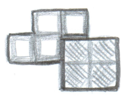
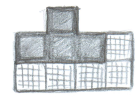
MATH 124: Matrixology (Linear Algebra)
Level Tetris (1984), 10 of 10
University of Vermont, Spring 2015

Dispersed: Wednesday, April 22, 2015.
Due: By start of lecture, Thursday, April 30, 2015.
Sections covered: 6.5, 6.7.

Some useful reminders:
Instructor: Prof. Peter Dodds
Office: Farrell Hall, second floor, Trinity Campus
E-mail: peter.dodds@uvm.edu
Office hours: 12:30 to 3:00 pm Mondays
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124
Textbook: "Introduction to Linear Algebra" (3rd or 4th edition) by Gilbert Strang (published by Wellesley-Cambridge Press).

- All questions are worth 3 points unless marked otherwise.
- Please use a cover sheet and write your name on the back and the front of your assignment.
- You must show all your work clearly.
- You may use Matlab to check your answers for non-Matlab questions (usually Qs. 1–8).
- Please list the names of other students with whom you collaborated.

Reminder: This assignment cannot be dropped.

1. (Q 4, 6.5) Show that the function $f(x_1, x_2) = x_1^2 + 4x_1x_2 + 3x_2^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients.

   Do this by rewriting $f(x_1, x_2)$ as $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and finding the pivots of $\mathbf{A}$ and noting their signs (and explaining why the signs of the pivots matter).

   Write f as a difference of squares and find a point (x_1, x_2) where f is negative.

   Note of caution: All of this signs matching for pivots and eigenvalues falls apart if we have to do row swaps in our reduction.

2. (Q 9, 6.5) Find the 3 by 3 matrix $\mathbf{A}$ and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

   Is this matrix positive definite, semi-positive definite, or neither?

3. (following set of questions based on Q 7, Section 6.7)

   Singular Value Decomposition = Happiness.

   Consider
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

   (a) What are m, n, and r for this matrix?

   (b) What are the dimensions of $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$?

   (c) Calculate $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$.

4. For the matrix $\mathbf{A}$ given above, find the eigenvalues and eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$, and thereby construct $\mathbf{V}$ and $\mathbf{\Sigma}$.

   See this tweet for some post-it based help:
   https://twitter.com/matrixologyvox/status/593540446845947904

5. For the same $\mathbf{A}$, now find the basis $\{\hat{u}_i\}$ using the essential connection $\mathbf{A}\hat{v}_i = \sigma_i \hat{u}_i$.

   Construct $\mathbf{U}$ from the basis you find.

   Again see this tweet for some post-it based help:
   https://twitter.com/matrixologyvox/status/593540446845947904

6. Next find the $\{\hat{u}_i\}$ in a different way by finding the eigenvalues and eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$.

7. (a) Put everything together and show that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$.

   (b) Draw the 'big picture' for this $\mathbf{A}$ showing which $\hat{v}_i$'s are mapped to which $\hat{u}_i$'s.

   (c) Which basis vectors, if any, belong to the two nullspaces?

8. Finally, for this same $\mathbf{A}$, perform the following calculation:
$$\sigma_1 \hat{u}_1 \hat{v}_1^{\mathrm{T}} + \sigma_2 \hat{u}_2 \hat{v}_2^{\mathrm{T}} + \ldots + \sigma_r \hat{u}_r \hat{v}_r^{\mathrm{T}}$$
   where r is the rank of $\mathbf{A}$.

   You should obtain $\mathbf{A}$...

9. Matlab question.

   Verify the signs you found for the pivots of $\mathbf{A}$ in question 1 by using Matlab to find $\mathbf{A}$'s eigenvalues.

10. Matlab question.

    Use Matlab to compute the SVD for the matrix $\mathbf{A}$ you explored in questions 3–8.

11. (The bonus one pointer)

    Where does the fearsome kiwi rank among among rattites and what's unusual about the kiwi egg?