P o C S What's The Story?

Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2014
Assignment 7 • code name: Quoth the Raven

Dispersed: Monday, October 20, 2014.
Due: By start of lecture, 1:00 pm, Thursday, October 30, 2014.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 2:30 pm to 3:45 pm on Tuesday, 12:30 pm to 2:00 pm on Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2014-08UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your workingses clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. (3 + 3):

   Consider a modified version of the Barabàsi-Albert (BA) model [1] where two possible mechanisms are now in play. As in the original model, start with m_0 nodes at time $t = 0$. Let's make these initial guys connected such that each has degree 1. The two mechanisms are:

   M1: With probability p, a new node of degree 1 is added to the network. At time $t + 1$, a node connects to an existing node j with probability

   $$P(\text{connect to node } j) = \frac{k_j}{\sum_{i=1}^{N(t)} k_i} \qquad (1)$$

   where k_j is the degree of node j and $N(t)$ is the number of nodes in the system at time t.

   M2: With probability $q = 1 - p$, a randomly chosen node adds a new edge, connecting to node j with the same preferential attachment probability as above.

   Note that in the limit $q = 0$, we retrieve the original BA model (with the difference that we are adding one link at a time rather than m here).

   In the long time limit $t \to \infty$, what is the expected form of the degree distribution P_k?

   Do we move out of the original model's universality class?

Different analytic approaches are possible including a modification of the BA paper, or a Simon-like one (see also Krapivsky and Redner [2]).

Hint: You can attempt to solve the problem exactly and you'll find an integrating factor story.

Another hint, moment of mercy: Approximate the differential equation by considering large t (this will simplify the denominators).

(3 points for set up, 3 for solving.)

2. (3 + 3)

   More on the peculiar nature of distributions of power law tails:

   Consider a set of N samples, randomly chosen according to the probability distribution $P_k = ck^{-\gamma}$ where $k \geq 1$ and $2 < \gamma < 3$. (Note that k is discrete rather than continuous.)

   (a) Estimate $\min k_{\max}$, the approximate minimum of the largest sample in the network, finding how it depends on N.

       (Hint: we expect on the order of 1 of the N samples to have a value of $\min k_{\max}$ or greater.)

       Hint—Some visual help on setting this problem up:
       `http://www.youtube.com/v/4tqlEuXA7QQ?rel=0`

   (b) Determine the average value of samples with value $k \geq \min k_{\max}$ to find how the expected value of $k_{\max}$ (i.e., $\langle k_{\max} \rangle$) scales with N.

       For language, this scaling is known as Heap's law.

3. (3 + 3)

   Let's see how well your answer for the previous question works.

   For $\gamma = 5/2$, generate $n = 1000$ sets each of $N = 10$, 10^2, 10^3, 10^4, 10^5, and 10^6 samples, using $P_k = ck^{-5/2}$ with $k = 1, 2, 3, \ldots$

   Question: how do we computationally sample from a discrete probability distribution? Does our approach differ for large and small spaces?

   Hint: A continuum approximation will help.

   (a) For each value of sample size N, plot the maximum value of the $n = 1000$ samples as a function of sample number (which is not the sample size N). So you should have $k_{\max}$ for $i = 1, 2, \ldots, n$ where i is sample number. These plots should give a sense of the unevenness of the maximum value of k, a feature of power-law size distributions.

(b) For each set, find the maximum value. Then find the average maximum value for each N. Plot $\langle k_{\mathrm{max}} \rangle$ as a function of N and calculate the scaling using least squares.

Does your scaling match up with your theoretical estimate?

References

[1] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286:509–511, 1999. pdf

[2] P. L. Krapivsky and S. Redner. Organization of growing random networks. *Phys. Rev. E*, 63:066123, 2001. pdf