Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2013
Assignment 6 • code name: Obelix (⊞)

Dispersed: Thursday, October 10, 2013.
Due: By start of lecture, 1:00 pm, Thursday, October 17, 2013.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 10:30 am to 11:30 am, Monday, and 1:00 pm to 3:00 pm, Wednesday
Course website: http://www.uvm.edu/~pdodds/teaching/courses/2013-08UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. (3 + 3)

   Determine the clustering coefficient for toy model small-world networks [1] as a function of the rewiring probability p. Find C_1, the average local clustering coefficient:

   $$C_1(p) = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i = \frac{1}{N} \sum_{i=1}^{N} \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

   where N is the number of nodes, $a_{ij} = 1$ if nodes i and j are connected, and $\mathcal{N}_i$ indicates the neighborhood of i.

   As per the original model, assume a ring network with each node connected to a fixed, even number m local neighbors ($m/2$ on each side). Take the number of nodes to be $N \gg m$.

   Start by finding $C_1(0)$ and argue for a $(1-p)^3$ correction factor to find an approximation of $C_1(p)$.

   Hint 1: you can think of finding C_1 as averaging over the possibilities for a single node.

   Hint 2: assume that the degree of individual nodes does not change with rewiring but rather stays fixed at m. In other words, take the average degree of individuals as the degree of a randomly selected individual.

   For what value of p is $C_1 \simeq 1/2$?

   (3 points for set up, 3 for solving.)

2. (3 + 3)

   More on the peculiar nature of distributions of power law tails:

   Consider a set of N samples, randomly chosen according to the probability distribution $P_k = ck^{-\gamma}$ where $k \geq 1$ and $2 < \gamma < 3$. (Note that k is discrete rather than continuous.)

   (a) Estimate $\min k_{\max}$, the approximate minimum of the largest sample in the network, finding how it depends on N.

   (Hint: we expect on the order of 1 of the N samples to have a value of $\min k_{\max}$ or greater.)
   Hint—Some visual help on setting this problem up:

   Direct link: `http://www.youtube.com/v/4tqlEuXA7QQ?rel=0`

   (b) Determine the average value of samples with value $k \geq \min k_{\max}$ to find how the expected value of $k_{\max}$ (i.e., $\langle k_{\max} \rangle$) scales with N.

   For language, this scaling is known as Heap's law.

3. (3 + 3)

   Let's see how well your answer for the previous question works.

   For $\gamma = 5/2$, generate $n = 1000$ sets each of $N = 10$, 10^2, 10^3, 10^4, 10^5, and 10^6 samples, using $P_k = ck^{-5/2}$ with $k = 1, 2, 3, \ldots$

   Question: how do we computationally sample from a discrete probability distribution?

   (a) For each value of sample size N, plot the maximum value of the $n = 1000$ samples as a function of sample number (which is not the sample size N).

So you should have $k_{\max}$ for $i = 1, 2, \ldots, n$ where i is sample number. These plots should give a sense of the unevenness of the maximum value of k, a feature of power-law size distributions.

(b) For each set, find the maximum value. Then find the average maximum value for each N. Plot $\langle k_{\max} \rangle$ as a function of N and calculate the scaling using least squares.

Does your scaling match up with your theoretical estimate?

References

[1] D. J. Watts and S. J. Strogatz. Collective dynamics of 'small-world' networks. Nature, 393:440–442, 1998.