Mechanisms for Generating Power-Law Size Distributions I

Principles of Complex Systems
CSYS/MATH 300, Fall, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics | Center for Complex Systems | Vermont Advanced Computing Center | University of Vermont

Outline

Mechanisms

A powerful story in the rise of complexity:

- structure arises out of randomness.
- Exhibit A: Random walks... (⊞)

Random walks

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Random walks

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

Random walks

Variances sum: (⊞)*

$$\mathrm{Var}(x_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \mathrm{Var}(\epsilon_i) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

Random walks

So typical displacement from the origin scales as

$$\sigma = t^{1/2}$$

$\Rightarrow$ A non-trivial power-law arises out of additive aggregation or accumulation.

Random walks

Random walks are weirder than you might think...

For example:

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.

See Feller, [2] Intro to Probability Theory, Volume I

Random walks

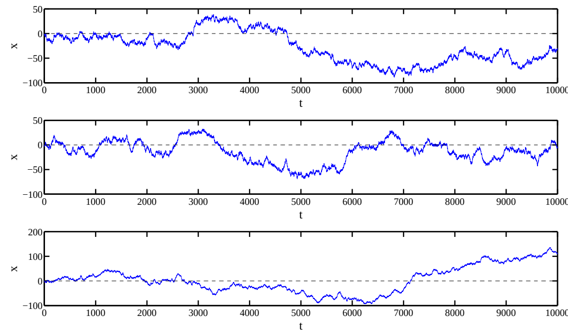
In fact:

$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

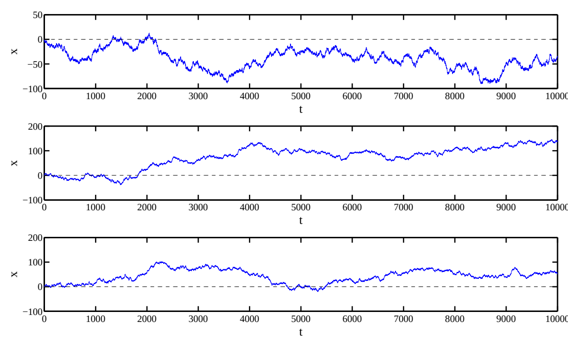
Even crazier:

The expected time between tied scores = ∞!

Random walks—some examples



Random walks—some examples



Random walks

The problem of first return:

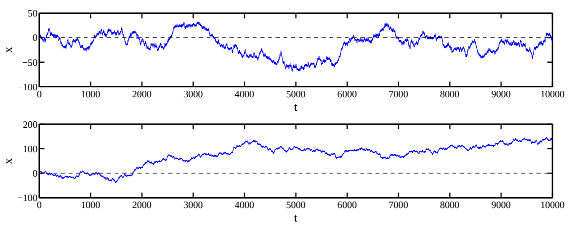
- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

First returns

Reasons for caring:

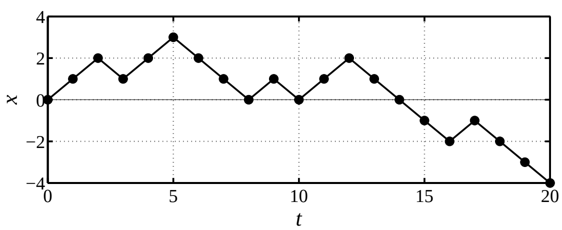
1. We will find a power-law size distribution with an interesting exponent
2. Some physical structures may result from random walks
3. We'll start to see how different scalings relate to each other

Random Walks



Again: expected time between ties = ∞...
Let's find out why...[2]

First Returns



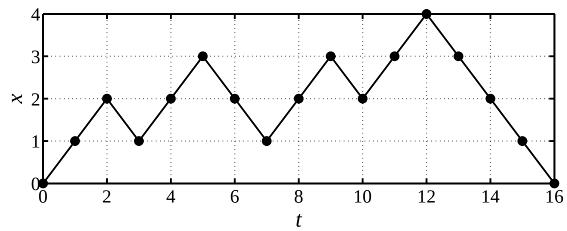
First Returns

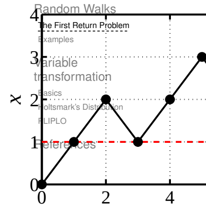
For random walks in 1-d:

- Return can only happen when $t = 2n$.
- Call $P_{\text{first return}}(2n) = P_{\text{fr}}(2n)$ probability of first return at $t = 2n$.
- Assume drunkard first lurches to $x = 1$.
- The problem

$$P_{\text{fr}}(2n) = 2Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_{2n} = 0)$$

First Returns





- A useful restatement: $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- Want walks that can return many times to $x = 1$.
- (The $\frac{1}{2}$ accounts for stepping to 2 instead of 0 at $t = 2n$.)

First Returns

- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define $N(i, j, t)$ as the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Subtract how many hit $x = 0$.

First Returns

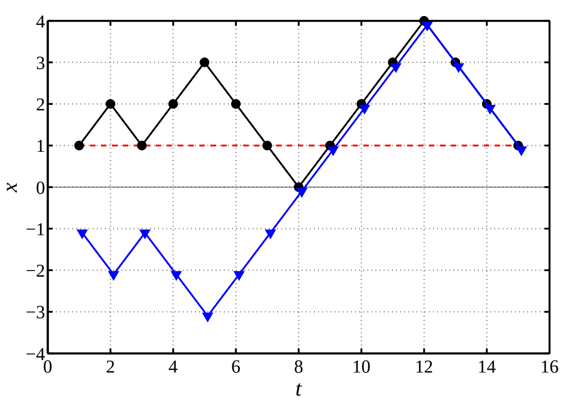
Key observation:
\# of t-step paths starting and ending at $x = 1$
and hitting $x = 0$ at least once
= # of t-step paths starting at $x = -1$ and ending at $x = 1$
= $N(-1, 1, t)$

So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

See this 1-1 correspondence visually...

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



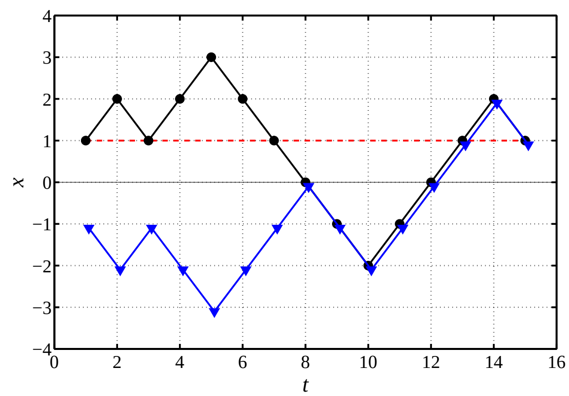
First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- For any path starting at $x = 1$ that hits 0, there is a unique matching path starting at $x = -1$.
- Matching path first mirrors and then tracks.

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- Next problem: what is $N(i, j, t)$?
- \# positive steps + # negative steps = t.
- Random walk must displace by $j - i$ after t steps.
- \# positive steps - # negative steps = $j - i$.
- \# positive steps = $(t + j - i)/2$.
- $$N(i, j, t) = \binom{t}{\#\text{ positive steps}} = \binom{t}{(t + j - i)/2}$$

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

We now have

$$N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$$

where

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

First Returns

Insert question from assignment 4 (⊞)

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$.

- Normalized Number of Paths gives Probability
- Total number of possible paths = 2^{2n}
- $$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$
$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$
$$= \frac{1}{\sqrt{2\pi}}(2n)^{-3/2}$$

First Returns

- Same scaling holds for continuous space/time walks.
- $$P(t) \propto t^{-3/2}, \; \gamma = 3/2$$
- $P(t)$ is normalizable
- Recurrence: Random walker always returns to origin
- Moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

First Returns

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...

Random walks

On finite spaces:

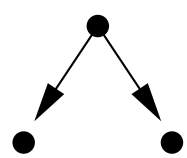
- In any finite volume, a random walker will visit every site with equal probability
- Random walking $\equiv$ Diffusion
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

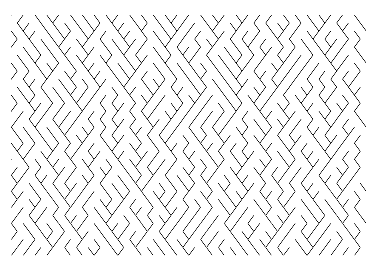
Random walks on

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree
- Equal probability still present:
walkers traverse edges with equal frequency.

Scheidegger Networks [4, 1]





- Triangular lattice
- ‘Flow’ is southeast or southwest with equal probability.

Scheidegger Networks

- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$

Connections between Exponents

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3} \mathrm{d}a$
- $Pr(\text{basin area} = a)\mathrm{d}a$
  $= Pr(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-3/2}\mathrm{d}\ell$
  $\propto (a^{2/3})^{-3/2} a^{-1/3}\mathrm{d}a$
  $= a^{-4/3}\mathrm{d}a$
  $= a^{-\tau}\mathrm{d}a$

Connections between Exponents

- Both basin area and length obey power law distributions
- Observed for real river networks
- Typically: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$
- Smaller basins more allometric ($h > 1/2$)
- Larger basins more isometric ($h = 1/2$)

Connections between Exponents

- Generalize relationship between area and length
- Hack's law [3]:

$$\ell \propto a^h$$

  where $0.5 \lesssim h \lesssim 0.7$
- Redo calc with γ, τ, and h.

Connections between Exponents

- Given

$$\ell \propto a^h, \ P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1}\mathrm{d}a$
- $Pr(\text{basin area} = a)\mathrm{d}a$
  $= Pr(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
- $$\boxed{\tau = 1 + h(\gamma - 1)}$$

Connections between Exponents

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

- Only one exponent is independent
- Simplify system description
- Expect scaling relations where power laws are found
- Characterize universality class with independent exponents

Other First Returns

Failure

- A very simple model of failure/death:
- x_t = entity's 'health' at time t
- x_0 could be > 0.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

More than randomness

Power-Law Mechanisms I
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- Can generalize to Fractional Random Walks
- Levy flights, Fractional Brownian Motion
- In 1-d,

$$\sigma \sim t^{\alpha}$$

  $\alpha > 1/2$ — superdiffusive
  $\alpha < 1/2$ — subdiffusive
- Extensive memory of path now matters...

Variable Transformation

Power-Law Mechanisms I
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Understand power laws as arising from

1. elementary distributions (e.g., exponentials)
2. variables connected by power relationships

Variable Transformation

Power-Law Mechanisms I
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x = \sum_{y|f(x)=y} P_x(f^{-1}(y))\frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$ Figure...
- Often easier to do by hand...

General Example

Power-Law Mechanisms I
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Assume relationship between x and y is 1-1.

- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$
  $$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$
  $$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$
  $$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$
  $$\mathrm{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$

General Example

Power-Law Mechanisms I
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

$$P_y(y)\mathrm{d}y = P_x\overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)}\overbrace{\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then
  $$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$
- If $P_x(x) \rightarrow x^{\beta}$ as $x \rightarrow 0$ then
  $$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness...
- More later when we cover robustness.

Gravity

- Select a random point in the universe $\vec{x}$
- (possible all of space-time)
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

Ingredients [5]

Matter is concentrated in stars:

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:
$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$
- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:
$$F \propto r^{-2}$$
- invert:
$$r \propto F^{-1/2}$$

Transformation

- $$\mathrm{d}F \propto \mathrm{d}(r^{-2})$$
- $$\propto r^{-3}\mathrm{d}r$$
- invert:
$$\mathrm{d}r \propto r^3\mathrm{d}F$$
- $$\propto F^{-3/2}\mathrm{d}F$$

Transformation

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$ and $\boxed{P_r(r) \propto r^2}$

- $$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$
- $$\propto P_r(F^{-1/2})F^{-3/2}\mathrm{d}F$$
- $$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$
- $$= F^{-1-3/2}\mathrm{d}F$$
- $$= F^{-5/2}\mathrm{d}F$$

Gravity

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite
- Variance = ∞
- A wild distribution
- Random sampling of space usually safe but can end badly...

Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (⊞)...
- Don't do this!!! (slap, slap)
- We need mechanisms!

References I

Power-Law Mechanisms I
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

[1] P. S. Dodds and D. H. Rothman.
Scaling, universality, and geomorphology.
Annu. Rev. Earth Planet. Sci., 28:571–610, 2000.
pdf (⊞)

[2] W. Feller.
An Introduction to Probability Theory and Its Applications, volume I.
John Wiley & Sons, New York, third edition, 1968.

[3] J. T. Hack.
Studies of longitudinal stream profiles in Virginia and Maryland.
United States Geological Survey Professional Paper, 294-B:45–97, 1957. pdf (⊞)

References II

Power-Law Mechanisms I
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

[4] A. E. Scheidegger.
The algebra of stream-order numbers.
United States Geological Survey Professional Paper, 525-B:B187–B189, 1967. pdf (⊞)

[5] D. Sornette.
Critical Phenomena in Natural Sciences.
Springer-Verlag, Berlin, 2nd edition, 2003.