

Mechanisms for Generating Power-Law Size Distributions I

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A powerful story in the rise of complexity:

- ▶ structure arises out of randomness.
- ▶ Exhibit A: Random walks... (田)



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The essential random walk:

- ▶ One spatial dimension.
- ▶ Time and space are discrete
- ▶ Random walker (e.g., a drunk) starts at origin $x = 0$.
- ▶ Step at time t is ϵ_t :

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$



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Displacement after t steps:

$$x_t = \sum_{i=1}^t \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^t \epsilon_i \right\rangle = \sum_{i=1}^t \langle \epsilon_i \rangle = 0$$



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Variations sum: $(\boxplus)^*$

$$\begin{aligned}\text{Var}(x_t) &= \text{Var}\left(\sum_{i=1}^t \epsilon_i\right) \\ &= \sum_{i=1}^t \text{Var}(\epsilon_i) = \sum_{i=1}^t 1 = t\end{aligned}$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.



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So typical displacement from the origin scales as

$$\sigma = t^{1/2}$$

⇒ A non-trivial power-law arises out of
additive aggregation or accumulation.



Random walks are weirder than you might think...

For example:

- ▶ $\xi_{r,t}$ = the probability that by time step t , a random walk has crossed the origin r times.
- ▶ Think of a coin flip game with ten thousand tosses.
- ▶ If you are behind early on, what are the chances you will make a comeback?
- ▶ The most likely number of lead changes is... **0**.

See Feller, ^[2] Intro to Probability Theory, Volume I



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In fact:

$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \dots$$

Even crazier:

The expected time between tied scores = ∞ !



Random walks—some examples

Random Walks

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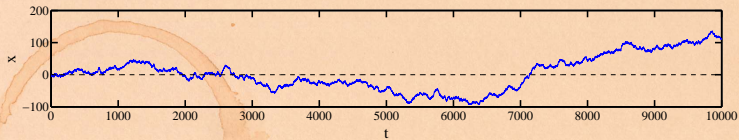
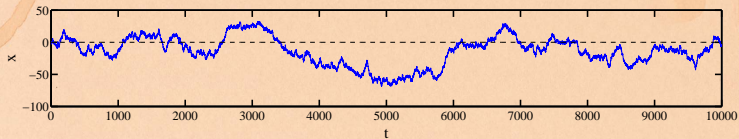
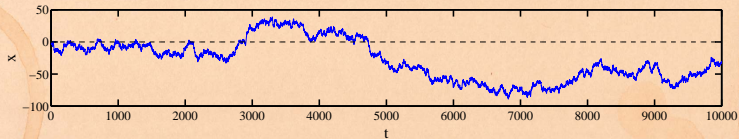
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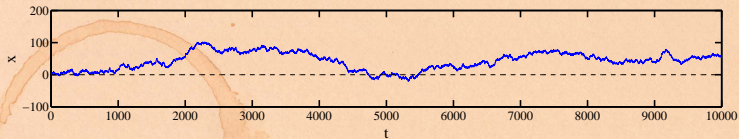
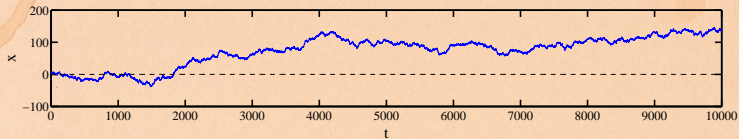
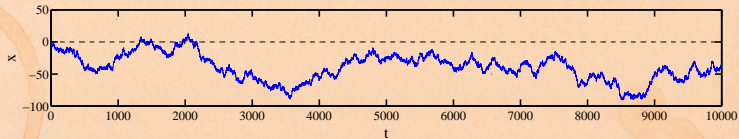
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The problem of first return:

- ▶ What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- ▶ Will our drunkard always return to the origin?
- ▶ What about higher dimensions?



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Reasons for caring:

1. We will find a power-law size distribution with an **interesting** exponent
2. Some physical structures may result from random walks
3. We'll start to see how different scalings relate to each other



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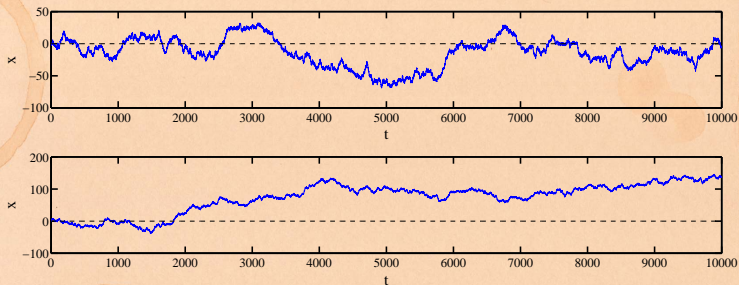
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Again: expected time between ties = ∞ ...

Let's find out why... [2]



First Returns

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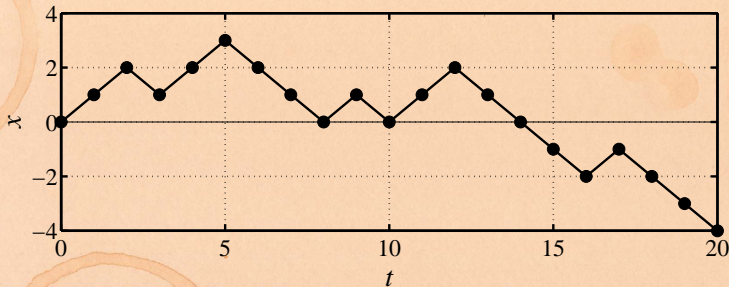
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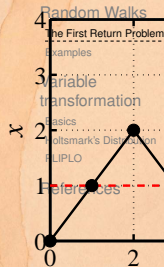
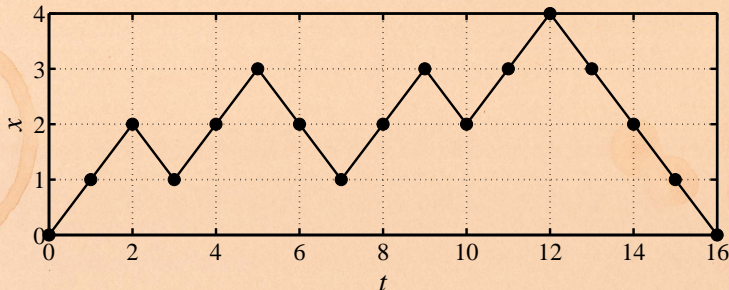
For random walks in 1-d:

- ▶ Return can only happen when $t = 2n$.
- ▶ Call $P_{\text{first return}}(2n) = P_{\text{fr}}(2n)$ probability of first return at $t = 2n$.
- ▶ Assume drunkard first lurches to $x = 1$.
- ▶ The problem

$$P_{\text{fr}}(2n) = 2Pr(x_t \geq 1, t = 1, \dots, 2n - 1, \text{ and } x_{2n} = 0)$$



First Returns



- ▶ A useful restatement: $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} \Pr(x_t \geq 1, t = 1, \dots, 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- ▶ Want walks that can return many times to $x = 1$.
- ▶ (The $\frac{1}{2}$ accounts for stepping to 2 instead of 0 at $t = 2n$.)

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- ▶ Counting problem (combinatorics/statistical mechanics)
- ▶ Use a method of images
- ▶ Define $N(i, j, t)$ as the # of possible walks between $x = i$ and $x = j$ taking t steps.
- ▶ Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- ▶ Subtract how many hit $x = 0$.



First Returns

Key observation:

of t -step paths starting and ending at $x = 1$
and hitting $x = 0$ at least once

= # of t -step paths starting at $x = -1$ and ending at $x = 1$

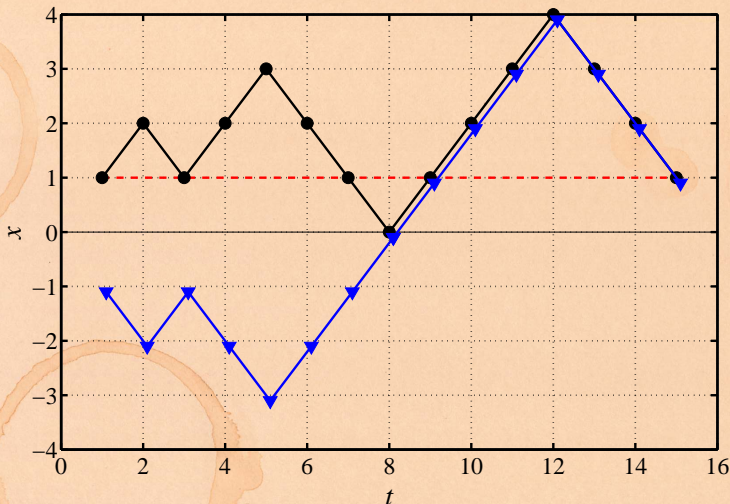
= $N(-1, 1, t)$

So $N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$

See this 1-1 correspondence visually...



First Returns



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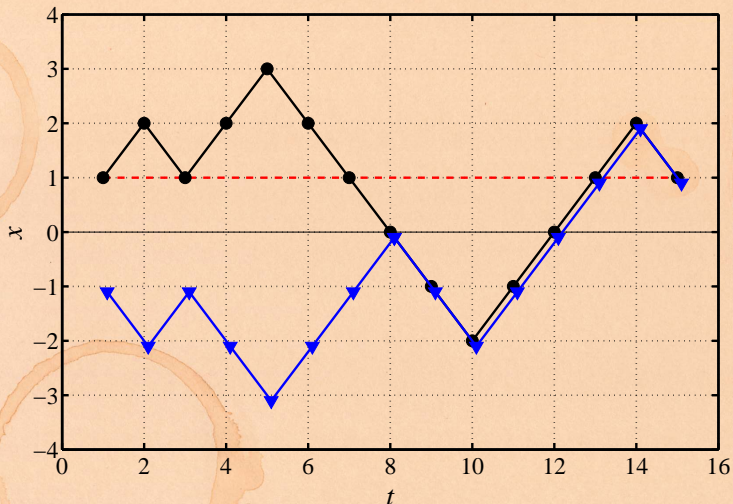
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- ▶ For any path starting at $x = 1$ that hits 0, there is a unique matching path starting at $x = -1$.
- ▶ Matching path first mirrors and then tracks.



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- ▶ Next problem: what is $N(i, j, t)$?
- ▶ # positive steps + # negative steps = t .
- ▶ Random walk must displace by $j - i$ after t steps.
- ▶ # positive steps - # negative steps = $j - i$.
- ▶ # positive steps = $(t + j - i)/2$.
- ▶

$$N(i, j, t) = \binom{t}{\# \text{ positive steps}} = \binom{t}{(t + j - i)/2}$$



We now have

$$N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$$

where

$$N(i, j, t) = \binom{t}{(t+j-i)/2}$$



Insert question from assignment 4 (田)

$$\text{Find } N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}.$$

- ▶ Normalized Number of Paths gives Probability
- ▶ Total number of possible paths = 2^{2n}
- ▶

$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}} (2n)^{-3/2}$$

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- ▶ Same scaling holds for continuous space/time walks.

$$P(t) \propto t^{-3/2}, \quad \gamma = 3/2$$

- ▶ $P(t)$ is normalizable
- ▶ **Recurrence**: Random walker always returns to origin
- ▶ **Moral**: Repeated gambling against an infinitely wealthy opponent must lead to ruin.



Higher dimensions:

- ▶ Walker in $d = 2$ dimensions must also return
- ▶ Walker may not return in $d \geq 3$ dimensions
- ▶ For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- ▶ Even though walker must return, expect a long wait...



On finite spaces:

- ▶ In any finite volume, a random walker will visit every site with equal probability
- ▶ Random walking \equiv Diffusion
- ▶ Call this probability the **Invariant Density** of a dynamical system
- ▶ Non-trivial Invariant Densities arise in chaotic systems.



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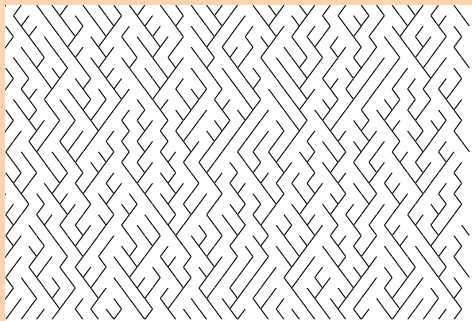
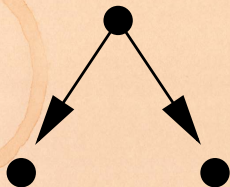
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On networks:

- ▶ On networks, a random walker visits each node with frequency \propto node degree
- ▶ Equal probability still present: walkers traverse **edges** with equal frequency.





- ▶ Triangular lattice
- ▶ 'Flow' is southeast or southwest with equal probability.

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- ▶ Creates basins with random walk boundaries
- ▶ **Observe** Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- ▶ Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$



Connections between Exponents

- ▶ For a basin of length l , width $\propto l^{1/2}$
- ▶ Basin area $a \propto l \cdot l^{1/2} = l^{3/2}$
- ▶ Invert: $l \propto a^{2/3}$
- ▶ $dl \propto d(a^{2/3}) = 2/3 a^{-1/3} da$
- ▶ $Pr(\text{basin area} = a) da$
 $= Pr(\text{basin length} = l) dl$
 $\propto l^{-3/2} dl$
 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$
 $= a^{-4/3} da$
 $= a^{-\tau} da$

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Connections between Exponents

- ▶ Both basin area and length obey power law distributions
- ▶ Observed for real river networks
- ▶ Typically: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$
- ▶ Smaller basins more allometric ($h > 1/2$)
- ▶ Larger basins more isometric ($h = 1/2$)

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Connections between Exponents

- ▶ Generalize relationship between area and length
- ▶ Hack's law^[3]:

$$l \propto a^h$$

where $0.5 \lesssim h \lesssim 0.7$

- ▶ Redo calc with γ , τ , and h .



Connections between Exponents

▶ Given

$$l \propto a^h, P(a) \propto a^{-\tau}, \text{ and } P(l) \propto l^{-\gamma}$$

▶ $dl \propto d(a^h) = ha^{h-1} da$

▶ $Pr(\text{basin area} = a) da$
 $= Pr(\text{basin length} = l) dl$
 $\propto l^{-\gamma} dl$
 $\propto (a^h)^{-\gamma} a^{h-1} da$
 $= a^{-(1+h(\gamma-1))} da$

$$\tau = 1 + h(\gamma - 1)$$



Connections between Exponents

With more detailed description of network structure,
 $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

- ▶ Only one exponent is independent
- ▶ Simplify system description
- ▶ Expect scaling relations where power laws are found
- ▶ Characterize universality class with independent exponents

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Failure

- ▶ A very simple model of failure/death:
- ▶ x_t = entity's 'health' at time t
- ▶ x_0 could be > 0 .
- ▶ Entity fails when x hits 0.

Streams

- ▶ Dispersion of suspended sediments in streams.
- ▶ Long times for clearing.



More than randomness

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- ▶ Can generalize to Fractional Random Walks
- ▶ Levy flights, Fractional Brownian Motion
- ▶ In 1-d,

$$\sigma \sim t^\alpha$$

$\alpha > 1/2$ — **superdiffusive**

$\alpha < 1/2$ — **subdiffusive**

- ▶ Extensive memory of path now matters...



Variable Transformation

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Understand power laws as arising from

1. elementary distributions (e.g., exponentials)
2. variables connected by power relationships



Variable Transformation

- ▶ Random variable X with known distribution P_X
- ▶ Second random variable Y with $y = f(x)$.
- ▶ $P_Y(y)dy = P_X(x)dx$
=
 $\sum_{y|f(x)=y} P_X(f^{-1}(y)) \frac{dy}{|f'(f^{-1}(y))|}$ Figure...
- ▶ Often easier to do by hand...

General Example

Assume relationship between x and y is 1-1.

- ▶ Power-law relationship between variables:

$$y = cx^{-\alpha}, \alpha > 0$$

- ▶ Look at y large and x small



$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

invert: $dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$

$$dx = \frac{-1}{c\alpha} \left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha} dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy$$



General Example

Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x \left(\overbrace{\left(\frac{y}{c} \right)^{-1/\alpha}}^{(x)} \right) \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha} dy}^{dx}$$

- ▶ If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- ▶ If $P_x(x) \rightarrow x^\beta$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- ▶ Exponentials arise from randomness...
- ▶ More later when we cover robustness.



Gravity

- ▶ Select a random point in the universe \vec{x}
- ▶ (possible all of space-time)
- ▶ Measure the force of gravity $F(\vec{x})$
- ▶ Observe that $P_F(F) \sim F^{-5/2}$.



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Matter is concentrated in stars:

- ▶ F is distributed unevenly
- ▶ Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)dr \propto r^2 dr$$

- ▶ Assume stars are distributed randomly in space (oops?)
- ▶ Assume only one star has significant effect at \vec{x} .
- ▶ Law of gravity:

$$F \propto r^{-2}$$

- ▶ **invert:**

$$r \propto F^{-1/2}$$

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Transformation



▶ invert:



$$dF \propto d(r^{-2})$$

$$\propto r^{-3} dr$$

$$dr \propto r^3 dF$$

$$\propto F^{-3/2} dF$$



Transformation

Using $r \propto F^{-1/2}$, $dr \propto F^{-3/2}dF$ and $P_r(r) \propto r^2$

$$P_F(F)dF = P_r(r)dr$$

$$\propto P_r(F^{-1/2})F^{-3/2}dF$$

$$\propto (F^{-1/2})^2 F^{-3/2}dF$$

$$= F^{-1-3/2}dF$$

$$= F^{-5/2}dF$$

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$$P_F(F) = F^{-5/2} dF$$

$$\gamma = 5/2$$

- ▶ Mean is finite
- ▶ Variance = ∞
- ▶ A **wild** distribution
- ▶ Random sampling of space usually safe but can end badly...



Caution!

- ▶ **PLIPLO** = Power law in, power law out
- ▶ Explain a power law as resulting from another unexplained power law.
- ▶ Yet another homunculus argument (田)...
- ▶ **Don't do this!!!** (slap, slap)
- ▶ We need mechanisms!



References I

- [1] P. S. Dodds and D. H. Rothman.
Scaling, universality, and geomorphology.
[Annu. Rev. Earth Planet. Sci., 28:571–610, 2000.](#)
[pdf](#) (田)
- [2] W. Feller.
An Introduction to Probability Theory and Its Applications, volume I.
John Wiley & Sons, New York, third edition, 1968.
- [3] J. T. Hack.
Studies of longitudinal stream profiles in Virginia and Maryland.
[United States Geological Survey Professional Paper, 294-B:45–97, 1957.](#) [pdf](#) (田)



References II

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References

- [4] A. E. Scheidegger.
The algebra of stream-order numbers.
[United States Geological Survey Professional Paper, 525-B:B187–B189, 1967. pdf \(田\)](#)
- [5] D. Sornette.
[Critical Phenomena in Natural Sciences.](#)
Springer-Verlag, Berlin, 2nd edition, 2003.

