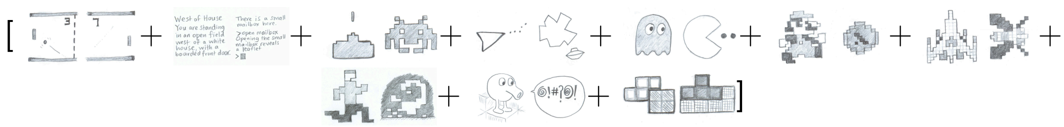

Solutions to **122 Matrixology (Linear Algebra)—Practice exam #4**
University of Vermont, Fall Semester

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1. Draw the 'big picture' of how $\mathbf{A}\vec{x} = \vec{b}$ works when $\mathbf{A}$ is an $m \times n$ matrix. Indicate on your diagram the following:

   (a) Which space is R^m and which is R^n.

   (b) Row space, column space, nullspace, and left nullspace.

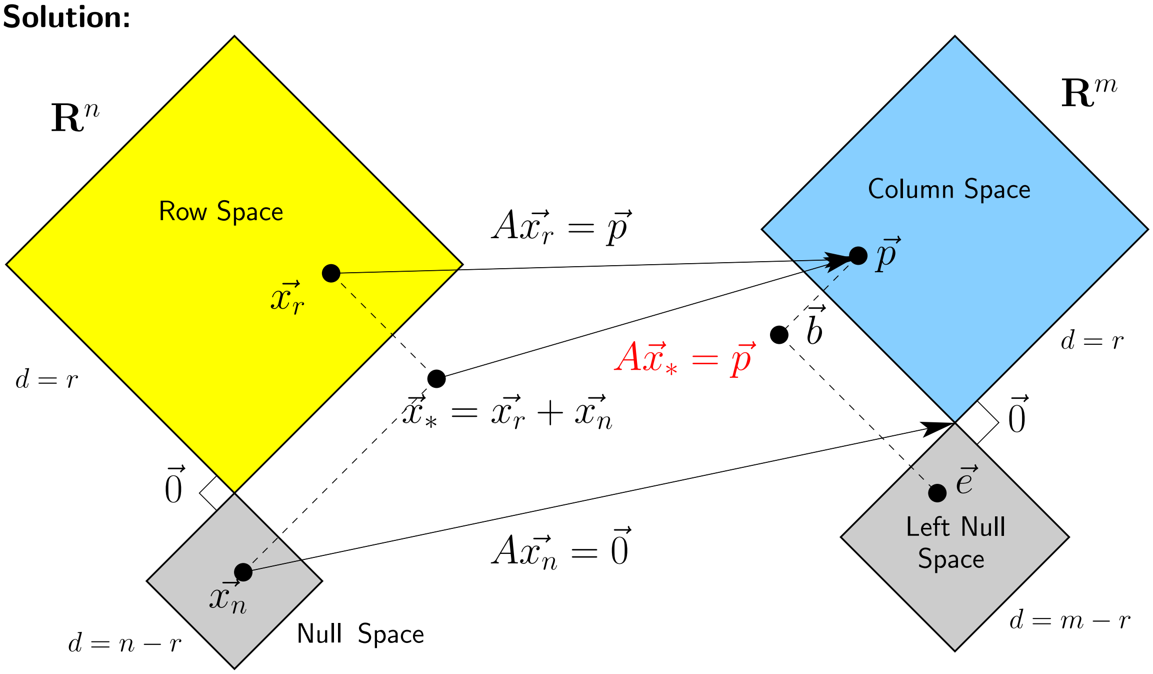
   (c) The dimensions of the above subspaces in terms of r, m, and n.

   (d) How $\mathbf{A}$ maps vectors.

   (e) Where the vectors $\vec{x} = \vec{x}_r + \vec{x}_n$ and $\vec{b} = \vec{p} + \vec{e}$ live.

   (f) The appropriate orthogonality of subspaces.

Solution:



□

2. For the four general cases of $\mathbf{A}\vec{x} = \vec{b}$ below:

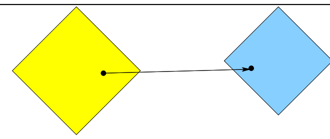
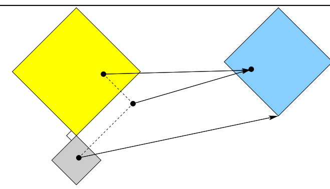
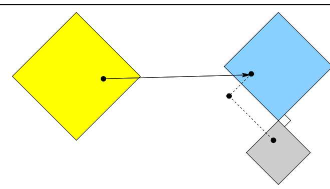
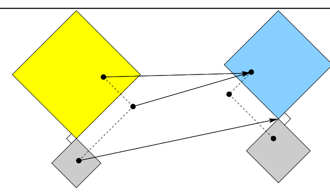
   (a) give an example reduced row echelon form matrix $\mathbf{R}_\mathbf{A}$;

(b) sketch the appropriate cartoon abstract ‘big pictures’;

(c) indicate the number of possible solutions (0, 1, or ∞);

(d) and note whether or not nullspace and left nullspace are equal to $\{\vec{0}\}$ (Y/N).

Solution:

case	example $\mathbf{R_A}$	big picture	# solutions	$N(\mathbf{A}) = \{\vec{0}\}$?	$N(\mathbf{A}^{\mathrm{T}}) = \{\vec{0}\}$?
$m = r$ $n = r$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		1 always	Y	Y
$m = r$, $n > r$	$\begin{bmatrix} 1 & 0 & 0 & \text{⚽}_1 \\ 0 & 1 & 0 & \text{⚽}_2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$		∞ always	N	Y
$m > r$, $n = r$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		0 or 1	Y	N
$m > r$, $n > r$	$\begin{bmatrix} 1 & 0 & \text{☯} & \text{✂} \\ 0 & 1 & \text{ϟ} & \text{☢} \\ 0 & 0 & 0 & 0 \end{bmatrix}$		0 or ∞	N	N

□

3. Given a matrix $\mathbf{A}$ and its transpose $\mathbf{A}^{\mathrm{T}}$ have the following reduced row echelon forms, respectively,

$$\mathbf{R_A} = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{R}_{\mathbf{A}^{\mathrm{T}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

answer the following questions.

(a) **Solution:** $m = 3$, $n = 5$, $r = 3$,

dim $C(\mathbf{A}^{\mathrm{T}}) = 3$, dim $C(\mathbf{A}) = 3$, dim $N(\mathbf{A}) = 2$, dim $N(\mathbf{A}^{\mathrm{T}}) = 0$. □

(b) Find bases for $\mathbf{A}$'s row space and column space.

Solution: We read these off the non-zero rows of $\mathbf{R_A}$ and $\mathbf{R_{A^T}}$:

A basis for row space is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and for column space:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

□

(c) Find a basis for $\mathbf{A}$'s nullspace

Solution: The free variables are x_2 and x_5 and the pivot variables are x_1, x_3 and x_4. We express the pivot variables in terms of the free variables: $x_1 = -2x_2 + x_5$, $x_3 = -3x_5$, $x_4 = 0$. We therefore have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_5 \\ x_2 \\ -3x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

where x_2 and $x_5 \in R$. One possible basis for nullspace is therefore

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

□

4. $\mathbf{LU}$ decomposition:

Find $\mathbf{U}$ for the following matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 4 & 1 & 4 \\ -4 & 11 & 0 \end{bmatrix}.$$

Write down each row operation, the multipliers l_{21}, l_{31}, and l_{32}, and the corresponding elimination matrices $\mathbf{E}_{21}$, $\mathbf{E}_{31}$, and $\mathbf{E}_{32}$.

Solution:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 4 & 1 & 4 \\ -4 & 11 & 0 \end{bmatrix} \underset{\substack{\text{R2'} \\ = \text{R2 -} \\ \text{2 R1}}}{\leadsto} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ -4 & 11 & 0 \end{bmatrix} \underset{\substack{\text{R3'} \\ = \text{R3 -} \\ \text{(-2) R1}}}{\leadsto} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 9 & 4 \end{bmatrix}$$

$$\underset{\substack{\text{R3'} \\ = \text{R3 -} \\ \text{3 R1}}}{\leadsto} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

So we have: $l_{21} = 2$, $l_{31} = -2$, and $l_{32} = 3$;

$$\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix};$$

and

$$\mathbf{U} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

□

5. This question carries on with the the preceding question's $\mathbf{A}$.

(a) What are the pivots of $\mathbf{A}$?

Solution: From $\mathbf{U}$: $d_1 = 2$, $d_2 = 3$, and $d_3 = 4$. □

(b) Write down a general formula for $|\mathbf{A}|$ in terms of its pivots (remembering that in general, row swaps may be needed to reduce $\mathbf{A}$ to $\mathbf{U}$), and compute the determinant of the $\mathbf{A}$ we have here.

Solution:

$$|\mathbf{A}| = \pm \prod_{i=1}^{n} d_i$$

where $\pm$ depends on the number of row swaps (+ if even, - if odd).

$|\mathbf{A}| = (2) \cdot (3) \cdot (4) = 24.$ □

(c) Write down the inverses of the elimination matrices and compute $\mathbf{L} = \mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{32}^{-1}$.

Solution: We flip the sign of the $-l_{ij}$'s to find the inverses of the $\mathbf{E}$'s:

$$\mathbf{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

The matrix $\mathbf{L}$ is lower triangular and built out of the multipliers we found in the previous question. We know that the inverses of the $\mathbf{E}$'s combine in a simple way:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$$

□

6. Least squares approximation:

(a) Given

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix},$$

solve the normal equation $\mathbf{A}^{\mathrm{T}}\mathbf{A}\vec{x}^* = \mathbf{A}^{\mathrm{T}}\vec{b}$.

Solution: First build the normal equation:

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Convert the right hand side:

$$\mathbf{A}^{\mathrm{T}}\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}.$$

Since $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is invertible, we can compute the solution as

$$\vec{x}^* = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\vec{b} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 11 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 16 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

□

(b) Find $\vec{p}$ and $\vec{e}$, the components of $\vec{b}$ that live in column space and left nullspace respectively.

Solution: The simplest way to find $\vec{p}$ is to use the fact that $\mathbf{A}\vec{x}^* = \vec{p}$. So

$$\vec{p} = \mathbf{A}\vec{x}^* = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}.$$

The error vector $\vec{e}$ is given by $\vec{b} - \vec{p}$:

$$\vec{e} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}.$$

A quick check shows that $\vec{e}$ is orthogonal to the columns of $\mathbf{A}$ and to $\vec{p}$ (gasp).
□

7. The Gram-Schmidt process:

   Consider the subspace $\mathbf{S}$ of R^3 that is spanned by the following two linearly independent vectors:

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

   Find an orthonormal basis vectors ($\hat{q}_1$ and $\hat{q}_2$) for $\mathbf{S}$ using the (exciting) Gram-Schmidt process.

   Solution:

$$\vec{q}_1 = \vec{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

$$\vec{q}_2 = \vec{a}_2 - \frac{\vec{q}_1^{\mathrm{T}}\vec{a}_2}{\vec{q}_1^{\mathrm{T}}\vec{q}_1}\vec{q}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{3}{9}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}.$$

   We normalize to find

$$\hat{q}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ and } \hat{q}_2 = \frac{1}{\sqrt{18}}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$$

$\square$

(b) Consequently, for the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$ find the factorization $\mathbf{A} = \mathbf{QR}$ (i.e., find $\mathbf{Q}$ and $\mathbf{R}$).

Solution:

$$\mathbf{Q} = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & \frac{-4}{\sqrt{18}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} \end{bmatrix}$$

and

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$$

$\square$

8. (a) Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$$

Solution: For the eigenvalues, we solve $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for λ:

$$|\mathbf{A} - \lambda\mathbf{I}| = (3 - \lambda)(1 - \lambda).$$

Let's assign these eigenvalues as $\lambda_1 = 3$ and $\lambda_2 = 1$. Eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\square$

(b) Write down $\mathbf{A}$'s diagonalized counterpart $\mathbf{\Lambda}$ and the transformation matrices $\mathbf{S}$ and $\mathbf{S}^{-1}$.

Solution:

$$\mathbf{\Lambda} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \text{ and } \mathbf{S}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}.$$

$\square$

(c) Hence determine $\mathbf{A}^n$ where n is arbitrary.

Solution:

$$\mathbf{A}^n = \mathbf{S}\mathbf{\Lambda}^n\mathbf{S}^{-1} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 \cdot 3^n & 0 \\ 3(3^n - 1) & 2 \end{bmatrix}.$$

□

9. Computing determinants: Given

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix},$$

(a) Write down the minor matrices $\mathbf{M}_{12}$, $\mathbf{M}_{22}$, and $\mathbf{M}_{32}$, compute the cofactors C_{12}, C_{22}, and C_{32}, and hence find $\det(\mathbf{A})$.

Solution:

$$\mathbf{M}_{12} = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}, \ \mathbf{M}_{22} = \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \mathbf{M}_{32} = \begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix}.$$

Using $C_{ij} = (-1)^{i+j}|\mathbf{M}_{ij}|$, we have $C_{12} = -8$, $C_{22} = 12$, and $C_{32} = -8$.
$|\mathbf{A}| = \sum_{i=1}^{3} a_{i2}C_{i2} = (2) \cdot (-8) + (4) \cdot (12) + (2)(-8) = 16.$

□

(b) Also find $|\mathbf{A}|$ by reducing $\mathbf{A}$ to an upper triangular matrix with 1's on the leading diagonal.

Solution:

$$|\mathbf{A}| = \begin{vmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{vmatrix} \underset{\left(\substack{R2' = \\ R2 - 1\,R1}\right)}{=} \begin{vmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 3 \end{vmatrix} \underset{\left(\substack{R3' = \\ R3 - \frac{1}{2}\,R1}\right)}{=} \begin{vmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$\underset{\left(\substack{R3' = \\ R3 - \frac{1}{2}\,R1}\right)}{=} \begin{vmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{vmatrix} = (4)(2)(2) \begin{vmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 16.$$

□

10. Positive Definite Matrices

Let $f(x, x_2, x_3) = 2x^2 + x_2^2 + 6x_3^2 + 2x_1x_2 - 4x_1x_3 + 4x_2x_3$.

(a) Rewrite $f(x_1, x_2, x_3)$ as $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $\mathbf{A}$ is a symmetric matrix.

Solution:

$$f(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 2 \\ -2 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

□

(b) Determine the signs of eigenvalues by finding the pivots.

Solution:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 2 \\ -2 & 2 & 6 \end{bmatrix} \text{ reduces to } \mathbf{U} = \begin{bmatrix} 2 & 1 & -2 \\ 0 & \frac{1}{2} & 3 \\ 0 & 0 & -14 \end{bmatrix}.$$

The pivots are thus $2, \frac{1}{2}, -14$ and we must have two positive eigenvalues and one negative eigenvalue. □

(c) Write down the definition of positive definiteness. Is this matrix positive definite?

Solution: A positive definite matrix is one that has all eigenvalues > 0.

Therefore, our $\mathbf{A}$ is not positive definite. □

11. Singular Value Decomposition

(a) Consider the matrix:

$$\mathbf{A} = \frac{1}{5} \begin{bmatrix} 9 & 12 \\ 8 & -6 \end{bmatrix}.$$

Determine the singular value decomposition of $\mathbf{A}$, i.e., find the three matrices $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}^{\mathrm{T}}$ such that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$.

(Reminder: $\mathbf{A}\hat{v}_i = \sigma_i \hat{u}_i$ and $\mathbf{A}^\top \mathbf{A}\hat{v}_i = \sigma_i^2 \hat{v}_i$.)

Solution:

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \frac{1}{5} \begin{bmatrix} 145 & 60 \\ 60 & 180 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 29 & 12 \\ 12 & 36 \end{bmatrix}.$$

Sneaky trick #37: Find eigenvalues for the matrix $5\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 29 & 12 \\ 12 & 36 \end{bmatrix}$ and then divide them by 5 to find the eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

$0 = |5\mathbf{A}^{\mathrm{T}}\mathbf{A} - \lambda\mathbf{I}| = (29-\lambda)(36-\lambda) - 12^2 = \lambda^2 - 65\lambda + 1044 - 144 = \lambda^2 - 65\lambda + 900 = (\lambda - 45)(\lambda - 20)$. One can see the factorization just by making some reasonable guess, or by using the quadratic formula. The eigenvalues for $5\mathbf{A}^{\mathrm{T}}\mathbf{A}$ are therefore 45 and 20 and for $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ we then have $\lambda_1 = 45/5 = 9$ and $\lambda_2 = 20/5 = 4$.

Therefore, $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{4} = 2$.

Next task: we find the eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ to obtain the v vectors. (We must be careful with the factor of 5.)

$\lambda_1 = 9$: we have the nullspace problem

$$\vec{0} = (\mathbf{A}^{\mathrm{T}}\mathbf{A} - 9\mathbf{I})\vec{v}_1 = \left(\frac{1}{5}\begin{bmatrix} 29 & 12 \\ 12 & 36 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right)\vec{v}_1$$

$$= \frac{1}{5}\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix}\vec{v}_1.$$

We can see that $\hat{v}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ where we have correctly normalized the vector.

And since $\hat{v}_1 \perp \hat{v}_2$, we can also simply see that $\hat{v}_2 = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

The above then gives us

$$\mathbf{\Sigma} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \mathbf{V} = \mathbf{V}^{\mathrm{T}} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

Last, the connection $\mathbf{A}\hat{v}_i = \sigma_i\hat{u}_i$ gives

$$\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

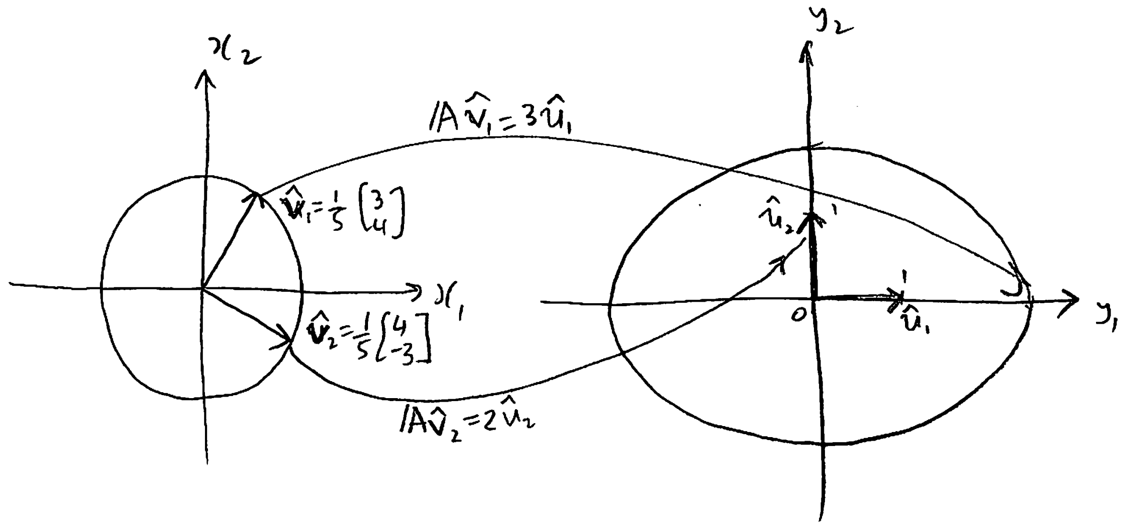
Multiplying everything together indeed gives $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$.

□

(b) The Big Picture: Illustrate how $\mathbf{A}$ maps between the happy basis vectors that are the $\hat{v}_i$'s and $\hat{u}_i$'s. (Please draw the particular Big Picture not the abstract Big picture.)

Complete your picture by adding a unit circle in row space and the ellipse that $\mathbf{A}$ creates in column space by transforming this circle.

Solution:



☐

12. (a) True or False (2 pts):

i. The nullspace of a nontrivial 1×3 matrix $\mathbf{A}$ is a 2-D plane in $\mathbb{R}^3$:
Solution: True. The equation of the plane is given by $\mathbf{A}\vec{x} = \vec{0}$. ☐

ii. The product $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is symmetric for any $n \times n$ matrix $\mathbf{A}$:
Solution: True. $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{\top} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}^{\top})^{\top} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$ ☐

iii. An $n \times n$ matrix cannot be diagonalized if one or more eigenvalues of $\mathbf{A}$ are 0:
Solution: False. Such a matrix has no inverse. Whether or not it is diagnolizable depends on its eigenvectors being a basis for R^n or not.
☐

iv. The matrix $\mathbf{M} = [\vec{v}_1|\vec{v}_2]$ transforms a vector's representation from the basis $\{\vec{v}_1, \vec{v}_2\}$ to the natural basis:
Solution: True. ☐

v. The determinant of a matrix $\mathbf{A}$ is equal to the sum of $\mathbf{A}$'s eigenvalues:
Solution: False. The determinant is equal to the product of the eigenvalues. ☐

vi. The matrices $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ have different eigenvalues:
Solution: False. $|\mathbf{A} - \lambda\mathbf{I}| = |(\mathbf{A} - \lambda\mathbf{I})^{\mathrm{T}}| = |\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}^{\mathrm{T}}| = |\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}|$.
☐

(b) Find the determinant of the following matrix (1 pt):

$$\mathbf{A}_n = \begin{bmatrix} \cos(1) & \cos(2) & \cdots & \cos(n) \\ \cos(n+1) & \cos(n+2) & \cdots & \cos(2n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(n(n-1)+1) & \cos(n(n-1)+2) & \cdots & \cos(n^2) \end{bmatrix}$$

Solution: We use multilinearity of determinants, the sum rule for cosines (i.e., $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$), and the fact that if two rows of a matrix are equal, then its determinant is 0.

We find that $\det(\mathbf{A}_n) = 0$ for $n \geq 3$. It's enough to see the general proof working with $\mathbf{A}_3$.

$$\det(\mathbf{A}_3) = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$$

$$= \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 3\cos 1 & \cos 3\cos 2 & \cos 3\cos 3 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix} - \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \sin 3\sin 1 & \sin 3\sin 2 & \sin 3\sin 3 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$$

$$= \cos 3 \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 1 & \cos 2 & \cos 3 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix} - \sin 3 \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \sin 1 & \sin 2 & \sin 3 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$$

(now work on the third row with the sum rule:)

$$= -\sin 3\cos 6 \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \sin 1 & \sin 2 & \sin 3 \\ \cos 1 & \cos 2 & \cos 3 \end{vmatrix} + \sin 3\sin 6 \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \sin 1 & \sin 2 & \sin 3 \\ \sin 1 & \sin 2 & \sin 3 \end{vmatrix}.$$

We can see that the above treatment works for all $n \geq 3$. Only the first three rows need to be manipulated to obtain the same result.

Also, we see that $|\mathbf{A}_1| = \cos 1 \neq 0$.

Therefore, $|\mathbf{A}_n| = 0$ for $n \geq 3$.

$\square$