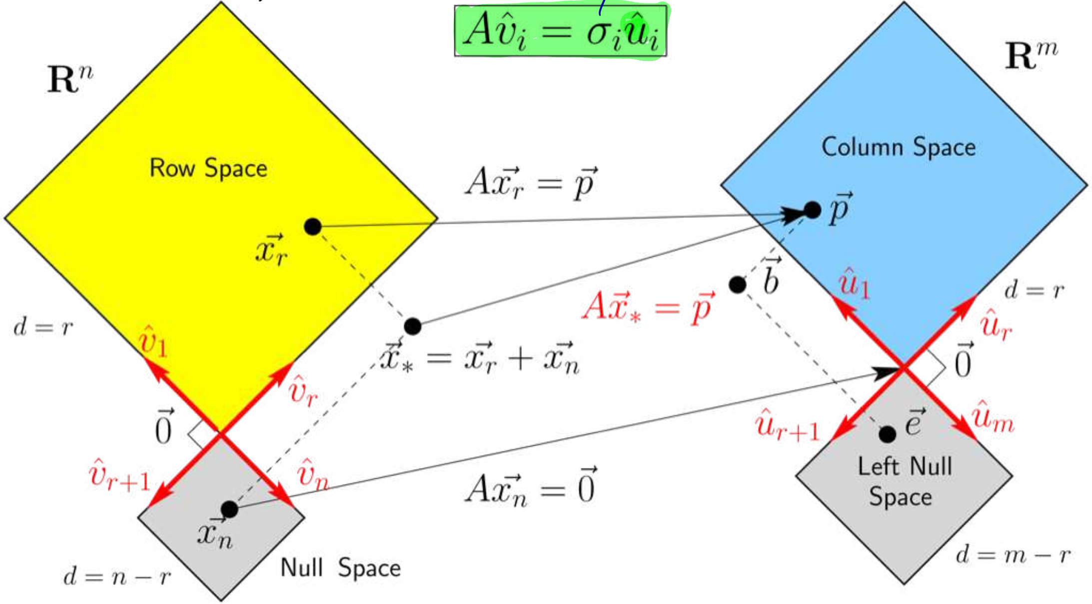
E25ap2





$\sigma_i > 0$

$$A\hat{v}_i = \sigma_i \hat{u}_i$$



Time for a nap:

E25ap3